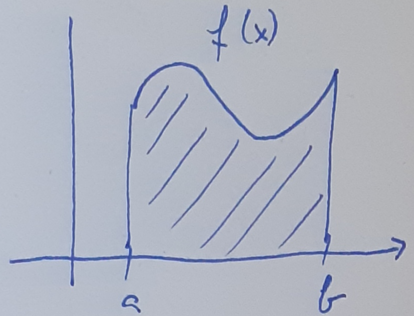
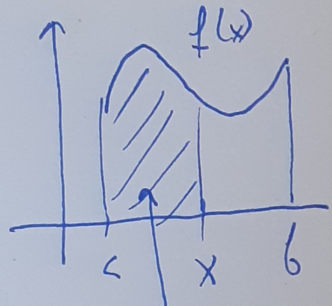
INTEGRÁLSZÁMÍTÁS

Feladat: Legyen $f(x)$ egy folytonos és nemnegatív fv $[a,b]$-n. Határozza meg az $f(x)$ grafikonja alatti területet!



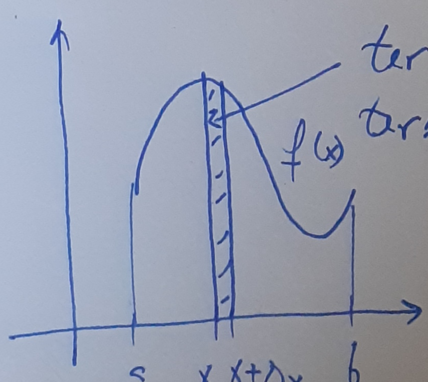
$\text{ter} = ?$

Legyen $a \le x \le b$. Jelölje $\text{ter}(x)$ az alábbi területet:



$\text{ter} = \text{ter}(x)$

Ekkor ha $\Delta x > 0$, Δx pici:



$\text{ter} = \text{ter}(x+\Delta x) - \text{ter}(x)$

$\text{ter} \approx f(x)\cdot\Delta x$, mert ter jól közelíthető egy Δx alapú $f(x)$ magasságú téglalap területével.

Így $\text{ter}(x+\Delta x) - \text{ter}(x) \approx f(x)\Delta x$, azaz $\dfrac{\text{ter}(x+\Delta x) - \text{ter}(x)}{\Delta x} \approx f(x)$.

Pontosan $\lim\limits_{\Delta x \to 0} \dfrac{\text{ter}(x+\Delta x) - \text{ter}(x)}{\Delta x} = f(x)$.

Így $ter'(x) = f(x)$.

__Def__ Az $f(x)$ fv primitív fve a $F(x)$, ha $F'(x) = f(x)$ minden $x \in [a,b]$.

__Megj.__ 1. Ha $F(x)$ primitív fve $f(x)$-nek és $c \in \mathbb{R}$, akkor $F(x)+c$ is primitív fv, mert $(F(x)+c)' = F(x)' = f(x)$.

2. Ha $F(x)$ és $G(x)$ is primitív fve $f(x)$-nek, akkor minden $x \in [a,b]$ esetén $(G(x)-F(x))' = f(x) - f(x) = 0$, ezért létezik $c \in \mathbb{R}$, hogy $G(x) - F(x) = c$, azaz $G(x) = F(x) + c$ minden $a \le x \le b$.

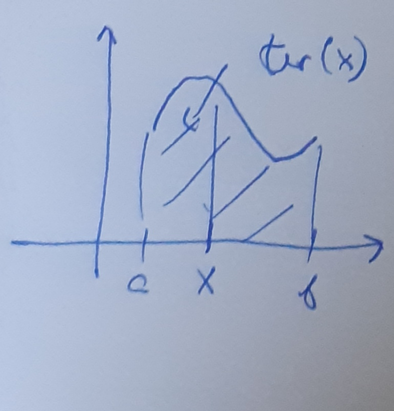
__Def__ Az $f(x)$ fv határozatlan integrálja a primitív fveinek halmaza. Jelölés: $\int f(x)dx$

Ha $f(x)$ egy primitív fve $F(x)$, akkor

$$\int f(x)dx = \{F(x)+c : c \in \mathbb{R}\} \overset{\text{röviden}}{=} F(x)+c$$

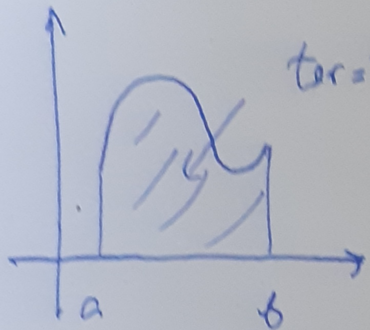
Pl. $\int x\,dx = \frac{x^2}{2} + c$, mert $\left(\frac{x^2}{2}\right)' = \frac{2x}{2} = x$.

Vissza a feladathoz: $ter(x)$ az $f(x)$ egy primitív fve. Legyen $F(x)$ az $f(x)$ egy primitív fve. Ekkor létezik $c \in \mathbb{R}$, amire $ter(x) = F(x) + c$. $c = ?$



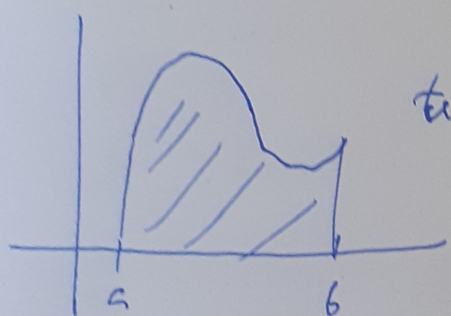
Nyilván $\left.\begin{array}{l} ter(a) = 0 \\ ter(a) = F(a) + c \end{array}\right\} \Rightarrow F(a) + c = 0 \Rightarrow c = -F(a)$

Tehát $ter(x) = F(x) - F(a)$



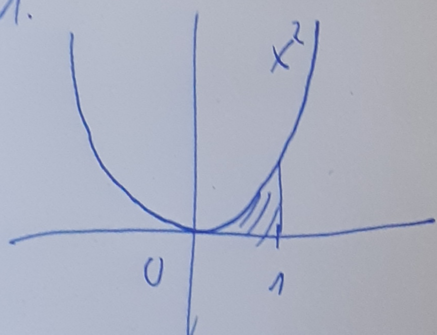
$\text{ter} = \text{ter}(b) = F(b) - F(a)$

Newton–Leibniz tétel: Legyen $f(x) \geq 0$ folytonos $[a,b]$-ben, $f(x)$ egy primitív fvе $F(x)$. Ekkor



$\text{ter} = F(b) - F(a) = \left[F(x)\right]_a^b$.

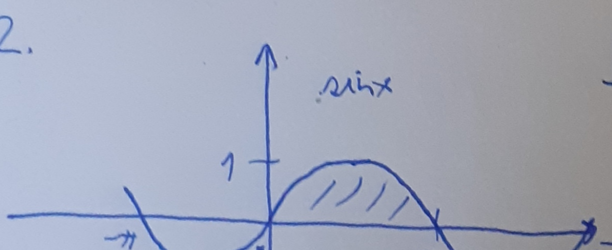
Pl. 1.



$\int x^2\,dx = \frac{x^3}{3} + c$

$\text{ter} = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$

2.



$\int \sin x\,dx = -\cos x + c$, mert

$(-\cos x)' = -(-\sin x) = \sin x$

$\text{ter} = \left[-\cos x\right]_0^\pi = -\overset{-1}{\cos\pi} - (-\overset{1}{\cos 0}) = 2$

Alapintegrálok

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, ha $n = -1$, mert $(x^{n+1})' = (n+1) \cdot x^n$.

2. $\int \frac{1}{x} dx = \ln|x| + c$, mert ha $\begin{cases} x>0: (\ln|x|)' = (\ln x)' = \frac{1}{x} \\ x<0: (\ln|x|)' = (\ln(-x))' = \frac{1}{-x} \cdot (-1) = \frac{1}{x} \end{cases}$

3. $\int a^x dx = \frac{a^x}{\ln a} + c$, ha $a>0$, $a \neq 1$, mert $(a^x)' = a^x \ln a$

4. $\int \operatorname{ch} x \, dx = \operatorname{sh} x + c$

5. $\int \operatorname{sh} x \, dx = \operatorname{ch} x + c$

6. $\int \sin x \, dx = -\cos x + c$

7. $\int \cos x \, dx = \sin x + c$

8. $\int \frac{1}{\cos^2 x} dx = \operatorname{tg} x + c$

9. $\int \frac{1}{\sin^2 x} dx = -\operatorname{ctg} x + c$

10. $\int \frac{1}{\operatorname{ch}^2 x} dx = \operatorname{th} x + c$

11. $\int \frac{1}{\operatorname{sh}^2 x} dx = \operatorname{cth} x + c$

12. $\int \frac{1}{1+x^2} dx = \operatorname{arctg} x + c$

13. $\int \frac{1}{1-x^2} dx = \begin{cases} \operatorname{arth} x & \text{ha } |x|<1 \\ \operatorname{arcth} x & \text{ha } |x|>1 \end{cases}$

14. $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$

15. $\int \frac{1}{\sqrt{1+x^2}} dx = \operatorname{arsh} x + c$

16. $\int \frac{1}{\sqrt{x^2-1}} dx = \operatorname{arch} x + c$

Integrálási szabályok

1.) Legyen $\int f(x)dx = F(x) + c$, $\int g(x)dx = G(x) + c \Rightarrow \int f(x) \pm g(x) dx = F(x) \pm G(x) + c$,
mert $(F(x) \pm G(x))' = F'(x) \pm G'(x) = f(x) \pm g(x)$

Pl. $\int x^3 + \sin x - 10^x dx = \frac{x^4}{4} - \cos x - \frac{10^x}{\ln 10} + c$

2.) Legyen $\int f(x)dx = F(x) + c$ és $A \in \mathbb{R} \Rightarrow \int A f(x) dx = A F(x) + c$,
mert $(A F(x))' = A \cdot F'(x) = A f(x)$.

Pl. $\int 2x^5 dx = 2 \cdot \frac{x^6}{6} + c$

$\int 6x^2 + 5x + 7 dx = 6 \cdot \frac{x^3}{3} + 5 \cdot \frac{x^2}{2} + 7x + c$

3. Legyen $\int f(x)dx = F(x)+c$, $a,b \in \mathbb{R}$, $a \neq 0$. Ekkor

$$\int f(ax+b)dx = \frac{F(ax+b)}{a} + c, \text{ mert}$$

$$\left(\frac{F(ax+b)}{a}\right)' = \frac{1}{a} F'(ax+b)\cdot a = f(ax+b)$$

Pl. 1. $\int (3x-2)^7 dx = \dfrac{\frac{(3x-2)^8}{8}}{3} + c = \dfrac{(3x-2)^8}{24} + c$

$\int x^7 dx = \frac{x^8}{8} + c$

2. $\int e^{5x+2} dx = \dfrac{e^{5x+2}}{5} + c$

$\int e^x dx = \dfrac{e^x}{\ln e} + c = e^x + C$

4. $\int f^{\mu}(x)\cdot f'(x)dx = \dfrac{f^{\mu+1}(x)}{\mu+1} + c$, ha $\mu \neq -1$, mert

$$\left(\frac{f^{\mu+1}(x)}{\mu+1}\right)' = \frac{1}{\mu+1}\cdot(\mu+1)\cdot f^{\mu}(x)\cdot f'(x) = f^{\mu}(x)\cdot f'(x).$$

Pl. 1. $\int \sin^8 x \cdot \cos x\, dx = \dfrac{\sin^9 x}{9} + c$

$f(x) = \sin x$, $f'(x) = \cos x$, $\mu = 8$

2. $\int x\sqrt{x^2+4}\, dx = \int \frac{1}{2}(2x)(x^2+4)^{1/2} dx = \frac{1}{2}\cdot\dfrac{(x^2+4)^{3/2}}{3/2} + c$

$f(x) = x^2+4$, $f'(x) = 2x$, $\mu = \frac{1}{2}$

3. $\int \dfrac{10^x}{(10^x+7)^{20}} dx = \int 10^x\cdot(10^x+7)^{-20} dx = \int \frac{1}{\ln 10}\cdot 10^x \ln 10\cdot(10^x+7)^{-20} dx$

$f(x) = 10^x+7$ $f'(x) = 10^x \ln 10$, $\mu = -20$

$= \dfrac{(10^x+7)^{-19}}{-19\cdot \ln 10} + c$

5. $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$, mert

$$(\ln|f(x)|)' = \begin{cases} \text{ha } f(x)>0: & (\ln f(x))' = \dfrac{1}{f(x)}\cdot f'(x) = \dfrac{f'(x)}{f(x)} \\ \text{ha } f(x)<0: & (\ln(-f(x)))' = \dfrac{1}{-f(x)}\,(-f'(x)) = \dfrac{f'(x)}{f(x)}. \end{cases}$$

Pl. 1. $\displaystyle\int \operatorname{ctg}x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c$

$$(\sin x)' = \cos x$$

2. $\displaystyle\int \frac{x}{x^2+1}\,dx = \int \frac{1}{2}\,\frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1|\,dx$

3. $\displaystyle\int \frac{2^x}{2^x+8}\,dx = \int \frac{1}{\ln 2}\cdot\frac{2^x\cdot\ln 2}{2^x+8}\,dx = \frac{1}{\ln 2}\ln|2^x+8| + c$

6. Parciális integrálás:

$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x) \Rightarrow$

$\int (u(x)v(x))' = \int u'(x)v(x) + u(x)v'(x)\,dx = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$

$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)$

$\Rightarrow \int u(x)v'(x) = u(x)v(x) - \int u'(x)v(x)\,dx$

Röviden: $\int u\cdot v' = u\cdot v - \int u'v$

Pl. 1. $\displaystyle\int \underset{u}{x}\,\underset{v'}{e^{2x}}\,dx = x\cdot\frac{e^{2x}}{2} - \int 1\cdot\frac{e^{2x}}{2}\,dx = x\,\frac{e^{2x}}{2} - \frac{1}{4}e^{2x} + c$

$u' = 1 \quad v = \dfrac{e^{2x}}{2}$

2. $\displaystyle\int \underset{u}{(x^2+2x+3)}\,\underset{v'}{\sin x}\,dx = (x^2+2x+3)(-\cos x) - \int \underset{u}{(2x+2)}\,\underset{v'}{(-\cos x)}\,dx =$

$u' = 2x+2 \quad v = -\cos x \qquad\qquad u' = 2 \quad v = -\sin x$

$= -(x^2+2x+3)\cos x - \left(-\sin x\cdot(2x+2) - \int 2(-\sin x)\,dx\right) =$

$= -(x^2+2x+3)\cos x + (2x+2)\sin x + 2\cos x + C$

3. $\int 2x \cdot \operatorname{arctg} x \, dx = x^2 \operatorname{arctg} x - \int \underbrace{x^2 \cdot \frac{1}{1+x^2}} dx = x^2 \operatorname{arctg} x - \int 1 - \frac{1}{1+x^2} dx =$

$v' \quad u$

$v = x^2 \quad u' = \frac{1}{1+x^2}$

$\frac{x^2}{1+x^2} = \frac{1+x^2-1}{1+x^2}$

$x^2 \operatorname{arctg} x - (x - \operatorname{arctg} x)$

4. $\int \ln x \, dx = \int 1 \cdot \ln x = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int 1 dx = x \ln x - x + C$

$v' \quad u$

$v = x \quad u' = \frac{1}{x}$

Általános szabály: Legyen $p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

I. $\int p_n(x) \cdot \begin{Bmatrix} e^{ax+b} \\ \sin(ax+b) \\ \cos(ax+b) \end{Bmatrix} dx$

$u \quad v'$

II. $\int p_n(x) \begin{Bmatrix} \text{logaritmus} \\ \text{arkus} \quad \text{fv} \\ \text{area} \end{Bmatrix} dx$

$v' \quad u$

5. $\int \arcsin x \, dx = \int 1 \cdot \arcsin x = x \arcsin x - \int \underbrace{x \cdot \frac{1}{\sqrt{1-x^2}}} dx =$

$v' \quad u$

$v = x \quad u' = \frac{1}{\sqrt{1-x^2}}$

$x(1-x^2)^{-1/2} = -\frac{1}{2}(-2x)(1-x^2)^{-1/2}$

$= x \arcsin x - \left(-\frac{1}{2}\right) \frac{(1-x^2)^{1/2}}{1/2} + C = x \arcsin x + \sqrt{1-x^2} + C$

7. Racionális törtfvek integrálása

$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \; a_n \neq 0$ (n-edfokú polinom)

$q_m(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0, \; b_m \neq 0$ (m-edfokú polinom)

$\int \frac{p_n(x)}{q_m(x)} dx = ?$

Pl. $\int \frac{3x^3 + 6x^2 - 7x + 2}{x^2 - x - 6} dx = ?$

Ha $n \geq m$, akkor $p_n(x)$-et elosztjuk $q_m(x)$-mel:

Pl. 1. $2x^3+4x^2-7x+3 : x+3 = \underbrace{2x^2-2x-1}_{\text{hányados}}$

$$\begin{array}{l} -\underline{2x^3+6x^2} \\ \quad -2x^2-7x \\ -\underline{\;-2x^2-6x\;} \\ \qquad -x+3 \\ \quad \underline{-x-3} \\ \qquad \underbrace{6}_{\text{maradék}} \end{array}$$

$$(2x^3+4x^2-7x+3) = (2x^2-2x-1)(x+3)+6$$

2. $x^4+2x^3-3x+2 : x^2+x-2 = x^2+x+1$

$$\begin{array}{l} \underline{x^4+x^3-2x^2} \\ \quad x^3+2x^2-3x \\ -\underline{\;x^3+x^2-2x\;} \\ \qquad x^2-x+2 \\ -\underline{\;x^2+x+2\;} \\ \qquad -2x \end{array}$$

$$x^4+2x^3-3x+2 = (x^2+x+1)(x^2+x-2)-2x$$

Csak azzal az esettel foglalkozunk, ahol $m \leq 2$, azaz a nevező vagy elsőfokú vagy másodfokú.

$\int \frac{p_n(x)}{q_m(x)}\,dx$ kiszámolása:

1. lépés: Ha $n \geq m$, akkor polinomosztás: $p_n(x) = s(x)\,q_m(x) + r(x)$, ahol $r(x)$ egy m-nél kisebb fokú polinom (a 0-adfokú polinom: $r(x) = a_0$)

2. $\int \frac{p_n(x)}{q_m(x)}\,dx = \int \frac{s(x)\,q_m(x)+r(x)}{q_m(x)}\,dx = \int s(x) + \frac{r(x)}{q_m(x)}\,dx$

Feladat: $\int \frac{r(x)}{q_m(x)}\,dx$

Ha $n=1$ ($q_n(x)$ elsőfokú polinom): $\int \frac{1}{x} dx = \ln|x|$ alapján számoljuk

Ha $n=2$: $\int \frac{d_1x+d_0}{b_2x^2+b_1x+b_0} dx$: először $\int \frac{f'(x)}{f(x)} dx$ -et alkalmazzuk

$\int \frac{k_0}{b_2x^2+b_1x+b_0}$ -ra vezetjük vissza.

Ha a ~~nevezőnek~~ nevezőnek nincs valós gyöke, akkor $\int \frac{1}{1+x^2} dx = \operatorname{arctg} x + c$ -t használjuk

Ha a nevezőnek két különböző valós gyöke van: α_1, α_2,

akkor $\frac{k_0}{b_2x^2+b_1x+b_0} = \frac{A}{x-\alpha_1} + \frac{B}{x-\alpha_2}$ A, B alakba írjuk

Ha a nevezőnek egy valós gyöke van: α : $\int \frac{1}{(x-\alpha)^2} dx = \frac{-1}{x-\alpha} + c$

alapján számoljuk.

Pl. 1. $\int \frac{2x^2+5x-3}{x-1} dx = \int \frac{(2x+7)(x-1)+4}{x-1} dx = \int 2x+7+\frac{4}{x-1} dx =$

$x^2+7x+4\ln|x-1|+c$

$2x^2+5x-3 : x-1 = 2x+7$

$\underline{-\ 2x^2-2x}$

$7x-3$

$\underline{-\ 7x-7}$

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$(x^2+10x+29)' = 2x+10$

2. $\int \frac{2x^2+6x-3}{x^2+10x+29} dx = \int \frac{2(x^2+10x+29)-14x-61}{x^2+10x+29} dx = \int 2 - \frac{14x+61}{x^2+10x+29} dx =$

$2x^2+6x-3 : x^2+10x+29 = 2$

$\underline{-\ 2x^2+20x+58}$

$-14x-61$

$2x - \int \frac{5(2x+10)+11}{x^2+10x+29} dx =$

$2x - 5\ln|x^2+10x+29| - \int \frac{11}{(x+5)^2+4} dx =$

$2x - 5\ln|x^2+10x+29| - \int \frac{11}{4} \cdot \frac{1}{1+\left(\frac{x+5}{2}\right)^2} dx =$

$2x - 5\ln|x^2+10x+29| - \frac{11}{4} \frac{\operatorname{arctg}\left(\frac{x+5}{2}\right)}{\frac{1}{2}} + c$

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3. $\int \frac{3x+2}{x^2+x-6}dx = \int \frac{1{,}5(2x+1)+0{,}5}{(x-2)(x+3)}dx = 1{,}5\ln|x^2+x-6| + \int \frac{0{,}5}{(x-2)(x+3)}dx =$

$x^2+x-6=0$

$x_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2}$ $\begin{array}{l} 2 \\ -3 \end{array}$

$\frac{0x+0{,}5}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3} = \frac{A(x+3)+B(x-2)}{(x-2)(x+3)} = \frac{(A+B)x+3A-2B}{(x-2)(x+3)}$

$A+B=0 \Rightarrow B=-A$

$3A-2B=0{,}5$

$3A-2(-A)=0{,}5$

$5A=0{,}5$

$A=0{,}1,\ B=-0{,}1$

$= 1{,}5\ln|x^2+x-6| + \int \frac{0{,}1}{x-2} - \frac{0{,}1}{x+3}dx = 1{,}5\ln|x^2+x-6| + 0{,}1\ln|x-2|$
$-0{,}1\ln|x+3|$

4. $\int \frac{6x+9}{4x^2-4x+1}dx = \cancel{\int \frac{6x+9}{(2x-1)^2}dx} = \int \frac{0{,}75(8x-4)+12}{\cancel{(2x-1)^2}}dx =$

$4x^2-4x+1$

$(4x^2-4x+1)' = 8x-4$

$0{,}75\ln|4x^2-4x+1| + \int \frac{12}{(2x-1)^2}dx = 0{,}75\ln|4x^2-4x+1| + 12 \cdot \frac{\frac{-1}{2x-1}}{2} + c$

$\int \frac{1}{x^2}dx = \int x^{-2}dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x}$

8. A $\sin x$ és $\cos x$ hatványait úgy integráljuk ki, hogy az eredetileg több tényezős szorzatot egyre kevesebb tényezős szorzatok összegére és különbségére alakítjuk át, amit végül az $\int \sin ax\, dx = -\frac{\cos ax}{a} + c$, $\int \cos ax\, dx = \frac{\sin ax}{a} + c$ szabályokkal számolunk ki.

A felhasznált szabályok:

$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$, $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$,

$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$,

$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$,

$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$

1. $\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx = \int \frac{1}{2} + \frac{1}{2}\cos 2x\, dx = \frac{1}{2}x + \frac{1}{2}\cdot\frac{\sin 2x}{2} + c$

2. $\int \sin^2 2x\, dx = \int \frac{1-\cos 4x}{2} = \int \frac{1}{2} - \frac{1}{2}\cos 4x\, dx = \frac{1}{2}x + \frac{1}{2}\cdot\frac{\sin 4x}{4} + c$

3. $\int \sin 2x \cos 5x\, dx = \int \frac{1}{2}\left(\sin 7x + \sin(-3x)\right) dx =$

$\frac{1}{2}\left(-\frac{\cos 7x}{7} + \left(-\frac{\cos(-3x)}{-3}\right)\right) + c$

4. $\int \sin \overset{\beta}{x} \cdot \sin \overset{\alpha}{3x}\, dx = \int \frac{1}{2}\left(\cos 2x - \cos 4x\right) dx =$

$\frac{1}{2}\cdot\frac{\sin 2x}{2} - \frac{1}{2}\cdot\frac{\sin 4x}{4} + c$

5. $\int \cos^3 x\, dx = \int \cos x \cdot \cos^2 x\, dx = \int \cos x \cdot \frac{1+\cos 2x}{2}\, dx =$

$\int \cos x\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \int \frac{1}{2}\cos x + \frac{1}{2}\cos \overset{\beta}{x} \cos \overset{\alpha}{2x}\, dx =$

$\int \frac{1}{2}\cos x + \frac{1}{2}\cdot\frac{1}{2}\left(\cos 3x + \cos x\right) dx = \int \frac{3}{4}\cos x + \frac{1}{4}\cos 3x\, dx = \frac{3}{4}\sin x + \frac{1}{4}\cdot\frac{\sin 3x}{3} + c$

vagy

$\int \cos^3 x \, dx = \int \cos x \cdot \cos^2 x \, dx = \int \cos x (1 - \sin^2 x) dx =$

$\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x$

$\int \cos x - \sin^2 x \cos x \, dx = \sin x - \frac{\sin^3 x}{3} + C$

6. $\int \sin^2 x \cos^2 x \, dx = \int \frac{1 - \cos 2x}{2} \cdot \frac{1 + \cos 2x}{2} dx = \int \frac{1}{4} - \frac{1}{4} \cos^2 2x \, dx =$

$\int \frac{1}{4} - \frac{1}{4} \cdot \frac{1 + \cos 4x}{2} dx = \int \frac{1}{8} - \frac{1}{8} \cdot \cos 4x \, dx = \frac{1}{8} x - \frac{1}{8} \cdot \frac{\sin 4x}{4} + C.$

9. Helyettesítés

Láncszabály: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

x szerint deriválunk

Ha $F'(x) = f(x)$, vagyis $f(x)$ primitív fv-e $F(x)$, akkor

$\frac{d}{dt} F(g(t)) = F'(g(t)) \cdot g'(t) = f(g(t)) g'(t)$.

t szerint deriválunk

Ha az $\int f(x) dx = F(x) + C$ határozatlan integrálban

$x = g(t)$ helyettesítést hajtunk végre, akkor

$\int f(\underline{x}) \underline{\underline{dx}} = F(x) + C = F(g(t)) + C = \int f(\underline{g(t)}) \cdot \underline{\underline{g'(t) dt}}$,

tehát ekkor $dx = g'(t) dt$.

Pl. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int \frac{e^t}{t} 2t \, dt = \int 2e^t dt = 2e^t + C = 2e^{\sqrt{x}} + C$

$\sqrt{x} = t$

$x = t^2$

$dx = 2t \, dt$

Speciális helyettesítések:

1. Ha $f(x)$ e^x-től függ, azaz $f(x)=R(e^x)$.

Akkor $t=e^x \Rightarrow x=\ln t \Rightarrow dx=\frac{1}{t}dt$

Pl. 1. $\int e^x \sin(e^x)\,dx = \int t\cdot \sin t \cdot \frac{1}{t}dt = \int \sin t\,dt =$

$t=e^x$

$-\cos t + c = -\cos(e^x)+c$

$x=\ln t$

$dx=\frac{1}{t}dt$

2. $\int \frac{e^x}{1+e^{2x}}dx = \int \frac{e^x}{1+(e^x)^2}dx = \int \frac{t}{1+t^2}\cdot\frac{1}{t}dt = \int \frac{1}{1+t^2}dt =$

$\operatorname{arctg} t + c = \operatorname{arctg}(e^x)+c$

$e^x=t$

$x=\ln t$

$dx=\frac{1}{t}dt$

3. $\int \frac{1}{e^x+1}dx = \int \frac{1}{t+1}\cdot\frac{1}{t}dt = \int \frac{1}{t^2+t}dt = \int \frac{1}{t(t+1)}dt =$

$e^x=t$

$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} = \frac{A(t+1)+Bt}{t(t+1)} =$

$x=\ln t$

$dx=\frac{1}{t}dt$

$\frac{(A+B)t+A}{t(t+1)} \Rightarrow$ $A+B=0$, $A=1 \Rightarrow B=-1$

$= \int \frac{1}{t} + \frac{-1}{t+1}dt = \int \frac{1}{t} - \frac{1}{t+1}dt = \ln|t| - \frac{\ln|t+1|}{1} + c =$

$\ln|e^x| - \ln|e^x+1| + c$

4. $\int e^{2x}\cdot e^{e^x}dx = \int (e^x)^2\cdot e^{e^x}dx = \int t^2\cdot e^t\cdot\frac{1}{t}dt = \int \underset{u}{t}\,\underset{v'}{e^t}\,dt =$

$t=e^x$

$u'=1 \quad v=e^t$

$x=\ln t$

$te^t - \int e^t dt = te^t - e^t + c = e^x\cdot e^{e^x} - e^{e^x} + c$

$dx=\frac{1}{t}dt$

2. $f(x) = R(\sqrt[n]{ax+b})$ spec. helyettesítés

$t = \sqrt[n]{ax+b}$

$t^n = ax+b$

$ax = t^n - b$

$x = \frac{1}{a} t^n - \frac{b}{a}$

$dx = \frac{1}{a} \cdot n t^{n-1} dt$

Pl. 1. $\int \frac{1}{x+\sqrt{x}} dx = \int \frac{1}{t^2+t} \cdot 2t\,dt = \int \frac{2t}{t^2+t} dt = \int \frac{2}{t+1} dt =$

$\frac{2\ln|t+1|}{1} + C = 2\ln|\sqrt{x}+1| + C$

$\sqrt{x} = t$

$x = t^2$

$dx = 2t\,dt$

2. $\int x\sqrt{2x+1}\,dx = \int \left(\frac{1}{2}t^2 - \frac{1}{2}\right) \cdot t \cdot t\,dt = \int \left(\frac{1}{2}t^2 - \frac{1}{2}\right) t^2 dt =$

$t = \sqrt{2x+1}$

$\int \frac{1}{2}t^4 - \frac{1}{2}t^2 dt = \frac{1}{2} \cdot \frac{t^5}{5} - \frac{1}{2} \cdot \frac{t^3}{3} + C =$

$t^2 = 2x+1$

$\frac{1}{10}\left(\sqrt{2x+1}\right)^5 - \frac{1}{6}\left(\sqrt{2x+1}\right)^3 + C$

$2x = t^2 - 1$

$x = \frac{1}{2}t^2 - \frac{1}{2}$

$dx = \frac{1}{2} \cdot 2t\,dt = t\,dt$

3. $\int \cos\sqrt[3]{x}\,dx = \int \underset{v'}{\cos t} \cdot \underset{u}{3t^2}\,dt = 3t^2 \sin t - \int \underset{u}{6t} \underset{v'}{\sin t}\,dt =$

$u' = 6 \quad v = -\cos t$

$\sqrt[3]{x} = t$

$v = \sin t \quad u' = 6t$

$x = t^3$

$dx = 3t^2 dt$

$3t^2 \sin t - \left(+6t\cos t\right) - \int 6(-\cos t\,dt) =$

$3t^2 \sin t + 6t\cos t - 6\sin t + C = 3(\sqrt[3]{x})^2 \sin\sqrt[3]{x} + 6\sqrt[3]{x}\cos\sqrt[3]{x} - 6\sin\sqrt[3]{x} +$

3. Spec helyettesítés : $f(x)=R(\sqrt{a^2-x^2})$

$x=a\sin t$

$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2 t}=\sqrt{a^2(1-\sin^2 t)}=\sqrt{a^2\cos^2 t}=|a\cos t|$

$\cos^2 t+\sin^2 t=1 \Rightarrow \cos^2 t=1-\sin^2 t$

$dx=a\cos t\,dt$

Pl. 1. $\int\sqrt{1-x^2}\,dx=\int\cos t\cdot\cos t\,dt=\int\cos^2 t\,dt=\int\frac{1+\cos 2t}{2}\,dt=$

$x=\sin t$ $\quad\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=\cos t$

$dx=\cos t\,dt$ $\quad 2\sin t\cos t,\quad \cos t=\sqrt{1-\sin^2 t}=\sqrt{1-x^2}$

$\int\frac{1}{2}+\frac{1}{2}\cos 2t\,dt=\frac{1}{2}t+\frac{1}{2}\cdot\frac{\sin 2t}{2}+c=\frac{1}{2}\arcsin x+\frac{1}{2}x\sqrt{1-x^2}$

2. $\int x\sqrt{4-x^2}\,dx=\int 2\sin t\,2\cos t\cdot 2\cos t\,dt=\int 8\sin t\cos^2 t\,dt=$

$x=2\sin t$

$dx=2\cos t\,dt$ $\quad\sqrt{4-x^2}=\sqrt{4-4\sin^2 t}=\sqrt{4(1-\sin^2 t)}=$

$\sqrt{4\cos^2 t}=2\cos t$

$\int-8\cdot\cos^2 t(-\sin t)\,dt=-8\cdot\frac{\cos^3 t}{3}+c=-8\cdot\frac{\left(\frac{1}{2}\sqrt{4-x^2}\right)^3}{3}+c=$

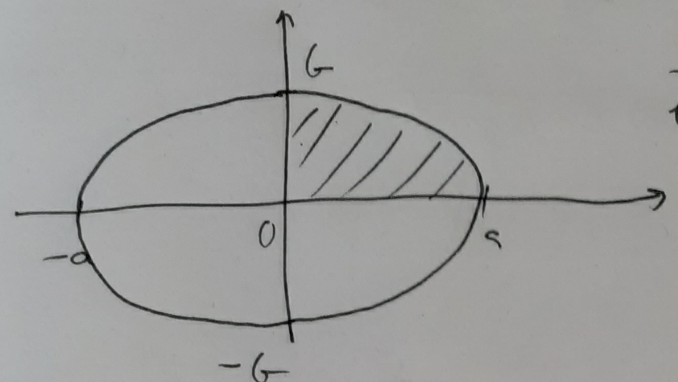
$-(4-x^2)^{3/2}+c$

Ez az $\int f^{\alpha}(x)\cdot f'(x)\,dx=\frac{f^{\alpha+1}(x)}{\alpha+1}+c$ szabályból közvetlenül is látszik:

$\int x\cdot(4-x^2)^{1/2}dx=\int-\frac{1}{2}(-2x)(4-x^2)^{1/2}dx=-\frac{1}{2}\cdot\frac{(4-x^2)^{3/2}}{3/2}=$

$-3(4-x^2)^{3/2}$

3. Határozza meg az $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipszis területét! (16)



$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2-\frac{b^2}{a^2}x^2$$

$$y=\sqrt{b^2-\frac{b^2}{a^2}x^2}$$

$$\text{ter} = 4t = 4\left[\frac{1}{2}ab\arcsin\frac{x}{a}+\frac{1}{2}\frac{b}{a}\cdot x\sqrt{a^2-x^2}\right]_0^{x=a} \overset{\circledast}{=}$$

$$\int\sqrt{b^2-\frac{b^2}{a^2}x^2}\,dx=\int\sqrt{\frac{b^2}{a^2}(a^2-x^2)}\,dx=\int\frac{b}{a}\sqrt{a^2-x^2}\,dx=$$

$x = a\sin t$ $\quad$ $\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2 t}=\sqrt{a^2(1-\sin^2 t)}=$

$dx = a\cos t\,dt$ $\quad$ $\sqrt{a^2\cos^2 t}= a\cos t \Rightarrow \cos t=\frac{\sqrt{a^2-x^2}}{a}$

$0\le t\le\frac{\pi}{2},\ a>0$

$$=\int\frac{b}{a}a\cos t\cdot a\cos t\,dt=\int ab\cos^2 t\,dt=\int ab\,\frac{1+\cos 2t}{2}\,dt=$$

$$\int\frac{1}{2}ab+\frac{1}{2}ab\cos 2t\,dt=\frac{1}{2}abt+\frac{1}{2}ab\cdot\frac{\sin 2t}{2}+C=\frac{1}{2}ab\arcsin\frac{x}{a}+$$

$$\frac{1}{2}ab\cdot\frac{\cancel{2}\cdot\frac{x}{\cancel{a}}\cdot\frac{\sqrt{a^2-x^2}}{a}}{\cancel{2}}+C$$

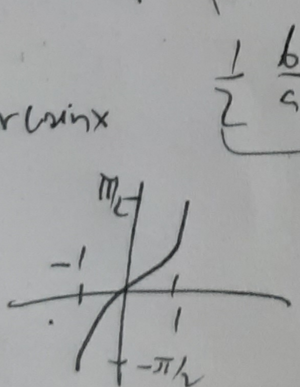
$\sin t=\frac{x}{a}\Rightarrow t=\arcsin\frac{x}{a}$

$\sin 2t=2\sin t\cos t=2\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}$

$$\overset{\circledast}{=}4\left(\frac{1}{2}ab\arcsin\frac{a}{a}+\overbrace{\frac{1}{2}\frac{b}{a}\cdot a\sqrt{a^2-a^2}}^{0}-\left(\overbrace{\frac{1}{2}ab\arcsin\frac{0}{a}}^{0}+\right.\right.$$

$$\left.\left.\underbrace{\frac{1}{2}\frac{b}{a}\cdot 0\cdot\sqrt{a^2-0^2}}_{0}\right)\right)=4\cdot\frac{1}{2}ab\cdot\frac{\pi}{2}=ab\pi$$

arcsinx



4. Speciális helyettesítés: a) $f(x) = R\left(\sqrt{a^2+x^2}\right)$

$x = a\,\mathrm{sh}\,t \qquad \sqrt{a^2+x^2} = \sqrt{a^2+a^2\mathrm{sh}^2 t} = \sqrt{a^2(1+\mathrm{sh}^2 t)} = \sqrt{a^2\mathrm{ch}^2 t} = a\,\mathrm{ch}\,t$

$dx = a\,\mathrm{ch}\,t\,dt \qquad$ Tudjuk: $\mathrm{ch}^2 t - \mathrm{sh}^2 t = 1$

$\mathrm{ch}^2 t = 1+\mathrm{sh}^2 t, \quad \mathrm{sh}^2 t = \mathrm{ch}^2 t - 1$

b) $f(x) = R\left(\sqrt{x^2-a^2}\right)$

$x = a\,\mathrm{ch}\,t \qquad \sqrt{x^2-a^2} = \sqrt{a^2\mathrm{ch}^2 t - a^2} = \sqrt{a^2(\mathrm{ch}^2 t-1)} = \sqrt{a^2\mathrm{sh}^2 t} = |a\,\mathrm{sh}\,t|$

$dx = a\,\mathrm{sh}\,t\,dt$

Pl. 1. $\int \sqrt{x^2-1}\,dx = \int \mathrm{sh}\,t \cdot \mathrm{sh}\,t\,dt = \int \mathrm{sh}^2 t\,dt = \int \frac{\mathrm{ch}\,2t-1}{2}\,dt =$

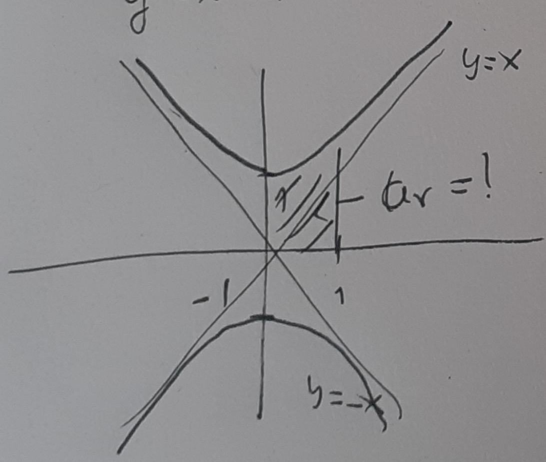
$x = \mathrm{ch}\,t \qquad \sqrt{x^2-1} = \sqrt{\mathrm{ch}^2 t-1} = \sqrt{\mathrm{sh}^2 t} = \mathrm{sh}\,t, \quad t = \mathrm{arch}\,x$

$dx = \mathrm{sh}\,t\,dt$

$\int \frac{1}{2}\mathrm{ch}\,2t - \frac{1}{2}\,dt = \frac{1}{2}\cdot\frac{\mathrm{sh}\,2t}{2} - \frac{1}{2}t + \cancel{c} = \frac{1}{2}\cdot\frac{\cancel{2}\,\mathrm{sh}\,t\,\mathrm{ch}\,t}{\cancel{2}} - \frac{1}{2}t =$

$\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\mathrm{arch}\,x + c$

2. $y^2 - x^2 = 1$:



$y^2 = 1+x^2$

$y = \sqrt{1+x^2}$

$\mathrm{Ter} = \left[\frac{1}{2}\mathrm{arsh}\,x + \frac{1}{2}x\sqrt{1+x^2}\right]_0^1 =$

$\frac{1}{2}\mathrm{arsh}\,1 + \frac{1}{2}\cdot 1\cdot\sqrt{1+1^2} - \left(\frac{1}{2}\underbrace{\mathrm{arsh}\,0}_{0} + \frac{1}{2}\cdot 0\cdot\sqrt{1+0^2}\right)$

$= \frac{1}{2}\mathrm{arsh}\,1 + \frac{\sqrt{2}}{2} = \frac{1}{2}\ln(1+\sqrt{2}) + \frac{\sqrt{2}}{2}$

$\mathrm{arsh}\,x = \ln\left(x+\sqrt{x^2+1}\right)$

$\int \sqrt{1+x^2}\,dx = \int \mathrm{ch}\,t\cdot\mathrm{ch}\,t\,dt = \int \mathrm{ch}^2 t\,dt = \int \frac{1+\mathrm{ch}\,2t}{2}\,dt = \int \frac{1}{2} + \frac{1}{2}\mathrm{ch}\,2t\,dt =$

$x = \mathrm{sh}\,t \qquad \sqrt{1+x^2} = \sqrt{1+\mathrm{sh}^2 t} = \sqrt{\mathrm{ch}^2 t} = \mathrm{ch}\,t, \quad t = \mathrm{arsh}\,x$

$dx = \mathrm{ch}\,t\,dt$

$\frac{1}{2}t + \frac{1}{2}\cdot\frac{\mathrm{sh}\,2t}{2} + c = \frac{1}{2}t + \frac{1}{2}\cdot\frac{2\,\mathrm{sh}\,t\,\mathrm{ch}\,t}{2} = \frac{1}{2}\mathrm{arsh}\,x + \frac{1}{2}x\sqrt{1+x^2} + c$