14. MATEMATIKA A1 FELADATSOR

1. Döntse el, hogy az alábbi improrius integrál konvergens vagy divergens. Amelyik konvergens, annak határozza meg az értékét!

   (a) $\int_1^\infty \frac{1}{x^2} dx$
   Megoldás:

$$\int_1^\infty \frac{1}{x^2} dx = \lim_{d\to\infty} \int_1^d \frac{1}{x^2} dx = \lim_{d\to\infty} [\frac{-1}{x}]_1^d = \lim_{d\to\infty} (\frac{-1}{d} - (-1)) = 1,$$

   tehát konvergens.

   (b) $\int_1^\infty \frac{1}{\sqrt{x}} dx$
   Megoldás:

$$\int_1^\infty \frac{1}{\sqrt{x}} dx = \lim_{d\to\infty} \int_1^d \frac{1}{\sqrt{x}} dx = \lim_{d\to\infty} [\frac{\sqrt{x}}{\frac{1}{2}}]_1^d = \lim_{d\to\infty} (2\sqrt{d} - 2) = +\infty,$$

   tehát divergens.

   (c) $\int_1^\infty \frac{1}{1+x^2} dx$
   Megoldás:

$$\int_1^\infty \frac{1}{1+x^2} dx = \lim_{d\to\infty} \int_1^d \frac{1}{1+x^2} dx = \lim_{d\to\infty} [\operatorname{arctg} x]_1^d = \lim_{d\to\infty} (\operatorname{arctg} d - \operatorname{arctg} 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$

   tehát konvergens.

   (d) $\int_{0,5}^\infty \frac{1}{1+4x^2} dx$
   Megoldás:

$$\int_{0,5}^\infty \frac{1}{1+4x^2} dx = \lim_{d\to\infty} \int_{0,5}^d \frac{1}{1+(2x)^2} dx = \lim_{d\to\infty} [\frac{\operatorname{arctg} 2x}{2}]_{0,5}^d = \lim_{d\to\infty} (\frac{\operatorname{arctg} d}{2} - \frac{\operatorname{arctg} 1}{2}) = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8},$$

   tehát konvergens

   (e) $\int_1^\infty \frac{1}{x^2+x} dx$
   Megoldás: $x^2 + x = x(x+1)$, ezért

$$\frac{1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)} = \frac{(A+B)x+A}{x^2+x},$$

   ezért $A + B = 0$ és $A = 1$, azaz $B = -1$. Igy

$$\int \frac{1}{x^2+x} dx = \int \frac{1}{x} - \frac{1}{x+1} dx = \ln|x| - \ln|x+1| + c,$$

   ezért

$$\int_1^\infty \frac{1}{x^2+x} dx = \lim_{d\to\infty} \int_1^d \frac{1}{x^2+x} dx = \lim_{d\to\infty} [\ln|x| - \ln|x+1|]_1^d = \lim_{d\to\infty} (\ln|d| - \ln|d+1|) - (\ln|1| - \ln|2|) =$$

$$\lim_{d\to\infty} \ln\frac{d}{d+1} - \ln\frac{1}{2} = \ln 1 - \ln\frac{1}{2} = \ln 2,$$

   tehát konvergens.

(f) $\int_{-\infty}^{0} \frac{1}{1+16x^2} dx$
Megoldás:

$$\int_{-\infty}^{0} \frac{1}{1+16x^2} dx = \lim_{c\to-\infty} \int_{c}^{0} \frac{1}{1+(4x)^2} dx = \lim_{c\to-\infty} [\frac{\operatorname{arctg} 4x}{4}]_c^0 =$$

$$\lim_{c\to-\infty} \frac{\operatorname{arctg} 0}{4} - \frac{\operatorname{arctg} 4c}{4} = 0 - (-\frac{\pi}{8}) = \frac{\pi}{8},$$

tehát konvergens.

(g) $\int_{-\infty}^{0} e^x dx$c
Megoldás:

$$\int_{-\infty}^{0} e^x dx = \lim_{c\to-\infty} \int_{c}^{0} e^x dx = \lim_{c\to-\infty} [e^x]_c^0 = \lim_{c\to-\infty} (e^0 - e^c) = 1,$$

tehát konvergens.

(h) $\int_{-\infty}^{1} xe^x dx$
Megoldás: Parciális integrálva $u = x$, $v' = e^x$ választással $u' = 1$ és $v = e^x$, ezért

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c,$$

ezért

$$\int_{-\infty}^{1} xe^x dx = \lim_{c\to-\infty} \int_{c}^{1} xe^x dx = \lim_{c\to-\infty} [xe^x - e^x]_c^1 = \lim_{c\to-\infty} \left((1e^1 - e^1) - (ce^c - e^c)\right),$$

mivel a L'Hospital-szabály alapján

$$\lim_{c\to-\infty} ce^c = \lim_{c\to-\infty} \frac{c}{e^{-c}} = \frac{+\infty}{+\infty} = \lim_{c\to-\infty} \frac{1}{-e^{-c}} = \frac{1}{+\infty} = 0,$$

és $\lim_{c\to-\infty} e^c = 0$, ezért

$$\int_{-\infty}^{1} xe^x dx = \lim_{c\to-\infty} \left((1e^1 - e^1) - (ce^c - e^c)\right) = 0,$$

tehát konvergens.

(i) $\int_{-\infty}^{\infty} \frac{1}{1+9x^2} dx$
Megoldás:

$$\int_{-\infty}^{\infty} \frac{1}{1+9x^2} dx = \lim_{c\to-\infty, d\to+\infty} \int_{c}^{d} \frac{1}{1+(3x)^2} dx = \lim_{c\to-\infty, d\to+\infty} [\frac{\operatorname{arctg} 3x}{3}]_c^d =$$

$$\lim_{c\to-\infty, d\to+\infty} \left(\frac{\operatorname{arctg} 3d}{3} - \frac{\operatorname{arctg} 3c}{3}\right) q = \frac{\pi/2}{3} - \frac{-\pi/2}{3} = \frac{\pi}{3},$$

tehát konvergens.

(j) $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2} dx$
Megoldás:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2} dx = \lim_{c\to-\infty, d\to+\infty} \int_{c}^{d} \frac{1}{1+(x+1)^2} dx = \lim_{c\to-\infty, d\to+\infty} [\frac{\operatorname{arctg} (x+1)}{1}]_c^d =$$

$$\lim_{c\to-\infty, d\to+\infty} (\operatorname{arctg} (d+1) - \operatorname{arctg} (c+1)) = \frac{\pi}{2} - \frac{-\pi}{2} = \pi,$$

theát konvergens.

2. Döntse el, hogy az alábbi improrius integrál konvergens vagy divergens. Amelyik konvergens, annak határozza meg az értékét!

(a) $\int_0^1 \frac{1}{\sqrt{x}} dx$
Megoldás:

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{a\to 0+} \int_a^1 x^{-1/2} dx = \lim_{a\to 0+} [\frac{x^{1/2}}{1/2}]_a^1 = \lim_{a\to 0+} 2 - 2\sqrt{a} = 2,$$

tehát konvergens.

(b) $\int_0^1 \frac{1}{x^3} dx$
Megoldás:

$$\int_0^1 \frac{1}{x^3} dx = \lim_{a\to 0+} \int_a^1 x^{-3} dx = \lim_{a\to 0+} [\frac{x^{-2}}{-2}]_a^1 = \lim_{a\to 0+} \frac{-0,5}{1^2} - \frac{-0,5}{a^2} = \infty,$$

tehát divergens.

(c) $\int_1^2 \frac{1}{x-1} dx$
Megoldás:

$$\int_1^2 \frac{1}{x-1} dxx = \lim_{a\to 1+} \int_a^1 \frac{1}{x-1} dx = \lim_{a\to 0+} [\ln|x-1|]_a^2 = \lim_{a\to 0+} \ln 1 - \ln|a-1| = \infty,$$

tehát divergens.

(d) $\int_{-1}^0 \frac{1}{\sqrt[3]{x}} dx$
Megoldás:

$$\int_{-1}^0 \frac{1}{\sqrt[3]{x}} dx = \lim_{b\to 0-} \int_{-1}^b x^{-1/3} dx = \lim_{b\to 0-} [\frac{x^{2/3}}{2/3}]_b^{-1} = \lim_{b\to 0-} \frac{3}{2}(-1)^{2/3} - \frac{3}{2} b^{2/3} = \frac{3}{2},$$

tehát konvergens.

(e) $\int_{-3}^0 \frac{1}{x^2} dx$
Megoldás:

$$\int_{-3}^0 \frac{1}{x^2} dx = \lim_{b\to 0-} \int_{-3}^b x^{-2} dx = \lim_{a\to 0+} [\frac{x^{-1}}{-1}]_{-3}^b = \lim_{b\to 0-} \frac{-1}{b} - \frac{-1}{-3} = +\infty,$$

tehát divergens.

(f) $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx$
Megoldás:

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = \lim_{a\to -1+, b\to 1-} \int_a^b \frac{1}{\sqrt{1-x^2}} dx = \lim_{a\to -1+, b\to 1-} [\arcsin x]_a^b$$

$$\lim_{a\to -1+, b\to 1-} (\arcsin b - \arcsin a) = \frac{\pi}{2} - \frac{-\pi}{2} = \pi,$$

tehát konvergens.

(g) $\int_{-2}^2 \frac{2}{\sqrt{4-x^2}} dx$
Megoldás:

$$\int_{-2}^2 \frac{1}{\sqrt{4-x^2}} dx = \lim_{a\to -2+, b\to 2-} \int_a^b \frac{1}{\sqrt{4-x^2}} dx = \lim_{a\to -2+, b\to 2-} \int_a^b \frac{1}{\sqrt{4\left(1-\frac{x^2}{4}\right)}} dx =$$

$$\lim_{a\to -2+, b\to 2-} \int_a^b \frac{1}{2} \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} dx = \lim_{a\to -1+, b\to 1-} [\frac{1}{2}\frac{\arcsin\frac{x}{2}}{\frac{1}{2}}]_a^b = \lim_{a\to -2+, b\to 2-} \left(\arcsin\frac{b}{2} - \arcsin\frac{a}{2}\right) =$$

$$\frac{\pi}{2} - \frac{-\pi}{2} = \pi,$$

tehát konvergens.