An upper bound for Hilbert cubes

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Abstract

In this note we give a new upper bound for the largest size of subset of \{1, 2, \ldots, n\} not containing a \(k\)-cube.

1. Introduction

We call a set \(H\) a Hilbert cube of dimension \(k\) or simply a \(k\)-cube if there are positive integers \(a_0, a_1, \ldots, a_k\) such that

\[H = \{a_0 + \sum_{i=1}^{k} \epsilon_i a_i : \epsilon_i \in \{0, 1\}\}\]

The positive integer \(k\) is the dimension of the Hilbert cube. Hilbert originally proved that if the positive integers are colored with finitely many colors then one color class contains a \(k\)-cube. The density version of theorem was proved by Szemerédi and has since become known as "Szemerédi's cube lemma" (see e.g. [3]):

**Theorem.** Let \(k \geq 2\) be a positive integer. If the sequence \(S_n\) satisfies \(|S_n| \geq (4n)^{1-\frac{1}{2k-1}}\) then \(S_n\) contains a \(k\)-cube.

Denote by \(H_k(n)\) be the largest size of subset of \{1, 2, \ldots, n\} not containing a \(k\)-cube. Gunderson and Rödl improved the above result to \(H_k(n) < 2^{1-\frac{1}{2k-1}}(\sqrt{n} + 1)^{2-\frac{1}{2k-2}}\) (see [2]).

A sequence \(S\) is called Sidon sequence if the sums \(s_1 + s_2, s_1, s_2 \in S, s_1 \leq s_2\) are distinct. Obviously a sequence is Sidon if and only if it does not contain any 2-cubes. It is well known that the maximal size of Sidon sequences can be selected from \{1, 2, \ldots, n\} is at most \(n^{1/2} + O(n^{1/4})\) (see [1]), that is \(H_2(n) < n^{1/2} + O(n^{1/4})\). A very short proof of this fact was given by Lindström (see [4]). Using his method we get the following result

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**Theorem**  For every \( k \geq 3 \) we have \( H_k(n) < n^{1 - \frac{1}{2k-1}} + O(n^{1 - \frac{1}{2k-2}}) \), where the constant depends on \( k \).

2. **Proof**

We will argue by induction. Let us suppose that either \( k = 3 \) or \( k > 3 \) and we have verified the statement for \( k - 1 \), that is \( H_{k-1}(n) < n^{1 - \frac{1}{2k-2}} + O(n^{1 - \frac{1}{2k-3}}) \) and we prove the theorem for \( k \). Let us suppose that the sequence \( 1 \leq a_1 < a_2 < \ldots < a_s \leq n \) does not contain any \( k \)-cubes. We have to prove that \( s < n^{1 - \frac{1}{2k-2}} + O(n^{1 - \frac{1}{2k-3}}) \). Let \( r = H_{k-1}(n) \). We will give lower and upper bound for the sum

\[
K = \sum_{1 \leq i-j \leq r} a_i - a_j.
\]

First we give a lower bound for \( K \). Since the above sequence does not contain any \( k \)-cubes, therefore a difference \( d \) occurs at most \( r \)-times in this sum. This sum contains \( rs - \frac{r(r+1)}{2} = rw \) \( (w = s - \frac{r+1}{2}) \) terms, hence \( K \) is at least \( r \)-times of the sum of the first \( \left[ \frac{rw}{r} \right] = [w] \) positive integers. Hence

\[
K \geq r \left( \left[ \frac{w}{r} \right] + 1 \right) \geq r \frac{w^2 - 0.25}{2}.
\]

In the following we give an upper bound for \( K \). The differences in the sum \( K \) can be arranged in sequences of type

\[
(a_{u+t} - a_t) + (a_{2u+t} - a_{u+t}) + \cdots + (a_{[\frac{n-t}{u}]u+t} - a_{[\frac{n-t}{u}]u+t+1}) \leq n,
\]

where \( 1 \leq u \leq r, 1 \leq t \leq u \). Hence

\[
K \leq n \frac{r(r+1)}{2}.
\]

Comparing the bounds we have \( r \frac{w^2 - 0.25}{2} \leq n \frac{r(r+1)}{2} \), that is \( w^2 \leq nr + n + 0.25 \). Hence

\[
s = w + \frac{r+1}{2} \leq \sqrt{nr + n + 0.25} + \frac{r+1}{2}
\]

For \( k = 3 \) we have \( r < n^{1/2} + O(n^{1/4}) \) which implies

\[
s < n^{0.75} + O(n^{0.5}).
\]

For \( k > 3 \) we have \( r < n^{1 - \frac{1}{2k-2}} + O(n^{1 - \frac{1}{2k-3}}) \), thus

\[
s \leq \sqrt{n^2 - \frac{1}{2k-2}} + O(n^{2 - \frac{1}{2k-3}}) + O(n^{1 - \frac{1}{2k-2}}) = n^{1 - \frac{1}{2k-1}} + O(n^{1 - \frac{1}{2k-2}}),
\]

which proves the theorem. ■
References


