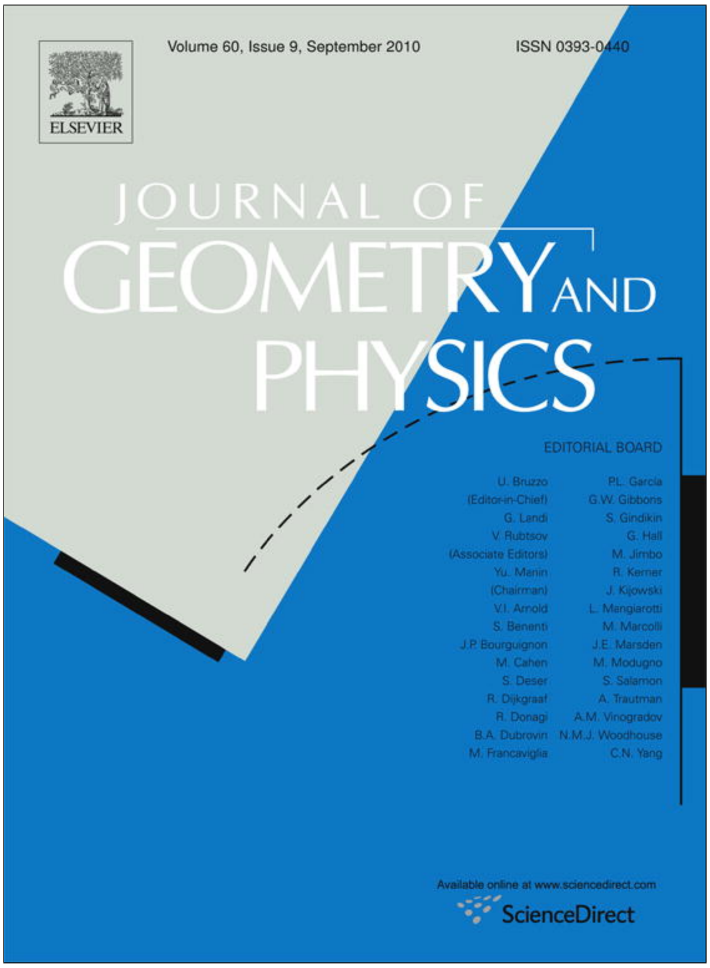
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Semi-indefinite inner product and generalized Minkowski spaces

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ABSTRACT

In this paper we develop the theories of normed linear spaces and of linear spaces with indefinite metric, for finite dimensions both of which are also called Minkowski spaces in the literature.

In the first part of this paper we collect the common properties of the semi- and indefinite inner products and define the semi-indefinite inner product as well as the corresponding semi-indefinite inner product space. We give a generalized concept of the Minkowski space embedded in a semi-indefinite inner product space using the concept of a new product, which contains the classical cases as special ones.

In the second part we investigate the real, finite-dimensional generalized Minkowski space and its sphere of radius i. We prove that it can be regarded as a so-called Minkowski–Finsler space, and if it is homogeneous with respect to linear isometries, then the Minkowski–Finsler distance of its points can be determined by the Minkowski product.

1. Introduction

1.1. Notation and terminology

concepts without definition: Real and complex vector spaces, basis, dimension, direct sum of subspaces, linear and bilinear mappings, quadratic forms, inner (scalar) product, hyperboloid, ellipsoid, hyperbolic space and hyperbolic metric, kernel and rank of a linear mapping.
i.p.: Inner (or scalar product) of a vector space.
s.i.p.: Semi-inner-product (see Definition 1).
continuous s.i.p.: The definition can be given after Definition 1.
differentiable s.i.p.: See Definition 3.
i.i.p.: Indefinite inner product (see Definition 4).
s.i.i.p: Semi-indefinite inner product (see Definition 6).
Minkowski product: See Definition 7.
generalized Minkowski space: See Definition 7.

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generalized space–time model: Finite-dimensional, real, generalized Minkowski space with one-dimensional time-like orthogonal direct components.
positive (resp. negative) subspace: It is a subspace in an i.i.p. space in which all vectors have positive (resp. negative) scalar square.
neutral or isotropic subspace: See Definition 5.
Auerbach basis: The corresponding definition with respect to a finite-dimensional real normed space can be seen before Theorem 8.
hypersurface: The definition in a generalized Minkowski space can be found before Lemma 3.
tangent vector, tangent hyperplane: These definitions can be seen before Lemma 3.
Minkowski–Finsler space: See Definition 15.
$\mathbb{C}, \mathbb{R}, \mathbb{R}^n, S^n$: The complex line, the real line, the n-dimensional Euclidean space and the n-dimensional unit sphere, respectively.
$\dim(V)$: The dimension of the vector space V.
$x \bot y$: The notion of (non-symmetric) orthogonality. We consider it with the meaning "y is orthogonal to x".
$[\cdot,\cdot]$: The notion of scalar product and all its suitable generalizations.
$[\cdot,\cdot]^-$: The notion of s.i.p. corresponding to a generalized Minkowski space.
$[\cdot,\cdot]^+$: The notion of Minkowski product of a generalized Minkowski space.
$[x,\cdot]'_z(y)$: The derivative map of an s.i.p. in its second argument, into the direction of z at the point (x,y). See Definition 3.
$\|\cdot\|'_x(y)$, $\|\cdot\|''_{x,z}(y)$: The derivative of the norm in the direction of x at the point y, and the second derivative of the norm in the directions x and z at the point y.
$\langle\{\cdot\}\rangle$: The linear hull of a set.
$\Re\{\cdot\}, \Im\{\cdot\}$: The real and imaginary part of a complex number, respectively.
T_v: The tangent space of a Minkowskian hypersurface at its point v.
$\mathcal{S}, \mathcal{T}, \mathcal{L}$: The set of space-like, time-like and light-like vectors respectively.
S, T: The space-like and time-like orthogonal direct components of a generalized Minkowski space, respectively.
$\{e_1,\ldots,e_k,e_{k+1},\ldots,e_n\}$: An Auerbach basis of a generalized Minkowski space with $\{e_1,\ldots,e_k\} \subset S$ and $\{e_{k+1},\ldots,e_n\} \subset T$, respectively.
H, H^+: The sphere of radius i and its upper sheet, respectively.

1.2. *Completion of the preliminaries*

In this introduction we give some notions explaining also how this paper came to be; these observations are needed for our investigations.

1.2.1. *Semi-inner-product spaces*

A generalization of the inner product and the inner product spaces (briefly i.p. spaces) was raised by G. Lumer in [1].

Definition 1 (*[1]*)**.** The *semi-inner-product (s.i.p.)* on a complex vector space V is a complex function $[x,y] : V \times V \longrightarrow \mathbb{C}$ with the following properties:

s1: $[x+y,z] = [x,z] + [y,z]$,
s2: $[\lambda x,y] = \lambda[x,y]$ for every $\lambda \in \mathbb{C}$,
s3: $[x,x] > 0$ when $x \neq 0$,
s4: $|[x,y]|^2 \leq [x,x][y,y]$.

A vector space V with a s.i.p. is an *s.i.p. space*.

Lumer proved that an s.i.p. space is a normed vector space with norm $\|x\| = \sqrt{[x,x]}$ and, on the other hand, that every normed vector space can be represented as an s.i.p. space. In [2] Giles showed that the following homogeneity property holds:

s5: $[x,\lambda y] = \bar{\lambda}[x,y]$ for all complex λ.

This can be imposed, and all normed vector spaces can be represented as s.i.p. spaces with this property. Giles also introduced the concept of **continuous s.i.p. space** as an s.i.p. space having the additional property

s6: For any unit vectors $x, y \in S$, $\Re\{[y, x+\lambda y]\} \to \Re\{[y,x]\}$ for all real $\lambda \to 0$.

The space is uniformly continuous if the above limit is reached uniformly for all points x, y of the unit sphere S.
A characterization of the continuous s.i.p. space is based on the differentiability property of the space.

Definition 2 (*[2]*). A normed space is *Gâteaux differentiable* if for all elements x, y of its unit sphere and real values λ, the limit

$$\lim_{\lambda\to 0}\frac{\|x+\lambda y\|-\|x\|}{\lambda}$$

exists. A normed vector space is *uniformly Frèchet differentiable* if this limit is reached uniformly for the pair x, y of points from the unit sphere.

Giles proved in [2] that

Theorem 1 ([2]). *An s.i.p. space is a continuous (uniformly continuous) s.i.p. space if and only if the norm is Gâteaux (uniformly Frèchet) differentiable.*

In the second part of this paper we need a stronger condition on differentiability of the s.i.p. space. Therefore we define the differentiable s.i.p. as follows:

Definition 3. A *differentiable s.i.p. space* is an continuous s.i.p. space where the s.i.p. has the additional property

s6′: For every three vectors x, y, z and real λ

$$[x,\cdot]'_z(y) := \lim_{\lambda\to 0}\frac{\Re\{[x,y+\lambda z]\}-\Re\{[x,y]\}}{\lambda}$$

does exist. We say that the s.i.p. space is *continuously differentiable*, if the above limit, as a function of y, is continuous.

First we note that the equality $\Im\{[x,y]\}=\Re\{[-\mathrm{i}x,y]\}$ together with the above property guarantees the existence and continuity of the complex limit:

$$\lim_{\lambda\to 0}\frac{[x,y+\lambda z]-[x,y]}{\lambda}.$$

Analogously to the theorem of Giles (see Theorem 3 in [2]) we combine this definition with the differentiability properties of the norm function generated by the s.i.p. First we introduce the notion of Gâteaux derivative of the norm. Let

$$\|\cdot\|'_x(y) := \lim_{\lambda\to 0}\frac{\|y+\lambda x\|-\|y\|}{\lambda}$$

be the derivative of the norm in the direction of x at the point y. Similarly, we use the notation

$$\|\cdot\|''_{x,z}(y) := \lim_{\lambda\to 0}\frac{\|\cdot\|'_x(y+\lambda z)-\|\cdot\|'_x(y)}{\lambda},$$

which is the second derivative of the norm in the directions x and z at the point y. We need the following useful lemma going back, with different notation, to McShane [3] or Lumer [4].

Lemma 1 ([4]). *If E is any s.i.p. space with $x, y \in E$, then*

$$\|y\|(\|\cdot\|'_x(y))^- \le \Re\{[x,y]\} \le \|y\|(\|\cdot\|'_x(y))^+$$

holds, where $(\|\cdot\|'_x(y))^-$ and $(\|\cdot\|'_x(y))^+$ denotes the left hand and right hand derivatives with respect to the real variable λ. In particular, if the norm is differentiable, then

$$[x,y]=\|y\|\{(\|\cdot\|'_x(y))+\|\cdot\|'_{-\mathrm{i}x}(y)\}.$$

Now our theorem is the following:

Theorem 2. *An s.i.p. space is a (continuously) differentiable s.i.p. space if and only if the norm is two times (continuously) Gâteaux differentiable. The connection between the derivatives is*

$$\|y\|(\|\cdot\|''_{x,z}(y)) = [x,\cdot]'_z(y)-\frac{\Re[x,y]\Re[z,y]}{\|y\|^2}.$$

The proof of Theorem 2 is a technical one, using Lemma 1 and some non-trivial, but not too hard calculations; so it can be omitted.

1.2.2. Further remarks on the theory of s.i.p.

Nath gave in [5] a straightforward generalization of an s.i.p., by replacing the Cauchy–Schwarz inequality by Hölder's inequality. He showed that this kind of generalized s.i.p. space induces a norm by setting $\|x\| = [x,x]^{\frac{1}{p}} 1 \le p \le \infty$, and that for every normed space a generalized s.i.p. space can be constructed. (For $p=2$, this theorem reduces to Theorem 2 of Lumer.) The connection between the Lumer–Giles s.i.p. and the generalized s.i.p. of Nath is simple. For any p, the s.i.p. $[x,y]$

defines a generalized s.i.p. by the equality

$$\widehat{[x,y]} = [y,y]^{\frac{p-2}{p}}[x,y].$$

The s.i.p. has the homogeneity property of Giles if and only if Nath's generalized s.i.p. satisfies the $(p-1)$-homogeneity property

s5″: $\widehat{[x,\lambda y]} = \bar{\lambda}|\lambda|^{p-2}\widehat{[x,y]}$ for all complex λ.

Thus, in this paper we will concentrate only to the original version of the s.i.p.

From the geometric point of view we know that if K is a 0-symmetric, bounded, convex body in the Euclidean n-space $\mathbb{R}^n$ (with fixed origin O), then it defines a norm whose unit ball is K itself (see [6]). Such a space is called (Minkowski or) normed linear space. Basic results on such spaces are collected in the surveys [7,8], and [9]. In fact, the norm is a continuous function which is considered (in geometric terminology, as in [6]) as a gauge function. Combining this with the result of Lumer and Giles we get that a normed linear space can be represented as an s.i.p. space. The metric of such a space (called Minkowski metric), i.e., the distance of two points induced by this norm, is invariant with respect to translations.

1.2.3. *Indefinite inner product spaces*

Another concept of Minkowski space was also raised by Minkowski and used in Theoretical Physics and Differential Geometry, based on the concept of indefinite inner product. (See, e.g., [10].)

Definition 4 (*[10]*)**.** The *indefinite inner product (i.i.p.)* on a complex vector space V is a complex function $[x,y]: V\times V \longrightarrow \mathbb{C}$ with the following properties:

i1 : $[x+y,z] = [x,z] + [y,z]$,
i2 : $[\lambda x,y] = \lambda[x,y]$ for every $\lambda \in \mathbb{C}$,
i3 : $[x,y] = \overline{[y,x]}$ for every $x,y \in V$,
i4 : $[x,y] = 0$ for every $y \in V$ then $x = 0$.

A vector space V with an i.i.p. is called an *indefinite inner product space*.

We recall, that a subspace of an i.i.p. space is positive (non-negative) if all of its nonzero vectors have positive (non-negative) scalar squares. The classification of subspaces of an i.i.p. space with respect to the positivity property is also an interesting question. First we pass to the class of subspaces which are peculiar to i.i.p. spaces, and which have no analogues in the spaces with a definite inner product.

Definition 5 (*[10]*)**.** A subspace N of V is called *neutral* if $[v,v] = 0$ for all $v \in N$.

In view of the identity

$$[x,y] = \frac{1}{4}\{[x+y,x+y] + i[x+\mathrm{i}y,x+\mathrm{i}y] - [x-y,x-y] - \mathrm{i}[x-\mathrm{i}y,x-\mathrm{i}y]\},$$

a subspace N of an i.i.p. space is neutral if and only if $[u,v] = 0$ for all $u,v \in N$. Observe also that a neutral subspace is non-positive and non-negative at the same time, and that it is necessarily degenerate. Therefore the following statement can be proved.

Theorem 3 (*[10]*)**.** *An non-negative (resp. non-positive) subspace is the direct sum of a positive (resp. negative) subspace and a neutral subspace.*

We note that the decomposition of a non-negative subspace U into a direct sum of a positive and a neutral component is, in general, not unique. However, the dimension of the positive summand is uniquely determined.

The standard mathematical model of space–time is a four-dimensional i.i.p. space with signature $(+,+,+,-)$, also called Minkowski space in the literature. Thus we have a well-known homonymism with the notion of Minkowski space!

1.3. *Results*

In the first part of this paper we introduce the concept of semi-indefinite inner product (s.i.i.p.) and the generalized notion of Minkowski space. We also define the concept of orthogonality of such spaces (Section 2).

In the second part we give the definition of a local Minkowski space in a generalized space–time model. This construction is somehow analogous to the definition of a Riemannian manifold (e.g., a geometric Minkowski space or hyperbolic space) by embedding it into an i.i.p. space (Section 3), and we will call it Minkowski–Finsler space.

We prove only those statements whose proof cannot be found in the literature. (These are: Statement 1, Theorems 7–11 and 13–15, and Lemmas 2–4.) The author uses the statements already proved in the literature without proof, but gives references to them. As non-proved statements we have Theorem 2 and Statement 1, respectively. Theorem 2 is used in this paper, but its proof is straightforward, while Statement 1 is such an interesting observation which is not used in the rest of this paper.

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2. Unification and geometrization

2.1. Semi-indefinite inner product spaces

In this section, let s1, s2, s3, s4, be the four defining properties of an s.i.p., and s5 be the homogeneity property of the second argument imposed by Giles, respectively. (As to the names: s1 is the additivity property of the first argument, s2 is the homogeneity property of the first argument, s3 means the positivity of the function, s4 is the Cauchy–Schwarz inequality.)

On the other hand, i1 = s1, i2 = s2, i3 is the antisymmetry property and i4 is the non-degeneracy property of the product. It is easy to see that s1, s2, s3, s5 imply i4, and if N is a positive (negative) subspace of an i.i.p. space, then s4 holds on N. In the following definition we combine the concepts of s.i.p. and i.i.p.

Definition 6. The *semi-indefinite inner product (s.i.i.p.)* on a complex vector space V is a complex function $[x, y] : V \times V \longrightarrow \mathbb{C}$ with the following properties:

1. $[x + y, z] = [x, z] + [y, z]$ (additivity in the first argument),
2. $[\lambda x, y] = \lambda[x, y]$ for every $\lambda \in \mathbb{C}$ (homogeneity in the first argument),
3. $[x, \lambda y] = \overline{\lambda}[x, y]$ for every $\lambda \in \mathbb{C}$ (homogeneity in the second argument),
4. $[x, x] \in \mathbb{R}$ for every $x \in V$ (the corresponding quadratic form is real valued),
5. if either $[x, y] = 0$ for every $y \in V$ or $[y, x] = 0$ for all $y \in V$, then $x = 0$ (non-degeneracy),
6. $|[x, y]|^2 \leq [x, x][y, y]$ holds on non-positive and non-negative subspaces of V, respectively (the Cauchy–Schwarz inequality is valid on positive and negative subspaces, respectively).

A vector space V with an s.i.i.p. is called an *s.i.i.p. space*.

The interest in s.i.i.p. spaces depends largely on the example spaces given by the s.i.i.p. space structure.

Example 1. We conclude that an s.i.i.p. space is a homogeneous s.i.p. space if and only if the property s3 holds, too. An s.i.i.p. space is an i.i.p. space if and only if the s.i.i.p. is an antisymmetric product. In this latter case $[x, x] = \overline{[x, x]}$ implies 4, and the function is also Hermitian linear in its second argument. In fact, we have: $[x, \lambda y + \mu z] = \overline{[\lambda y + \mu z, x]} = \overline{\lambda}\overline{[y, x]} + \overline{\mu}\overline{[z, x]} = \overline{\lambda}[x, y] + \overline{\mu}[x, z]$. It is clear that both of the classical "Minkowski spaces" can be represented either by an s.i.p or by an i.i.p., so automatically they can also be represented as an s.i.i.p. space.

Example 2. Let now $V = \langle\{e_1, \ldots, e_n\}\rangle$ be a finite-dimensional vector space, and C be the surface of a cross-polytope of dimension 3, which is defined by

$$C = \cup\{\mathrm{conv}\,\{\varepsilon_i e_i | i = 1, 2, 3\} \text{ for all choices of } \varepsilon_i = \pm 1\}.$$

It is clear that for a real vector $v \in C$ there exists at least one linear functional, and we choose exactly one $v^\star$ of the dual space with the property $v^\star(v) = (-1)^k$, where k is the combinatorial dimension of that combinatorial face F_v of C which contains the point v in its relative interior. (It is easy to see that $k + 1$ is the cardinality of the nonzero coefficients of the representation of v.) For $\lambda v \in V$, where $v \in C$, and any real λ (by Giles' method) we choose $(\lambda v)^\star = \lambda v^\star$. Given such a mapping from V into $V^\star$, it is readily verified that the product

$$[u, v] = v^\star(u)$$

satisfies the properties 1–4. Property 5 also holds, since there is no vector v for which $v^\star(v) = 0$. Finally, every two-dimensional subspace has vectors v and w by $v^\star(v) > 0$ and $w^\star(w) < 0$, and there are neither positive nor negative subspaces with dimension at least two, implying that also property 6 holds.

Example 3. In an arbitrary complex normed linear space V we can define an s.i.i.p. which is a generalization of a representing s.i.p. of the norm function. Let now C be the unit sphere of the space V. By the Hahn–Banach theorem there exists at least one continuous linear functional, and we choose exactly one such that $\|\widetilde{v}^\star\| = 1$ and $\widetilde{v}^\star(v) = 1$. Consider a sign function $\varepsilon([v])$ with value ± 1 on $C/\sim$, where $C/\sim$ means the factorization of C by the equivalence relation

$$\text{“}x \sim y \Leftrightarrow x = \lambda y \text{ with a nonzero } \lambda\text{”}.$$

If now $\varepsilon([v]) = 1$ let it be denoted by $v^\star = \widetilde{v}^\star$, and $\varepsilon([v]) = -1$ defines $v^\star = -\widetilde{v}^\star$. Finally, extend it homogeneously to V by the equality $(\lambda v)^\star = \overline{\lambda} v^\star$, as in the previous example. Of course, for an arbitrary vector v of V the corresponding linear functional satisfies the equalities $v^\star(v) := \varepsilon([v])\|v\|^2$ and $\|v\| = \|v^\star\|$. Now the function

$$[u, v] = v^\star(u)$$

satisfies 1–5. If U is a non-negative subspace, then it is positive and we have for all nonzero $u, v \in U$ that

$$|[u, v]| = |v^\star(u)| = \frac{|v^\star(u)|}{\|u\|}\|u\| \leq \|v^\star\|\|u\| = \|v\|\|u\|,$$

proving 6.

2.2. The generalized Minkowski space

Before the definition we prove an important lemma.

Lemma 2. *Let* $(S, [\cdot,\cdot]_S)$ *and* $(T, -[\cdot,\cdot]_T)$ *be two s.i.p. spaces. Then the function* $[\cdot,\cdot]^- : (S+T)\times(S+T) \longrightarrow \mathbb{C}$ *defined by*

$$[s_1+t_1, s_2+t_2]^- := [s_1,s_2] - [t_1,t_2]$$

is an s.i.p. on the vector space $S+T$.

Proof. The function $[\cdot,\cdot]^-$ is non-negative, as we can easily see from its definition. First we prove the linearity in the first argument. We have

$$\begin{aligned}[\lambda'(s'+t')+\lambda''(s''+t''), s+t]^- &= [\lambda's'+\lambda''s'', s]_S - [\lambda't'+\lambda''t'', t]_T \\ &= \lambda'[s',s]_S + \lambda''[s'',s]_S - \lambda'[t',t]_T - \lambda''[t'',t]_T \\ &= \lambda'[s'+t', s+t]^- + \lambda''[s''+t'', s+t]^-.\end{aligned}$$

The homogeneity in the second argument is trivial. In fact, we have

$$[s'+t', \lambda(s+t)]^- = [s',\lambda s]_S - [t',\lambda t]_T = \overline{\lambda}[s'+t', s+t]^-.$$

Finally we check the Cauchy–Schwarz inequality. We have

$$\begin{aligned}|[s_1+t_1, s_2+t_2]^-|^2 &= [s_1+t_1,s_2+t_2]^-\overline{[s_1+t_1,s_2+t_2]^-} \\ &= ([s_1,s_2]_S - [t_1,t_2]_T)(\overline{[s_1,s_2]_S} - \overline{[t_1,t_2]_T}) \\ &= [s_1,s_2]_S\overline{[s_1,s_2]_S} + [t_1,t_2]_T\overline{[t_1,t_2]_T} + [s_1,s_2]_S(-\overline{[t_1,t_2]_T}) + (-[t_1,t_2]_T)\overline{[s_1,s_2]_S} \\ &\le [s_1,s_1]_S[s_2,s_2]_S + [t_1,t_1]_T[t_2,t_2]_T + 2\Re\{[s_1,s_2]_S(-\overline{[t_1,t_2]_T})\} \\ &\le [s_1,s_1]_S[s_2,s_2]_S + [t_1,t_1]_T[t_2,t_2]_T + 2|[s_1,s_2]_S||[t_1,t_2]_T| \\ &\le [s_1,s_1]_S[s_2,s_2]_S + [t_1,t_1]_T[t_2,t_2]_T + 2\sqrt{[s_1,s_1]_S[s_2,s_2]_S[t_1,t_1]_T[t_2,t_2]_T},\end{aligned}$$

and by the inequality between the arithmetic and geometric means we get that

$$\begin{aligned}&[s_1,s_1]_S[s_2,s_2]_S + [t_1,t_1]_T[t_2,t_2]_T + 2\sqrt{[s_1,s_1]_S[s_2,s_2]_S[t_1,t_1]_T[t_2,t_2]_T} \\ &\quad\le [s_1,s_1]_S[s_2,s_2]_S + [t_1,t_1]_T[t_2,t_2]_T + [s_1,s_1]_S(-[t_2,t_2]_T) + (-[t_1,t_1]_T)[s_2,s_2]_S \\ &\quad= ([s_1,s_1]_S - [t_1,t_1]_T)([s_2,s_2]_S - [t_2,t_2]_T) = [s_1+t_1, s_1+t_1]^-[s_2+t_2,s_2+t_2]^-. \quad \square\end{aligned}$$

It is possible that the s.i.i.p. space V is a direct sum of its two subspaces where one of them is positive and the other one is negative. Then we have two more structures on V, an s.i.p. structure (by Lemma 2) and a natural third one, which we will call Minkowskian structure. More precisely, we have

Definition 7. Let $(V, [\cdot,\cdot])$ be an s.i.i.p. space. Let $S, T \le V$ be positive and negative subspaces, where T is a direct complement of S with respect to V. Define a product on V by the equality $[u,v]^+ = [s_1+t_1, s_2+t_2]^+ = [s_1,s_2] + [t_1,t_2]$, where $s_i \in S$ and $t_i \in T$, respectively. Then we say that the pair $(V, [\cdot,\cdot]^+)$ is a *generalized Minkowski space with Minkowski product* $[\cdot,\cdot]^+$. We also say that V is a *real generalized Minkowski space* if it is a real vector space and the s.i.i.p. is a real valued function.

Remark. 1. The Minkowski product defined by the above equality satisfies properties 1-5 of the s.i.i.p. But in general, property 6 does not hold. To see this, define an s.i.i.p. space in the following way:

Consider a two-dimensional L^∞ space S of the embedding three-dimensional Euclidean space E^3. Choose an orthonormed basis $\{e_1,e_2,e_3\}$ of E^3 for which $e_1,e_2 \in S$, and give an s.i.p. associated to the L^∞ norm as follows:

$$[x_1e_1+x_2e_2, y_1e_1+y_2e_2]_S := x_1y_1 \lim_{p\to\infty} \frac{1}{\left(1+\left(\frac{y_2}{y_1}\right)^p\right)^{\frac{p-2}{p}}} + x_2y_2 \lim_{p\to\infty} \frac{1}{\left(1+\left(\frac{y_1}{y_2}\right)^p\right)^{\frac{p-2}{p}}}.$$

By Lemma 2 the function

$$[x_1e_1+x_2e_2+x_3e_3, y_1e_1+y_2e_2+y_3e_3]^- := [x_1e_1+x_2e_2, y_1e_1+y_2e_2]_S + x_3y_3$$

is an s.i.p. on E^3 associated to the norm

$$\sqrt{[x_1e_1+x_2e_2+x_3e_3, x_1e_1+x_2e_2+x_3e_3]^-} := \sqrt{\max\{|x_1|,|x_2|\}^2 + x_3^2}.$$

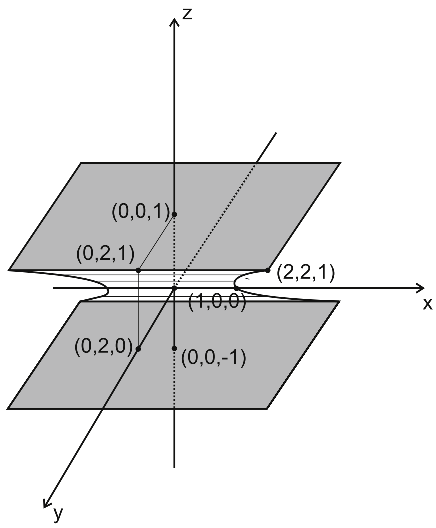

Fig. 1. The unit sphere of the positive subspace $z = \frac{1}{2}y$ in Remark 1.

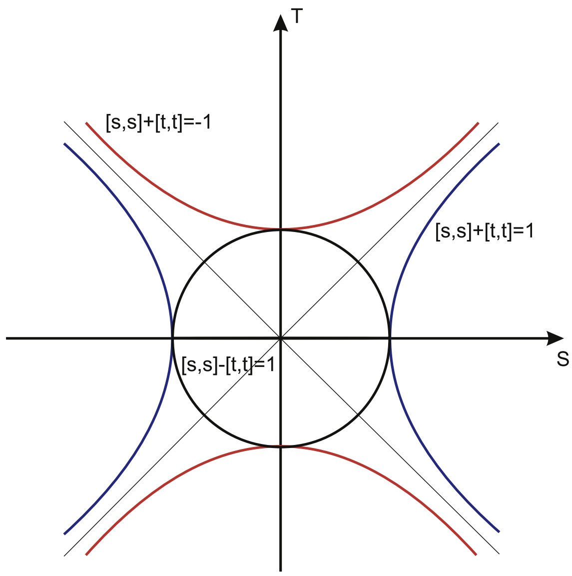

Fig. 2. The real and imaginary unit spheres in dimension two.

By the method of Example 3 consider such a sign function for which $\varepsilon(v)$ is equal to 1 if v is in $S \cap C$, and is equal to -1 if $v = e_3$ holds. (C denotes the unit sphere, as in the previous examples.) This sign function determines an s.i.i.p. $[\cdot,\cdot]$ and thus generates a Minkowski product $[\cdot,\cdot]^+$, for which the corresponding square root function is

$$f(v) := \sqrt{[x_1e_1 + x_2e_2 + x_3e_3, x_1e_1 + x_2e_2 + x_3e_3]^+}$$
$$= \sqrt{\max\{|x_1|, |x_2|\}^2 - x_3^2}.$$

As it can be easily seen, the plane $x_3 = \alpha x_2$ for $0 < \alpha < 1$ is a positive subspace with respect to the Minkowski product, but its unit ball is not convex (see Fig. 1). But $f(v)$ is homogeneous, correspondingly it is not subadditive. Since the Cauchy–Schwarz inequality implies subadditivity, this inequality remains false in this positive subspace.

2. By Lemma 2 the s.i.p. $\sqrt{[v,v]^-}$ is a norm function on V which can give an embedding space for a generalized Minkowski space. This situation is analogous to the situation when a pseudo-Euclidean space is obtained from a Euclidean space by the action of an i.i.p. (see Fig. 2).

2.3. Further examples for non-trivial s.i.i.p. and generalized Minkowski spaces

2.3.1. C^2 normsquare function and the associated s.i.i.p. space

In this section (by Theorem 4) we give a method to construct s.i.i.p. spaces with more differentiability properties.

A C^2 Minkowski space is an n-dimensional affine space with metric $d(x, y) = F(y - x)$, where F is the (Minkowskian) norm function of the associated vector space and the following conditions are satisfied:

n1 $F(x) > 0$ for $x \neq 0$,
n2 $F(\lambda x) = |\lambda|F(x)$ for all real λ,
n3 $F(x + y) \leq F(x) + F(y)$, where equality holds for $x, y \neq 0$ if and only if $y = \lambda x$ for some real $\lambda > 0$,
n4 $F(x)$ is of class C^2 in each of its n arguments, the components of the vector x.

In his paper [2], Giles proved that there is a natural form for an s.i.p. in the associated vector space for which it is a uniform s.i.p. space. The importance of uniform s.i.p. spaces is based on the fact that in such a space the representation theory of Riesz holds and its dual space is also uniform. Now we define a similar class of s.i.i.p. spaces associated to the concept of C^2 normsquare function.

Definition 8. Consider $\mathbb{R}^n$ as a real vector space V and let $G : V \longrightarrow \mathbb{R}$ be a function on it. If it satisfies the two properties

pn1 $G(\lambda x) = \lambda^2 G(x)$ for real λ,
pn2 if $G|_W \geq (\leq) 0$ on a subspace W of V, then for the positive function $\sqrt{G|_W}$ ($\sqrt{-G|_W}$) the convexity property [n3] holds,

then we say that G is a *normsquare function* on V. If we also require the differentiability property [n4] for G, then we say that the normsquare function is a C^2 one.

It is easy to see that the square of a norm function is a normsquare function, and every i.i.p. defines a normsquare function by $G(x) = [x, x]$. For C^2 normsquares we have

Theorem 4. *If G is a C^2 normsquare function on the real vector space V, then there is an associated s.i.i.p. which gives uniform s.i.p. structures on positive (resp. negative) subspaces of V.*

Proof. From the derivatives of a homogeneous function of order 2, we have for G, that

$$DG|_{\lambda x}x = 2\lambda G(x) \quad \text{and} \quad x^T D^2 G|_{\lambda x}x = 2G(x)$$

where $D(G)_x$ means the total (Frèchet) derivative of the function G at the point x. Substituting $\lambda = 1$ into these formulas, we get

$$G(x) = \frac{1}{2}x^T D^2(G|_x)x = \frac{1}{2}DG|_x x.$$

Let the associated s.i.i.p. be defined by the equality

$$[x, y] = \frac{1}{2}x^T D^2 G|_y y.$$

It is easy to see that this function satisfies properties 1, 2, 4, 5 of a s.i.i.p. Property 3 follows from the fact that $D^2G|_{\lambda x}$ is not depending on the value of λ. Finally, property 6 is established by the imposed differentiability property and the convexity property pn2 as follows: It is clear that the function $\sqrt{G|_W} : W \longrightarrow \mathbb{R}^+$ is a homogeneous C^2 function. So we have

$$D\sqrt{G}|_x x = \sqrt{G}(x) \quad \text{and} \quad x^T D^2\sqrt{G}|_x x = 0.$$

From the identity

$$D^2 G|_x = 2(\sqrt{G}(x) D^2(\sqrt{G}|_x) + D\sqrt{G}\,|_x^T\, D\sqrt{G}|_x)$$

we get that

$$\begin{aligned}\frac{1}{2}D^2 G|_y y &= \sqrt{G}(y)D^2\sqrt{G}|_y y + D\sqrt{G}\,|_y^T\, D\sqrt{G}|_y y\\ &= D\sqrt{G}\,|_y^T\,\sqrt{G}(y) = \sqrt{G}(y)D\sqrt{G}\,|_y^T\,.\end{aligned}$$

Thus

$$\begin{aligned}|[x, y]| &= \left|\frac{1}{2}x^T D^2 G|_y y\right| = \left|x^T\sqrt{G}(y)D\sqrt{G}\,|_y^T\right|\\ &= \sqrt{G}(y)|x^T D\sqrt{G}\,|_y^T\,| = \sqrt{G}(y)|D\sqrt{G}|_y x|.\end{aligned}$$

But by the second Mean Value Theorem we have that

$$\sqrt{G}(x) = \sqrt{G}(y) + D\sqrt{G}|_y(x - y) + (x - y)^T D^2\sqrt{G}|_{y+\theta(x-y)}(x - y),$$

where $0 < \theta < 1$. Since for a convex C^2 function the last summand is non-negative, we have that

$$D\sqrt{G}|_y x \leq \sqrt{G}(x),$$

implying that

$$|D\sqrt{G}|_y x| \leq \sqrt{G}(x).$$

Thus

$$|[x, y]| \leq \sqrt{G}(y)\sqrt{G}(x) = \sqrt{[x, x][y, y]},$$

as we stated. Now the last statement is a consequence of Giles results in [2]. □

If we have a normed vector space with an associated symmetric, bilinear function, then the positive semi-definiteness of the function implies the Cauchy–Schwarz inequality. If the associated function is linear in its first argument and homogeneous in its second one, the semi-definiteness property alone does not imply the Cauchy–Schwarz inequality, as we can see in the following example.

Example 4. Let V be a two-dimensional vector space with the Euclidean norm

$$\|(x, y)^T\| := \sqrt{x^2 + y^2},$$

where the coordinates can be computed with respect to a fixed orthonormed basis. It is easy to see that

$$[u_1, u_2] = (x_1 \cdot x_2 + 2y_1 \cdot y_2)\frac{x_2^2 + y_2^2}{x_2^2 + 2y_2^2}$$

is an associated product, where $u_i = (x_i, y_i)^T$. This function is linear in its first argument, homogeneous in its second one and associated to the norm. On the other hand, for $u_1 = (1, 2)^T$ and $u_2 = (1, 1)^T$ we get

$$[(1, 2)^T, (1, 1)^T] = \frac{10}{3} > \sqrt{10} = \sqrt{[(1, 2)^T, (1, 2)^T]}\sqrt{[(1, 1)^T, (1, 1)^T]},$$

counterexamples to the Cauchy–Schwarz inequality. The reason for this situation is that the norm of the linear functional associated to the first argument of the product and the fixed vector u_2 is greater than the norm of the vector u_2.

2.3.2. Generalized Minkowski spaces generated by L_p norms

In [2] Giles gives an associated s.i.p. for L_p spaces. Using the method of our Example 3, we can define s.i.i.p. spaces based on the L_p structure. Let $(S, [\cdot, \cdot]_S)$ be the s.i.p. space, where S is the real Banach space $L_{p_1}(X, \mathcal{S}, \mu)$ and T is the real Banach space $L_{p_2}(Y, \mathcal{S}', \nu)$, respectively. If $1 < p_1, p_2 \leq \infty$, then these spaces can be readily expressed, as a uniform s.i.p. space with s.i.p. defined by

$$[s_1, s_2]_S = \frac{1}{\|s_2\|_{p_1}^{p_1-2}} \int_X s_1|s_2|^{p_1-1}\mathrm{sgn}\,(s_2)\mathrm{d}\mu$$

and

$$[t_1, t_2]_T = \frac{1}{\|t_2\|_{p_2}^{p_2-2}} \int_Y t_1|t_2|^{p_2-1}\mathrm{sgn}\,(t_2)\mathrm{d}\nu,$$

respectively. Consider the real vector space $S + T$ with the s.i.p.

$$[u, v]^- := [s_1, s_2]_S + [t_1, t_2]_T.$$

This is also a uniform s.i.p. space, since in Lemma 2 we proved that it is an s.i.p. space and

$$\begin{aligned} |[z, x] - [z, y]| &= |([s_3, s_1]_S - [s_3, s_2]_S) + ([t_3, t_1]_T - [t_3, t_2]_T)| \\ &\leq |[s_3, s_1]_S - [s_2, s_1]_S| + |[t_3, t_1]_T - [t_2, t_1]_T| \\ &\leq 2(p_1 - 1)\|s_1 - s_2\|_{p_1} + 2(p_2 - 2)\|t_1 - t_2\|_{p_2}, \end{aligned}$$

implying that the space is uniformly continuous. It has been established that such spaces are uniformly convex (see [11], p. 403). By the method of Example 3 we can define an s.i.i.p. space on $S + T$ such that the subspace S is positive and T is a negative one, and a Minkowski space by the Minkowski product

$$[u, v]^+ := [s_1, s_2]_S - [t_1, t_2]_T,$$

respectively. (In Fig. 3 one can see the case when $\dim S = \dim T + 1 = 2$ and the norm of S is L_∞.)

It is easy to see that by this method, starting with arbitrary two normed spaces S and T, one can mix a generalized Minkowski space. Of course its smoothness property is basically determined by the analogous properties of S and T.

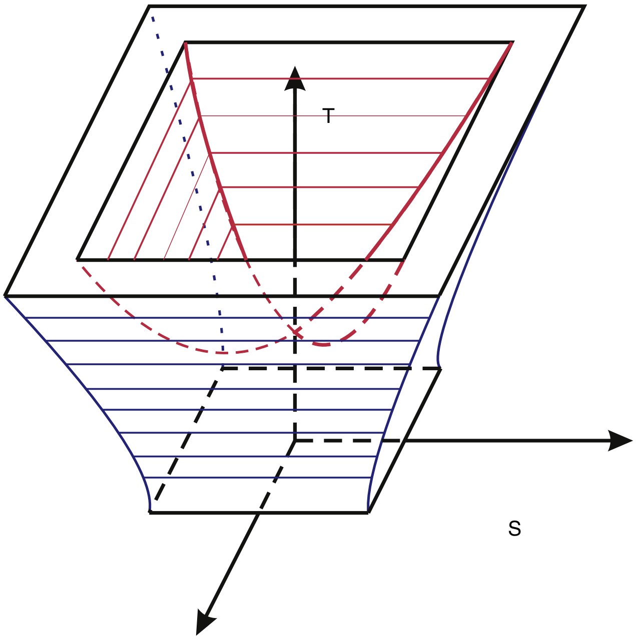

Fig. 3. The case of the norm L_∞.

2.4. Orthogonality

We now investigate an interesting classical topic, namely the types of orthogonalities. There are several definitions of orthogonality in a normed linear space which is not an inner product space (i.p. space), and one cannot find a concept which is more natural than the others. First we note that the generalization of the usual i.p. concept of orthogonality is not unique, that is, every concept of orthogonality in an s.i.p. space can be regarded to be reasonable if it gives back the usual orthogonality in the i.p. sense. Thus we have a lot of possibilities to define orthogonality. Some of these can be found in the papers [12–18] and [19]. We recall only the most important concepts.

Let $(V, \|\cdot\|)$ be a normed space and $x, y \in V$. Denote by $x\perp y$ the expression "y orthogonal to x".

B-J $x\perp y$ iff $\|x\| \leq \|x + \lambda y\|$ for any $\lambda \in \mathbb{C}$ (Birkhoff–James, 1935);
I $x\perp y$ iff $\|x + y\| = \|x - y\|$ (James or isosceles orthogonality, 1945);
P $x\perp y$ iff $\|x\|^2 + \|y\|^2 = \|x - y\|^2$ (Pythagorean, 1945).

If now we consider the theory of s.i.p. in the sense of Lumer–Giles, we have a natural concept of orthogonality. For the unified terminology we change the original notation of Giles and use instead

Definition 9. The vector *y is orthogonal to the vector* x if $[y, x] = 0$.

Since s.i.p. is neither antisymmetric in the complex case nor symmetric in the real one, this definition of orthogonality is not symmetric in general.

Giles proved that in a continuous s.i.p. space x is orthogonal to y in the sense of the s.i.p. if and only if x is orthogonal to y in the sense of B-J. We note that the s.i.p. orthogonality implies the B-J orthogonality in every normed space. Lumer pointed out that a normed linear space can be transformed into an s.i.p. space in a unique way if and only if its unit sphere is smooth (i.e., there is a unique supporting hyperplane at each point of the unit sphere). In this case the corresponding (unique) s.i.p. has the homogeneity property [s5]. Imposing the additivity property of the second argument, namely

s5′ : For every $x, y, z \in V [x, y + z] = [x, y] + [x, z]$,

the s.i.p. will be a bilinear function. But if the s.i.p. is the unique representation of a given norm and if it is bilinear, then it is antisymmetric (resp. symmetric) in the complex (resp. real) case. In fact, define the function $[x, y]' : V \times V \longrightarrow \mathbb{C}$ by the equality $[x, y]' = \overline{[y, x]}$. The properties s1, s2, s3, s5 trivially hold for this function, and the inequality

$$[x, y]'\overline{[x, y]'} = \overline{[y, x]}[y, x] \leq [y, y][x, x] = [y, y]'[x, x]'$$

shows s4. By the unicity of the s.i.p. $[\cdot, \cdot]'$ is equal to the original one, so the s.i.p. is antisymmetric (resp. symmetric), consequently the space is a Hilbert space (i.e., an i.p. space). Summarizing, we can say that a unique s.i.p. which is not an i.p. is not additive in its second argument.

There are many known results and open problems related to types of orthogonalities, but as we saw, the s.i.p. orthogonality of a pair of vectors essentially coincides with their B-J orthogonality in the represented normed space. In this paper we would like to generalize s.i.p.; so we have to concentrate only on B-J orthogonality.

Another interesting problem is the orthogonality of subspaces. It is clear that each of the orthogonality relations gives an orthogonality for the subspaces of V.

Definition 10. Let $X, Y \leq V$ be two subspaces. We say that X *is orthogonal to* Y if, for every pairs of vectors $x \in X$ and $y \in Y$, x is orthogonal to y.

It can be proved (and we now mention without proof) that the strongest subspace orthogonality criterion is the Pythagorean one.

Statement 1. *With respect to subspaces, Pythagorean orthogonality implies any other orthogonality relation.*

On the other hand, in an i.i.p. space there is a natural definition of orthogonality.

Definition 11 ([10])**.** Let $(V, [\cdot,\cdot])$ be an i.i.p. space and U be any subset of V. Define the *orthogonal companion* of U in V by

$$U^\perp = \{v \in V | [v, u] = 0 \text{ for all } u \in U\}.$$

Clearly, $U^\perp$ is a subspace in V, and we are particularly interested in the case when U itself is a subspace of V. In the latter case, it is not generally true that $U^\perp$ is a direct complement of U. In contrast it is true that, for any subspace U, the sum of the dimensions of the subspaces U and $U^\perp$ is equal to the dimension of V. The exact answer for this problem uses the concept of non-degeneracy of a subspace, which means that the i.i.p., restricted to this subspace, is also non-degenerate. The precise statement is the following one.

Theorem 5 ([10])**.** *The subspace $U^\perp$ is a direct complement to U in V if and only if U is non-degenerate.*

In particular, the orthogonal companion of a non-degenerate subspace is again non-degenerate.

In an i.p. space, the construction of a mutually orthogonal set of vectors $u_1, \ldots, u_n$, for which each subset $u_1, \ldots, u_k$ $(k \leq n)$ spans the same subspace as a subset of a given linearly independent set, plays a fundamental role. The well-known Gram–Schmidt process is of this type. Motivated by applications, attention will be given to sets of vectors $u_1, \ldots, u_n$ for which $[u_i, u_i] \neq 0$ holds, for each i. (Such a vector is called non-neutral.) First of all note that any set of non-neutral vectors, which is orthogonal, is necessarily linearly independent. This yields the concept of regular orthogonalization.

A system of vectors $u_1, \ldots, u_n$, which are mutually orthogonal is said to be a **regular orthogonalization** of $v_1, \ldots, v_n$ if it contains only non-neutral vectors with the property: $\langle\{u_1, \ldots, u_k\}\rangle = \langle\{v_1, \ldots, v_k\}\rangle$ for $k = 1, \ldots, n$. For any system of vectors $\{v_1, \ldots, v_k\}$, the Gram matrix is defined as a $k \times k$ matrix containing the pairwise scalar products of the vectors of the system. The basic statement on regular orthogonalization is the following

Theorem 6 ([10])**.** *The system of vectors $\{v_1, \ldots, v_n\}$ admits a regular orthogonalization if and only if the determinant of its Gram matrix is nonzero. This orthogonalization is essentially unique. If we have two such orthogonal systems of vectors, then their elements are distinct only in a scalar factor (with respect to the complex field $\mathbb{C}$).*

Now the pair $(V, [\cdot,\cdot])$ represents an s.i.i.p. space, where V is a complex (real) vector space. We define the orthogonality of such a space by a definition analogous to the definition of the orthogonality of an i.i.p. or s.i.p. space.

Definition 12. The vector v *is orthogonal to the vector* u if $[v, u] = 0$. If U is a subspace of V, define the orthogonal companion of U in V by

$$U^\perp = \{v \in V | [v, u] = 0 \text{ for all } u \in U\}.$$

We note that, as in the i.i.p. case, the orthogonal companion is always a subspace of V. The following theorem is analogous to Theorem 5 for i.i.p. spaces.

Theorem 7. *Let V be an n-dimensional s.i.i.p. space. Then the orthogonal companion of a non-neutral vector u is a subspace having a direct complement of the linear hull of u in V. The orthogonal companion of a neutral vector v is a degenerate subspace of dimension $n - 1$ containing v.*

Proof. First we observe that if the vector u is non-neutral and its subspace is $U = \langle\{u\}\rangle$, then

$$U^\perp = \{v | [v, \lambda u] = 0 \text{ for all } \lambda \in \mathbb{C}\} = \{v | [v, u] = 0\}.$$

Thus $U^\perp \cap U = \emptyset$. On the other hand, let the transformation $A : V \longrightarrow V$ be defined by $A : x \mapsto [x, u]u$. Obviously it is linear, because of the linearity in the first argument of an s.i.i.p. Its kernel is

$$\operatorname{Ker} A = \{x | [x, u]u = 0\} = \{x | [x, u] = 0\} = U^\perp,$$

and its image is

$$\operatorname{Im} A = \{[x, u]u | x \in V\}.$$

Clearly Im A is a subset of U. Since it is a subspace, and not a trivial one (e.g., $[u, u]u \neq 0$ by our assumption), it is equal to U. By the rank theorem on linear mappings we have that the dimension of $U^\perp$ is $n-1$, and that V is a direct sum of $U^\perp$ and U.

For a neutral vector v the above argument says that the kernel of A contains also v. Thus we get $\langle\{v\}\rangle \subset \langle\{v\}\rangle^\perp$. On the other hand, taking into consideration the non-degeneracy of V, dim Im $A \neq 0$. Thus again dim Im $A = 1$ and $\dim\langle\{v\}\rangle^\perp = n-1$, as we stated. □

Remark. Observe that this proof does not use the property 6 of the s.i.i.p. So this statement is true for any concepts of product satisfying properties 1–5. As we saw, the Minkowski product is also such a product.

The following theorem will be a common generalization of the theorem on diameters conjugated to each other in a real, finite-dimensional normed linear space, and of Theorem 6 on the existence of an orthogonal system in an i.i.p. space. A set of n diameters of the unit ball of an n-dimensional real normed space is considered to be a set of conjugate diameters if their normalized vectors have the following property: Choosing one of them, each vector in the linear span of the remaining direction vectors is orthogonal to it. An **Auerbach basis** of a normed space is a set of direction vectors having this property. Any real normed linear space has at least two Auerbach bases. One is induced by a cross-polytope inscribed in the unit ball of maximal volume (see [20]), and the other one by the midpoints of the facets of a circumscribed parallelotope of minimum volume (see [21]). These two ways of finding Auerbach bases are dual in the sense that if an Auerbach basis is induced by an inscribed cross-polytope of maximum volume, then any dual basis is induced by a circumscribed parallelotope of minimum volume, and vice versa (cf. [22]). If any minimum volume basis and maximum volume basis coincide, then by a result of Lenz (see [23]) we have that the space is a real i.p. space of finite dimension.

For generalized Minkowski spaces we have an analogous theorem:

Theorem 8. *In a finite-dimensional, real, generalized Minkowski space there is a basis with the Auerbach property. In other words, its vectors are orthogonal to the $(n-1)$-dimensional subspace spanned by the remaining ones. For this basis there is a natural number k, less or equal to n, for which $\{e_1, \ldots, e_k\} \subset S$ and $\{e_{k+1}, \ldots, e_n\} \subset T$. Finally, this basis has also the Auerbach property in the s.i.p. space $(V, [\cdot,\cdot]^-)$.*

Proof. Consider an Auerbach basis in $\{e_1, \ldots, e_k\} \subset S$ in the real normed space generated by the s.i.i.p. in S, and another one $\{e_{k+1}, \ldots, e_n\} \subset T$ in the other normed space generated by the negative of the s.i.i.p. on T. The union of these bases is an Auerbach basis for the Minkowski product and the s.i.p. $[\cdot,\cdot]^-$, respectively. In fact, e.g. the vectors of the linear hull of $e_2, \ldots, e_n$ are orthogonal to e_1, since

$$[\alpha_2 e_2 + \cdots + \alpha_k e_k + \beta_{k+1} e_{k+1} + \cdots + \beta_n e_n, e_1]^+ = [\alpha_2 e_2 + \cdots + \alpha_k e_k, e_1] + [\alpha_{k+1} e_{k+1} + \cdots + \alpha_n e_n, 0] = 0$$

is valid by the Auerbach property of $e_1, \ldots, e_k$. On the other hand we have the equalities

$$[e_i, e_j]^- = [e_i, e_j] = 0 \quad \text{for } 1 \leq i, j \leq k,$$
$$[e_i, e_j]^- = -[e_i, e_j] = 0 \quad \text{for } k+1 \leq i, j \leq n,$$

and

$$[e_i, e_j]^- = 0 \text{ otherwise}.$$

This proves the last statement of the theorem. □

Corollary 1. *In a generalized Minkowski space, the positive and negative components S and T are Pythagorean orthogonal to each other. In fact, for every pair of vectors $s \in S$ and $t \in T$, by definition we have $[s-t, s-t]^+ = [s, s] + [-t, -t] = [s, s]^+ + [t, t]^+$.*

3. Generalized space–time model and its imaginary unit sphere

In this section we consider a special subset, the imaginary unit sphere of a finite-dimensional, real, generalized Minkowski space. (Some steps of our investigation are also valid in a complex generalized Minkowski space. If we do not use the attribute "real", then we think about a complex Minkowski space.) We give a metric on it, and thus we will get a structure similar to the hyperboloid model of the hyperbolic space embedded in a space–time model. A similar construction of the hyperboloid model of the hyperbolic geometry can be found e.g. in [24].

Definition 13. Let V be a generalized Minkowski space. Then we call a vector *space-like, light-like, or time-like* if its scalar square is positive, zero, or negative, respectively. Let $\mathcal{S}$, $\mathcal{L}$ and $\mathcal{T}$ denote the sets of the space-like, light-like, and time-like vectors, respectively.

In a finite-dimensional, real generalized Minkowski space with dim $T = 1$ we can geometrically characterize these sets of vectors. Such a space is called **generalized space–time model**. In this case $\mathcal{T}$ is a union of its two parts, namely

$$\mathcal{T} = \mathcal{T}^+ \cup \mathcal{T}^-,$$

where

$$\mathcal{T}^+ = \{s + t \in \mathcal{T} \mid \text{where } t = \lambda e_n \text{ for } \lambda \geq 0\} \quad \text{and}$$
$$\mathcal{T}^- = \{s + t \in \mathcal{T} \mid \text{where } t = \lambda e_n \text{ for } \lambda \leq 0\}.$$

Theorem 9. *Let V be a generalized space–time model. Then $\mathcal{T}$ is an open double cone with boundary $\mathcal{L}$, and the positive part $\mathcal{T}^+$ (resp. negative part $\mathcal{T}^-$) of $\mathcal{T}$ is convex.*

Proof. The conic property immediately follows from the equality

$$[\lambda v, \lambda v]^+ = \lambda\bar{\lambda}[v, v]^+ = |\lambda|^2[v, v]^+.$$

Consider now the affine subspace of dimension $n-1$ which is of the form $U = S + t$, where $t \in T$ is arbitrary, but nonzero. Then, for an element of $\mathcal{T} \bigcap U$, we have

$$0 \geq [s + t, s + t]^+ = [s, s] + [t, t],$$

and therefore $[s, s] \leq -[t, t]$. This implies that the above intersection is a convex body in the $(n-1)$-dimensional real vector space S. The s.i.i.p. in S induces a norm whose unit ball is a centrally symmetric convex body. So $\mathcal{T}$ is a double cone and its positive (resp. negative) part is convex, as we stated. For the vectors of its boundary equality holds, and so these are light-like vectors. Since those vectors of the space, for which the inequality does not hold, are space–time vectors, we also get the remaining statement of the theorem. □

3.1. *The imaginary unit sphere H.*

We note that if $\dim T > 1$ or the space is complex, then the set of time-like vectors cannot be divided into two convex components. So we have to consider that our space is a generalized space–time model.

Definition 14. The set

$$H := \{v \in V \mid [v, v]^+ = -1\},$$

is called the *imaginary unit sphere*.

With respect to the embedding real normed linear space $(V, [\cdot, \cdot]^-)$ (see Lemma 2) H is, as we saw, a generalized two sheets hyperboloid corresponding to the two pieces of $\mathcal{T}$, respectively. Usually we deal only with one sheet of the hyperboloid, or identify the two sheets projectively. In this case the space–time component $s \in S$ of v determines uniquely the time-like one, namely $t \in T$. Let $v \in H$ be arbitrary. Let T_v denote the set $v + v^{\perp}$, where $v^{\perp}$ is the orthogonal complement of v with respect to the s.i.i.p., thus a subspace.

Theorem 10. *The set T_v corresponding to the point $v = s + t \in H$ is a positive, $(n-1)$-dimensional affine subspace of the generalized Minkowski space $(V, [\cdot, \cdot]^+)$.*

Proof. By the definition of H the component t of v is nonzero. As we saw in the Remark after Theorem 7, if $[v, v] \neq 0$, then $v^{\perp}$ is an $(n-1)$-dimensional subspace of V. Let now $w \in T_v - v$ be an arbitrary vector. We have to prove that if $[v, v] = -1$ and w is orthogonal to v, then $[w, w] > 0$. Let now $w = s' + t'$ and assume that $[t', t'] = 0$. Then, by the definition of T, $t' = 0$ and thus $[w, w] = [s, s] > 0$ holds. In this case, we may assume that $[t', t'] \neq 0$, and so $t' = \lambda t$. On the other hand, we have

$$0 = [w, v]^+ = [s', s] + [t', t].$$

We can use the Cauchy–Schwarz inequality for the space–time components, and we have

$$[s, s][s', s'] \geq |[s', s]|^2 = |-[t', t]|^2 = |\lambda|^2| - [t, t]|^2 = |\lambda|^2[t, t]^2.$$

Since

$$[s, s][t', t'] = \lambda\bar{\lambda}[s, s][t, t] = |\lambda|^2[s, s][t, t],$$

we get the inequality

$$[s, s][w, w]^+ = [s, s]([s', s'] + [t', t']) \geq |\lambda|^2([t, t]^2 + [s, s][t, t]).$$

By the definition of H we also have

$$-1 = [v, v]^+ = [s, s] + [t, t]$$

and

$$[s, s][w, w]^+ \geq |\lambda|^2([t, t]^2 + (-1 - [t, t])[t, t]) = -|\lambda|^2[t, t] > 0.$$

Consequently, if s is nonzero then $[w, w] > 0$, as we stated.

If now $[s, s] = 0$ then $[t, t] = -1$, and $0 = [s' + t', t] = [s', t] + [t', t] = [t', t]$ implies that $t' = 0$ and $w \in S$. Thus we proved the statement. □

Each of the affine spaces T_v of H can be considered as a semi-metric space, where the semi-metric arises from the Minkowski product restricted to this positive subspace of V. We recall that the Minkowski product does not satisfy the Cauchy–Schwarz inequality. Thus the corresponding distance function does not satisfy the triangle inequality. Such a distance function is called in the literature semi-metric (see [25]). Thus, if the set H is sufficiently smooth, then a metric can be adopted for it, which arises from the restriction of the Minkowski product to the tangent spaces of H. Let us discuss this more precisely.

The directional derivatives of a function $f : S \longmapsto \mathbb{R}$ with respect to a unit vector e of S can be defined in the usual way, by the existence of the limits for real λ:

$$f'_e(s) = \lim_{\lambda \mapsto 0} \frac{f(s + \lambda e) - f(s)}{\lambda}.$$

Let now the generalized Minkowski space be a generalized space–time model, and consider a mapping f on S to $\mathbb{R}$ and the basis $\{e_1, \ldots, e_n\}$ of Theorem 8. The set of points $F := \{(s + f(s)e_n) \in V \text{ for } s \in S\}$ is a so-called **hypersurface** of this space. Tangent vectors of a hypersurface F in a point p are the vectors associated to the directional derivatives of the coordinate functions in the usual way. So u is a **tangent vector** of the hypersurface F in its point $v = (s + f(s)e_n)$, if it is of the form

$$u = \alpha(e + f'_e(s)e_n) \text{ for real } \alpha \text{ and unit vector } e \in S.$$

The linear hull of the tangent vectors translated into the point s is the tangent space of F in s. If the tangent space has dimension $n - 1$, we call it **tangent hyperplane**.

Lemma 3. *Let V be a generalized Minkowski space and assume that the s.i.p. $[\cdot, \cdot]|_S$ is continuous. (So the property s6 holds.) Then the directional derivatives of the real valued function*

$$f : s \longmapsto \sqrt{1 + [s, s]}$$

are

$$f'_e(s) = \frac{\Re[e, s]}{\sqrt{1 + [s, s]}}.$$

Proof. The considered derivative is

$$\frac{f(s + \lambda e) - f(s)}{\lambda} = \frac{\sqrt{1 + [s + \lambda e, s + \lambda e]} - \sqrt{1 + [s, s]}}{\lambda} = \frac{\sqrt{1 + [s + \lambda e, s + \lambda e]}\sqrt{1 + [s, s]} - (1 + [s, s])}{\lambda\sqrt{1 + [s, s]}}.$$

Since $s + \lambda e, s \in S$, and S is a positive subspace, we have

$$0 \leq (\sqrt{[s + \lambda e, s + \lambda e]} - \sqrt{[s, s]})^2 = [s + \lambda e, s + \lambda e] - 2\sqrt{[s + \lambda e, s + \lambda e]}\sqrt{[s, s]} + [s, s],$$

and so

$$[s + \lambda e, s + \lambda e] + [s, s] \geq 2\sqrt{[s + \lambda e, s + \lambda e]}\sqrt{[s, s]} \geq 2|[s + \lambda e, s]|,$$

yielding also

$$[s + \lambda e, s + \lambda e] + [s, s] \geq 2|[s, s + \lambda e]|.$$

Using these inequalities, we get that

$$\frac{f(s + \lambda e) - f(s)}{\lambda} \geq \frac{\sqrt{1 + 2|[s + \lambda e, s]| + |[s + \lambda e, s]|^2} - (1 + [s, s])}{\lambda\sqrt{1 + [s, s]}} = \frac{1 + |[s + \lambda e, s]| - 1 - [s, s]}{\lambda\sqrt{1 + [s, s]}} \geq \frac{\Re\{[s, s] + \lambda[e, s]\} - [s, s]}{\lambda\sqrt{1 + [s, s]}} = \frac{\Re[e, s]}{\sqrt{1 + [s, s]}}.$$

But also

$$\frac{f(s + \lambda e) - f(s)}{\lambda} = \frac{(1 + [s + \lambda e, s + \lambda e]) - \sqrt{1 + [s, s]}\sqrt{(1 + [s + \lambda e, s + \lambda e])}}{\lambda\sqrt{1 + [s + \lambda e, s + \lambda e]}} \leq \frac{(1 + [s + \lambda e, s + \lambda e]) - 1 - |[s, s + \lambda e]|}{\lambda\sqrt{1 + [s + \lambda e, s + \lambda e]}}$$

$$= \frac{\Re\{[s+\lambda e, s+\lambda e]\} - |[s, s+\lambda e]|}{\lambda\sqrt{1+[s+\lambda e, s+\lambda e]}}$$
$$= \frac{\Re\{[s, s+\lambda e] + \lambda[e, s+\lambda e]\} - |[s, s+\lambda e]|}{\lambda\sqrt{1+[s+\lambda e, s+\lambda e]}}$$
$$\leq \frac{|[s, s+\lambda e]| + \Re\{\lambda[e, s+\lambda e]\} - |[s, s+\lambda e]|}{\lambda\sqrt{1+[s+\lambda e, s+\lambda e]}} = \frac{\Re\{[e, s+\lambda e]\}}{\sqrt{1+[s+\lambda e, s+\lambda e]}}.$$

Now the continuity property s6 implies that the examined limit exists, and that the differential is

$$\frac{\Re[e, s]}{\sqrt{1+[s, s]}},$$

as we stated. □

Now we apply our investigation to H. As can be seen easily, the explicit form of this hypersurface arises from the above function

$$f : s \longmapsto \sqrt{1+[s, s]}.$$

Since its directional derivatives can be concretely determined, we can give a connection between the differentiability properties and the orthogonality one.

Lemma 4. *Let H be the imaginary unit sphere of a generalized space–time model. Then the tangent vectors of the hypersurface H in its point*

$$v = s + \sqrt{1+[s, s]}e_n$$

form the orthogonal complement $v^\perp$ of v.

Proof. A tangent vector of this space is of the form

$$u = \alpha(e + f'_e(s)e_n),$$

where by the previous lemma

$$f'_e(s) = \frac{\Re[e, s]}{\sqrt{1+[s, s]}} = \frac{[e, s]}{\sqrt{1+[s, s]}}.$$

Thus we have

$$\left[\alpha\left(e + \frac{[e, s]}{\sqrt{1+[s, s]}}e_n\right), s+t\right]^+ = \alpha[e, s] + \alpha\left[\frac{[e, s]}{\sqrt{1+[s, s]}}e_n, \sqrt{1+[s, s]}e_n\right] = \alpha([e, s] - [e, s]) = 0.$$

So the tangent vectors are orthogonal to the vector v. Conversely, if for a vector $u = s' + t' = s' + \lambda e_n$

$$0 = [u, v] = [s', s] + [t', t]$$

holds, then

$$[s', s] = -[\lambda e_n, t] = \lambda\sqrt{1+[s, s]},$$

since $-[t, t] = 1 + [s, s]$ by the definition of H. Introducing the notion

$$e = \frac{s'}{\sqrt{[s', s']}},$$

we get that

$$[e, s] = \left[\frac{s'}{\sqrt{[s', s']}}, s\right] = \frac{\lambda}{\sqrt{[s', s']}}\sqrt{1+[s, s]},$$

implying that

$$\frac{\lambda}{\sqrt{[s', s']}} = \frac{[e, s]}{\sqrt{1+[s, s]}} = f'_e(s).$$

In this way

$$u = \sqrt{[s', s']}\left(\frac{s'}{\sqrt{[s', s']}} + \frac{\lambda}{\sqrt{[s', s']}}e_n\right) = \alpha(e + f'_e(s)e_n).$$

This last equality shows that a vector of the orthogonal complement is a tangent vector, as we stated. □

We define now the Finsler space type structure for a hypersurface of a generalized space–time model.

Definition 15. Let F be a hypersurface of a generalized space–time model for which the following properties hold:

(i) In every point v of F, there is a (unique) tangent hyperplane T_v for which the restriction of the Minkowski product $[\cdot,\cdot]_v^+$ is positive, and

(ii) the function $ds_v^2 := [\cdot,\cdot]_v^+ : F \times T_v \times T_v \longrightarrow \mathbb{R}^+$

$$ds_v^2 : (v, u_1, u_2) \longmapsto [u_1, u_2]_v^+$$

varies differentiably with the vectors $v \in F$ and $u_1, u_2 \in T_v$.

Then we say that the pair (F, ds^2) *is a Minkowski–Finsler space* with semi-metric ds^2 embedding into the generalized space–time model V.

Naturally "varies differentiably with the vectors v, u_1, u_2" means that for every $v \in T$ and pairs of vectors $u_1, u_2 \in T_v$ the function $[u_1, u_2]_v$ is a differentiable function on F.

Theorem 11. *Let V be a generalized space–time model. Let S be a continuously differentiable s.i.p. space, then (H^+, ds^2) is a Minkowski–Finsler space.*

Proof. If the s.i.p. of S is a continuously differentiable one, then the norm is twice differentiable (see Theorem 2). This also implies the continuity of the s.i.p., and so we know by Lemma 4 that there is a unique tangent hyperplane at each point of H. By Theorem 10 we get that the Minkowski product restricted to a tangent hyperplane is positive. So the first assumption of the definition is valid.

To prove the second condition, consider the product $[u_1, u_2]_v^+$, where v is a point of H and u_1,u_2 are two vectors on its tangent hyperplane. Then, by Lemma 4, we have:

$$u_i = \alpha_i \left(s_i + \frac{[s_i, s_v]}{\sqrt{1 + [s_v, s_v]}} e_n \right) \quad \text{for } i = 1, 2.$$

Here the vectors s_1, s_2, s_v are in S and $v = s_v + \sqrt{1 + [s_v, s_v]} e_n$. Thus the examined product is

$$[u_1, u_2]_v^+ = \alpha_1 \alpha_2 \frac{[s_1, s_2](1 + [s_v, s_v]) - [s_1, s_v][s_2, s_v]}{(1 + [s_v, s_v])}.$$

Since the function

$$[s_v, s_v] = ([v, e_n]^+)^2 - 1$$

is a continuously differentiable function of v, and $[s_1, s_2]$ is (by our assumption) also a continuously differentiable function of its arguments, we only have to prove, that the map sending u_i to s_i also has this property. But this latter fact is a consequence of the observation that the map $u \mapsto s$ is a projection, and so it is linear. □

3.2. The geometry of H^+

Our next goal is to give a characterization of the isometries of the Minkowski-Finsler manifold H^+. For this we need some further definitions. The following concept of linear isometry is usable in any generalized Minkowski space.

Definition 16. A *linear isometry* $f : H^+ \longrightarrow H^+$ of H^+ is the restriction of a linear map $F : V \longrightarrow V$ to H^+ which preserves the Minkowski product and which sends H^+ onto itself.

We note that in this definition a linear mapping F restricted to S gives an isometry between S and its image $F(S)$ implying that this image is a normed space with respect to those s.i.p. which raised from the s.i.p. of S. This isometry is stronger than the usual one, in which we need only the equality of the norm of the corresponding vectors. As we can see in the paper [26] of Koehler, the following theorem holds.

Theorem 12 ([26])**.** *A mapping in a smooth Banach space is an isometry if and only if it preserves the (unique) s.i.p.*

Thus, if the norm is at least smooth, then the two types of linear isometry coincide. Koehler also proved that if the generalized Riesz–Fischer representation theorem is valid in a normed space, then every bounded linear operator A has a generalized adjoint A^T defined by the equality

$$[A(x), y] = [x, A^T(y)] \quad \text{for all } x, y \in V.$$

This mapping is the usual Hilbert space adjoint if the space is an i.p. space. In this more general setting this map is not usually linear but it still has some interesting properties. The assumption for the s.i.p. in Koehler paper [26] is that the space should be a smooth and uniformly convex Banach space. It is well known that uniform convexity implies strict convexity. On the other hand, we now take also into consideration (see [27], p. 111) that every, strictly convex, finite-dimensional normed vector space is uniformly convex. So for the rest of the section we shall assume that the normed space S with respect to its s.i.p. is strictly convex and smooth. It is convenient to characterize strict convexity of the norm in terms of s.i.p. properties. E. Berkson [28] states what can be simply proved, namely

Lemma 5 (*[28]*). *An s.i.p. space is strictly convex if and only if* $[x, y] = \|x\|\|y\|$ *with* $x, y \neq 0$ *implies* $y = \lambda x$ *for some real* $\lambda > 0$.

Now we prove the following theorem:

Theorem 13. *Let V be a generalized space–time model. Assume that the subspace S is a strictly convex, smooth normed space with respect to the norm associated to the s.i.i.p. Then the s.i.p. space* $\{V, [\cdot, \cdot]^-\}$ *is also smooth and strictly convex. Let* F^T *be the generalized adjoint of the linear mapping F with respect to the s.i.p. space* $\{V, [\cdot, \cdot]^-\}$, *and define the involutory linear mapping* $J : V \longrightarrow V$ *by the equalities* $J|S = id|S, J|T = -id|_T$. *The map* $F|_H = f : H \longrightarrow H$ *is a linear isometry of the upper sheet* H^+ *of H if and only if it is invertible, satisfies the equality*

$$F^{-1} = JF^TJ,$$

and, moreover, takes e_n *into a point of* H^+.

Proof. First we prove that the embedding normed space $\{V, [\cdot, \cdot]^-\}$ is also smooth and strictly convex. The equality $1 = [s+t, s+t]^- = [s, s] - [t, t] = [s, s] + \|t\|^2$ shows that the unit balls of the two norms are smooth at the same time. To prove strict convexity, consider

$$[s+t, s'+t']^- = \|s+t\|^-\|s'+t'\|^-.$$

Since $\dim T = 1$, we can assume that $t' = \lambda t$ for some real λ. Thus we get the equality

$$[s, s][s', s'] = [s, s']^2 + [t, t]([s', s'] - 2\lambda[s, s'] + \lambda^2[s, s]).$$

By the Cauchy–Schwarz inequality we have

$$[s', s'] - 2\lambda[s, s'] + \lambda^2[s, s] \geq \left(\sqrt{[\lambda s, \lambda s]} - \sqrt{[s', s']}\right)^2 \geq 0,$$

and so

$$0 \leq [s, s']^2 \leq [s, s][s', s'] = [s, s']^2 + [t, t]([s', s'] - 2\lambda[s, s'] + \lambda^2[s, s]) \leq [s, s']^2,$$

implying that

$$[t, t]([s', s'] - 2\lambda[s, s'] + \lambda^2[s, s]) = 0.$$

If $[t, t] = 0$, then $t = t' = 0$, and from the strict convexity of S we get that there is a real $\mu > 0$ with $s' = \mu s$. For this μ we have also $s' + t' = \mu(s + t)$. So we can assume that $[t, t] \neq 0$, and thus both

$$[s, s][s', s'] = [s, s']^2 \quad \text{and} \quad ([s', s'] - 2\lambda[s, s'] + \lambda^2[s, s]) = 0$$

hold. But S is a strictly convex space. Therefore, again for a nonzero s there is a real $\mu > 0$ with $s' = \mu s$. But this also implies

$$0 = (\mu - \lambda)^2[s, s],$$

showing that $\mu = \lambda$ and $s' + t' = \mu(s + t)$. Using Lemma 5, we get the strict convexity of the embedding normed space.

Let F be a linear isometry of H. It is clear that the linear operator J transforms the Minkowski product into the s.i.p. of the embedding space. Precisely we have

$$[v, w]^+ = [v, Jw]^-.$$

Now using the existence of the adjoint operator, the calculation

$$[v, Jw]^- = [v, w]^+ = [Fv, Fw]^+ = [Fv, JFw]^- = [v, F^TJFw]^-$$

holds for each pair of vectors v and w. But the embedding space is a non-degenerate one; thus we get the equality

$$J = F^TJF$$

or, equivalently,

$$F^{-1} = JF^TJ.$$

By its definition the last condition on F also holds.

Conversely, if F is a linear mapping satisfying the condition of the theorem, then it preserves the Minkowski product. In fact,

$$[Fv, Fw]^+ = [Fv, JFw]^- = [v, F^TJFw]^- = [v, Jw]^- = [v, w]^+.$$

It takes the hyperboloid H homeomorphically onto itself, implying that it takes a sheet onto a sheet. Our last condition guarantees that $F(H^+) = H^+$ and F is a linear isometry of H^+ as we stated. □

As it can be seen from the formula in Theorem 13, the generalized adjoint of a linear isometry is a linear transformation. We also note that Theorem 13 in the i.p. case gives the characterization of the isometries of the hyperbolic space of dimension $n-1$.

It is not clear whether there is a non-pseudo-Euclidean generalized Minkowski space for which the group of linear isometries acts transitively on H^+. But if the answer is yes, the Minkowski–Finsler geometry of H^+ would be linearly homogeneous, and we could compute the Minkowski–Finsler distance. Now we determine the distance function $d : H^+ \times H^+ \longrightarrow \mathbb{R}^+$ of a linearly homogeneous Minkowski–Finsler space H^+.

Before the calculation we recall some known concept on classical Finsler spaces. We assume that the s.i.i.p. restricted into S is continuously differentiable. In a connected Finsler space any point has a distance from any other point of the space (see, e.g., [25]). By our terminology the distance can be computed in the following analogous way.

Definition 17. Denote by p, q a pair of points in H^+ and consider the set $\Gamma_{p,q}$ of equally oriented piecewise differentiable curves $c(t)$, $a \leq t \leq b$, of H^+ emanating from p and terminating at q. Then the *Minkowskian–Finsler distance* of these points is

$$\rho(p,q) = \inf\left\{\int_a^b \sqrt{[\dot{c}(x), \dot{c}(x)]^+_{c(x)}}\mathrm{d}x \text{ for } c \in \Gamma_{p,q}\right\},$$

where $\dot{c}(x)$ means the tangent vector of the curve c at its point $c(x)$.

We would like to examine the influence of a linear isometry to the Minkowski–Finsler distance. It is easy to see that this distance satisfies the triangle inequality; thus it is a metric on H^+ (see [25]).

Definition 18. A *topological isometry* $f : H \longrightarrow H$ of H is a homeomorphism of H which preserves the Minkowski–Finsler distance between each pair of points of H.

First we reformulate the length of a path as follows. The Minkowski–Finsler semi-metric on H^+ is the function ds^2 which assigns at each point $v \in H^+$ the Minkowski product which is the restriction of the Minkowski product to the tangent space T_v. This positive Minkowski product varies differentiably with v. Let $U \leq V$ be a subspace and consider a map $f : U \longrightarrow V$. If it is a totally differentiable map (with respect to the norm of the embedding n-space in the sense of Frèchet) then $f(T_v) = T_{f(v)}$ for the tangent spaces at v and $f(v)$, respectively, and one can define the pullback semi-metric $f^\star(ds^2)$ at the point v by the following formula:

$$f^\star(ds^2)_v(u_1,u_2) = ds^2_{f(v)}(Df(u_1), Df(u_2)) = [Df(u_1), Df(u_2)]^+_{f(v)}.$$

The square root ds of the semi-metric function defined by $\sqrt{ds^2_v(u,u)}$ is the so-called length element and the length of a path is the integral of the pullback length element by the differentiable map $c : \mathbb{R} \longrightarrow V$. This implies that if a linear isometry leaves the Minkowski–Finsler semi-metric invariant by the pullback, then it preserves the integrand, and thus preserves the integral as well. Let now F be a linear isomorphism, and its restriction to H^+ be f. Compute the pullback metric as follows:

$$\begin{aligned} f^\star(ds^2)_v(u_1,u_2) &= ds^2_{f(v)}(Df(u_1), Df(u_2)) = [Df(u_1), Df(u_2)]^+_{f(v)} \\ &= [DF(u_1), DF(u_2)]^+_{F(v)} = [F(u_1), F(u_2)]^+_{F(v)}, \end{aligned}$$

because F is linear. But it preserves the Minkowski product, and therefore we conclude that

$$[F(u_1), F(u_2)]^+_{F(v)} = [u_1,u_2]^+_v = (ds^2)_v(u_1,u_2).$$

This proves the following theorem.

Theorem 14. *A linear isometry of H^+ is also a topological isometry on it.*

In the proof of this theorem we also proved that a linear isometry is a Finsler isometry, in the sense that it is a diffeomorphism of H onto H which preserves the Minkowski–Finsler metric function. In a Riemann space the two types of isometries (the topological and the Riemannian one) are equivalent. This is a result of Myers and Steenrod (see in [29]). The analogous theorem on Finsler spaces was proved by Deng and Hou in [30]. In the latter paper it is also stated that the two concepts of isometry are equivalent for a Finsler space.

In the following theorem we impose the condition of linear homogeneity of H^+.

Theorem 15. *Let V be a generalized space–time model. Assume that the space S is strictly convex and smooth and the group of linear isometries of H^+ acts transitively on H^+. Denote the Minkowski–Finsler distance of H^+ by $d(\cdot,\cdot)$. Then the following statement is true:*

$$[a,b]^+ = -ch(d(a,b)) \quad \textit{for } a,b \in H^+.$$

Proof. In a Finsler space a function preserving the distance transforms geodesics to geodesics (see in [31]). In our case this is also true, since this fact is basically determined by the definition of the distance and the smoothness properties which are the same in both cases. Since our space is homogeneous and linear isometry preserves the distance by Theorem 14, we can assume that $a = e_n$. Let now $b \neq a$ and consider the 2-plane $\langle a, b\rangle$ spanned by the vectors a and b. The restriction of the s.i.i.p. to the plane $\langle a, b\rangle$ is an i.i.p.; thus the restricted Finsler function is a Riemannian one. So the intersection $H \cap \langle a, b\rangle$ is a hyperbola in the embedding Euclidean 2-space. Thus we can parameterize the points of a path from a to b by

$$c(t) = sh(\tau)e + ch(t)e_n \quad \text{for } t \in [0, 1],$$

with $c(0) = a$ and $c(1) = b$. The length of an arc from 0 to x is

$$\int_0^x \sqrt{ch^2(\tau) - sh^2(\tau)}\mathrm{d}\tau = x,$$

showing that the points of this arc satisfy the triangle inequality with equality. Consequently it is a geodesic on H^+, and therefore its arc length is the distance of the points a and $c(x)$. On the other hand, we also have

$$\begin{aligned}[a, b]^+ &= [e_n, sh(1)e + ch(1)e_n]^+ = [e_n, ch(1)e_n] = -ch(1)\\ &= -ch(d(a, c(1))) = -ch(d(a, b)). \quad \square\end{aligned}$$

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