RELATIVITY THEORY IN TIME-SPACE.

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Abstract. We give the basic formulas of special relativity in a time-space defined by an earlier paper of the author. We also give the concept of time-space manifold especially the concept of homogeneous time-space manifold. A homogeneous time-space manifold is a topological manifold allowed its tangent spaces with the same fixed time-space. In a homogeneous time-space manifold we give the concepts of global relativity theory, the concepts of affine connection, parallel transport, curvature tensor and Einstein equation, respectively.

1. Introduction

In [4] the author constructed a model on the basis of two known concepts of Minkowski spaces, the space with indefinite inner product (*Lorentzian-Minkowski space* see e.g. [3], [14]) and the space with a semi-inner product (finite-dimensional separable Banach space see in [2], [11], [12] and [13]), respectively. Among other concepts it was given a special flat-manifold which generalizes the Minkowski-Lorentz manifold (*generalized Minkowski space*). For differential-geometric point of view it was investigated in [5] giving the generalization of the known spaces of constant curvature the hyperbolic (anti-de Sitter), de Sitter, and Euclidean spaces, respectively embedded in this space. In the own right in the generalized Minkowski space there is a theory of special relativity which did not investigate in these theoretical papers. In [7] the concept of generalized Minkowski space extracted with changing space-like sections into a model called by *generalized Minkowski space with changing shape* (briefly *time-space*), and investigated by mathematical point of view. It was given two types of this model, a non-deterministic (random) variation and a deterministic one, respectively. Proved that in a finite range of time the random model can be approximated by a deterministic model, well. Thus, in practical point of view the deterministic model has the more important role. The measure of the examined random model based on the observation that on the space of norms it can be defined a geometric measure which push-forward onto the line of absolute-time has normally distribution (see [6]).

More conveniently, it can be defined a time-space via the concept of a *shape function.* In Section 2 we give the basic formulas of special relativity in a time-space (these are depend on the shape function). In Section 3 we embed some known metrics (holding the Einstein's equation) into a suitable time-space. We see that time-space is a good place to visualize of these ones. Of course, basically time-space has a direct product character thus a lot of metrics there are no natural embedding into it. In the last subsection of Section 3 we define a generalization of the Lorentzian manifold, our tangent spaces are time-spaces with linear shape-function. A *homogeneous time-space manifold* is a time-space manifold allowed the tangent spaces with the same fixed time-space. In a homogeneous time-space

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manifold we give all of the concepts of global relativity theory, the concepts of affine connection, parallel transport, curvature tensor and Einstein equation, respectively.

The first paragraph contains those basic definitions, notations and statements which are necessary to the understanding of the present paper.

1.1. **Deterministic and random time-space models.** We assume that there is an absolute coordinate system of dimension n in which we are modeling the universe by a time-space model. The origin is a generalized space-time model (see in [4]) in which the time axis plays the role of the absolute time. Its points are unattainable and immeasurable for me and the corresponding line is also in the exterior of the modeled universe. We note that in the Minkowskian space-time it was assumed only on the axes determining the space-coordinates. This means that in our model, even though the axis of time belongs to the double cone of time-like points, its points do not belong to the modeled universe. In a fixed moment (with respect to this absolute time) the collection of the points of the space can be regarded as an open ball of the embedding normed space centered at the origin that does not contain the origin. The omitted point is the origin of a coordinate system giving the space-like coordinates of the world-points with respect to our time-space system. Since the points of the axis of the absolute-time are not in our universe there is no reference system in our modeled world which determines the absolute time.

In our probabilistic model (based on a generalized space-time model) the absolute coordinates of points are calculated by a fixed basis of the embedding vector space. The vector $s(\tau)$ means the collection of the space-components with respect to the absolute time τ, the quantity τ has to be measured on a line T which orthogonal to the linear subspace S of the vectors $s(\tau)$. (The orthogonality was considered as the Pythagorean orthogonality of the embedding normed space.) Consider a fixed Euclidean vector space with unit ball B_E on S and use its usual functions e.g. volume, diameter, width, thinness and Hausdorff distance. With respect to the moment τ of the absolute time we have a unit ball $K(\tau)$ in the corresponding normed space $\{S, \|\cdot\|^\tau\}$. The modeled universe at τ is the ball $\tau K(\tau) \subset \{S, \|\cdot\|^\tau\}$. The shape of the model at the moment τ depends on the shape of the centrally symmetric convex body $K(\tau)$. The center of the model is on the axis of the absolute time, it cannot be determined. For calculations on time-space we need further smoothness properties on $K(\tau)$. These are

- $K(\tau)$ is a centrally symmetric, convex, compact, C^2 body of volume $\mathrm{vol}(B_E)$.
- For each pairs of points s', s'' the function
$$K : \mathbb{R}^+ \cup \{0\} \to \mathcal{K}_0 \ , \ \tau \mapsto K(\tau)$$
holds the property that $[s', s'']^\tau : \tau \mapsto [s', s'']^\tau$ is a C^1-function.

Definition 1. *We say that a generalized space-time model endowed with a function $K(\tau)$ holding the above properties is a* deterministic time-space model.

The main subset of a deterministic time-space model contains the points of negative norm-square. This is the set of time-like points and the upper connected sheet of the time-like points is the modeled universe. The points of the universe have positive time-components. We denote this model by $(M, K(\tau))$.

Of course, we should choose the function $K(\tau)$ “randomly”. To this purpose we use Kolmogorov’s extension theorem (or theorem on consistency, see in [10]). This says that a suitably ”consistent” collection of finite-dimensional distributions will define a probability measure on the product space. The sample space here is $\mathcal{K}_0$ with the Hausdorff distance. It is a locally compact, separable (second-countable) metric space. By Blaschke’s selection theorem $\mathcal{K}$ is a boundedly compact space so it is also complete. It is easy to check that $\mathcal{K}_0$ is also a complete metric space if we assume that the non-proper bodies (centrally symmetric convex compact sets with empty interior) also belong to it. In the remaining part we regard such a body as the unit ball of a normed space of smaller dimension. Finally, let P be a probability measure. In every moment we consider the same probability space $(\mathcal{K}_0, P)$ and also consider in each of the finite collections of moments the corresponding product spaces $((\mathcal{K}_0)^r, P^r)$. The consistency assumption of Kolmogorov's theorem now automatically holds. By the extension theorem we have a probability measure $\hat{P}$ on the measure space of the functions on T to $\mathcal{K}_0$ with the σ-algebra generated by the cylinder sets of the space. The distribution of the projection of $\hat{P}$ to the probability space of a fix moment is the distribution of P.

Definition 2. *Let $(K_\tau\ ,\ \tau \geq 0)$ be a random function defined as an element of the Kolmogorov's extension $\left(\Pi\mathcal{K}_0, \hat{P}\right)$ of the probability space $(\mathcal{K}_0, P)$. We say that the generalized space-time model with the random function*

$$\hat{K}_\tau := \sqrt[n]{\frac{\operatorname{vol}(B_E)}{\operatorname{vol}(K_\tau)}} K_\tau$$

is a random time-space model. *Here $\alpha_0(K_\tau)$ is a random variable with truncated normal distribution and thus $(\alpha_0(K_\tau)\ ,\ \tau \geq 0)$ is a stationary Gaussian process. We call it the* shape process *of the random time-space model.*

It is clear that a deterministic time-space model is a special trajectory of the random time-space model. The following theorem is essential.

Theorem 1 ([7]). *For a trajectory $L(\tau)$ of the random time-space model, for a finite set $0 \leq \tau_1 \leq \cdots \leq \tau_s$ of moments and for a $\varepsilon > 0$ there is a deterministic time-space model defined by the function $K(\tau)$ for which*

$$\sup_i\{\rho_H\left(L(\tau_i), K(\tau_i)\right)\} \leq \varepsilon.$$

An important consequence of Theorem 1 is then that without loss of generality we can assume, that the time-space model is deterministic.

Definition 3. *For two vectors $s_1 + \tau_1$ and $s_2 + \tau_2$ of the deterministic time-space model define their product with the equality*

$$[s_1 + \tau_1, s_2 + \tau_2]^{+,T} := [s_1, s_2]^{\tau_2} + [\tau_1, \tau_2] =$$
$$= [s_1, s_2]^{\tau_2} - \tau_1\tau_2.$$

Here $[s_1, s_2]^{\tau_2}$ means the s.i.p defined by the norm $\|\cdot\|^{\tau_2}$. This product is not a Minkowski product, as there is no homogeneity property in the second variable. On the other hand the additivity and homogeneity properties of the first variable, the properties on non-degeneracy of the product are again hold, and the continuity and differentiability properties of this product also remain the same as of a Minkowski product. The calculations in a generalized space-time model basically depend on a rule on the differentiability of the second variable of the Minkowski product. As a basic tools of investigations we proved in [7] that

Theorem 2 ([7]). *If $f_1, f_2 : S \longrightarrow V = S + T$ are two C^2 maps and $c : \mathbb{R} \longrightarrow S$ is an arbitrary C^2 curve then*

$$([(f_1 \circ c)(t)), (f_2 \circ c)(t))]^{+,T})' =$$

$$= [D(f_1 \circ c)(t), f_2(c(t))]^{+,T} + \left([f_1(c(t)), \cdot]^{+,T}\right)'_{D(f_2 \circ c)(t)} (f_2(c(t)))+$$
$$+\frac{\partial[(f_1)_S(c(t)), (f_2)_S(c(t))]^{\tau}}{\partial \tau}((f_2)_T(c(t))) \cdot ((f_2)_T \circ c)'(t)$$

The theory of generalized space-time model can be used in a generalization of special relativity theory, if we change some previous formulas using also the constant c. (It is practically can be considered as the speed of the light in vacuum.) The formula of the product in such a deterministic (random) time-space was

$$[x', x'']^{+,T} := [s', s'']^{\tau''} + c^2 [\tau', \tau''] .$$

Parallel we used the assumption that the dimension n is equal to 4. A particle is a random function $x : I_x \to S$ holding two conditions:

- the set $I_x \subset T^+$ is an interval
- $[x(\tau), x(\tau)]^{\tau} < 0$ if $\tau \in I_x$.

The particle lives on the interval I_x, is born at the moment $\inf I_x$ and dies at the moment $\sup I_x$. Since all time-sections of a time-space model is a normed space of dimension n the Borel sets of the time-sections are independent from the time. This means that we could consider the physical specifies of a particle as a trajectory of a stochastic process. A particle “realistic” if it holds the “known laws of physic” and “idealistic” otherwise. This is only a terminology for own use, the mathematical contain of the expression “known laws of physics” is indeterminable. First we introduced an inner metric $\delta_{K(\tau)}$ on the space at the moment τ.

Definition 4. *Let $X(\tau) : T \to \tau K(\tau)$ be a continuously differentiable (by the time) trajectory of the random function $(x(\tau)\ ,\ \tau \in I_x)$. We say that the particle $x(\tau)$ is* realistic in its position *if for every $\tau \in I_x$ the random variable $\delta_{K(\tau)}(X(\tau), x(\tau))$ has normal distribution on $\tau K(\tau)$. In other words the stochastic process $\left(\delta_{K(\tau)}(X(\tau), x(\tau))\ ,\ \tau \in I_x\right)$ has stationary Gaussian process with respect to a given continuously differentiable function $X(\tau)$. We call the function $X(\tau)$ the* world-line *of the particle $x(\tau)$.*

We note that the concept of ”realistic in its position” is independent from the choice of $\delta_{K(\tau)}$. As a refinement of this concept we defined another one, which can be considered as a generalization of the principle on the maximality of the speed of the light.

Definition 5. *We say that a particle* realistic in its speed *if it is realistic in its position and the derivatives of its world-line $X(\tau)$ are time-like vectors.*

For such two particles x', x'' which are realistic in their position we can define a momentary distance by the equality:

$$\delta(x'(\tau), x''(\tau)) = \|X'(\tau) - X''(\tau)\|^{\tau} = \sqrt{[X'(\tau) - X''(\tau), X'(\tau) - X''(\tau)]^{+,T}}.$$

We could say that two particles x' and x'' are agree if the expected value of their distances is equal to zero. Let $I = I_{x'} \cap I_{x''}$ be the common part of their domains. The required equality is:

$$E(\delta_{K(\tau)}(x'(\tau), x''(\tau))) = \int\limits_I \delta_{K(\tau)}(x'(\tau), x''(\tau))\mathrm{d}\tau =$$
$$= \int\limits_I \|X'(\tau) - X''(\tau)\|^{\tau}\mathrm{d}\tau = 0.$$

In a deterministic time-space we have a function $K(\tau)$, and we have more possibilities to define orthogonality in a concrete moment τ. We shall fixe a concept of orthogonality and we will consider it in every normed space. In the case when the norm induced by the Euclidean inner product this method should give the same result as the usual concept of orthogonality. The most natural choice is the concept of Birkhoff orthogonality (see in [4]). Using it, in every normed space we could consider an Auerbach basis (see in [4]) which can play the role of a basic coordinate frame. We could determine the coordinates of the point with respect to this basis. We said that a frame is *at rest with respect to the absolute time* if its origin (as a particle) is at rest with respect to the absolute time τ and the unit vectors of its axes are at rest with respect to a fixed Euclidean orthogonal basis of S. In S we fix an Euclidean orthonormal basis and give the coordinates of a point (vector) of S with respect to this basis. We get curves in S parameterized by the time τ. We defined the concept of a frame as follows.

Definition 6. *The system* $\{f_1(\tau), f_2(\tau), f_3(\tau), o(\tau)\} \in (S, \|\cdot\|^{+\tau}) \times \tau K(\tau)$ *is a* frame, *if*

- $o(\tau)$ *is a particle realistic in its speed, with such a world-line*
$$O(\tau) : T \to \tau K(\tau)$$
which does not intersect the absolute time axis T,
- *the functions*
$$f_i(\tau) : T \to \cup \{(S, \|\cdot\|^{\tau}) \ , \ \tau \in T\}$$
are continuously differentiable, for all fixed τ,
- *the system* $\{f_1(\tau), f_2(\tau), f_3(\tau)\}$ *is an Auerbach basis with origin* $O(\tau)$ *in the space* $(S, \|\cdot\|^{\tau})$.

Note, that for a good model we have to guarantee that Einstein's convention on the equivalence of the inertial frames can be remained for us. However at this time we have no possibility to give the concepts of "frame at rest" and the concept of "frame which moves constant velocity with respect to another one". The reason is that when we changed the norm of the space by the function $K(\tau)$ we concentrated only the change of the shape of the unit ball and did not use any correspondence between the points of the two unit balls. Obviously, in a concrete computation we should proceed vice versa, first we should give a correspondence between the points of the old unit ball and the new one and this implies the change of the norm. To this purpose we may define a homotopic mapping $\mathbf{K}$ which describes the deformation of the norm.

Definition 7. *Consider a homotopic mapping* $\mathbf{K}(x, \tau) : (S, \|\cdot\|_E) \times T \to (S, \|\cdot\|_E)$ *holding the assumptions:*

- $\mathbf{K}(x, \tau)$ *is homogeneous in its first variable and continuously differentiable in its second one,*
- $\mathbf{K}(\{e_1, e_2, e_3\}, \tau)$ *is an Auerbach basis of* $(S, \|\cdot\|^{\tau})$ *for every* τ,
- $\mathbf{K}(B_E, \tau) = K(\tau)$.

Then we say that the function $\mathbf{K}(x, \tau)$ *is the* shape-function *of the time-space.*

The mapping $\mathbf{K}(x, \tau)$ determines the changes at all levels. For example we can consider a frame is "at rest" if its change arises only from this globally determined change, and "moves with constant velocity" if its origin has this property and the directions of its axes are "at rest". Precisely, we said, that

Definition 8. *The frame* $\{f_1(\tau), f_2(\tau), f_3(\tau), o(\tau)\}$ moves with constant velocity with respect to the time-space *if for every pairs* τ, τ' *in* T^+ *we have*
$$f_i(\tau) = \mathbf{K}(f_i(\tau'), \tau) \text{ for all } i \text{ with } 1 \leq i \leq 3$$

and there are two vectors $O = o_1e_1 + o_2e_2 + o_3e_3 \in S$ *and* $v = v_1e_1 + v_2e_2 + v_3e_3 \in S$ *that for all values of* τ *we have*

$$O(\tau) = \mathbf{K}(O, \tau) + \tau\mathbf{K}(v, \tau).$$

A frame is at rest with respect to the time-space *if the vector* v *is the zero vector of* S.

Consider the derivative of the above equality by τ. We get that

$$\dot{O}(\tau) = \frac{\partial \mathbf{K}(O, \tau)}{\partial \tau} + \mathbf{K}(v, \tau) + \tau\frac{\partial \mathbf{K}(v, \tau)}{\partial \tau},$$

showing that for such a homotopic mapping, which is constant in the time $O(\tau)$, is a line with direction vector v through the origin of the time space. Similarly in the case when v is the zero vector it is a vertical (parallel to T) line-segment through O.

We can re-define the concept of time-axes,too.

Definition 9. *The* time-axis *of the time-space model is the world-line* $O(\tau)$ *of such a particle which moves with constant velocity with respect to the time-space and starts from the origin. More precisely, for the world-line* $(O(\tau), \tau)$ *we have* $\mathbf{K}(O, \tau) = 0$ *and hence with a given vector* $v \in S$*,*

$$O(\tau) = \tau\mathbf{K}(v, \tau).$$

Remark. Note that the shape-function is linear in its first variable then all sections define by $\tau =$ const. are Euclidean spaces. This is the case when the shape-function is of the form:

$$\mathbf{K}(v, \tau) = f(\tau)A(s),$$

where f is a continuously differentiable function and $A : S \longrightarrow S$ is a linear mapping.

2. On the formulas of special relativity theory

In this section we assume that the shape-function is a two-times continuously differentiable function, so it is a C^2 function. We need two further axioms to interpret in time-space of the usual axioms of special relativity theory. First we assume that:

Axiom 1. *The laws of physics are invariant under transformations between frames. The laws of physics will be the same whether you are testing them in frame "at rest", or a frame moving with a constant velocity relative to the "rest" frame.*

Axiom 2. *The speed of light in a vacuum is measured to be the same by all observers in frames.*

These two axioms can be transformed into the language of the time-space by the method of Minkowski [14]. To this we use the imaginary sphere H_c of parameter c introduced in the previous subsection and the group G_c as the set of those isometries of the space which leave invariant this sphere of parameter c. Such an isometry can be interpreted as a coordinate transformation of the time-space which sends the axis of the absolute time into another time-axis t', and also maps the intersection point of the absolute time-axis with the imaginary sphere H_c into the intersection point of the new time-axis and H_c. An isometry of the time-space is also a homeomorphism thus it maps the subspace S into a topological hyperplane S' of the embedding normed space. S' is orthogonal to the new time-axis in the sense that its tangent hyperplane at the origin is orthogonal to t' with respect to the product of the space. Of course the new space-axes are continuously differentiable curves in S' which tangents at the origin are orthogonal to each other. Since the absolute time-axis is orthogonal to the imaginary sphere H_c the new time-axis t' must holds this property, too. Thus the investigations in the previous section are essential from this point of view. Assuming that the definition of the time-space implies this property we can get some formulas similar to of special relativity. We note that the function $\mathbf{K}(v,\tau)$ holds the orthogonality property of vectors of S and by the equality

$$[\mathbf{K}(v,\tau),\mathbf{K}(v,\tau)]^{\tau}=\|v\|_E^2$$

we can see also that the formulas on time-dilatation and length-contraction are valid, too. This implies that using the well-known notations

$$\beta=\frac{\|v\|_E}{c}$$

$$\gamma=\frac{1}{\sqrt{1-\beta^2}}$$

we get that the connection between the time τ_0 and τ of an event measuring by two observers one of at rest and the other moves with an constant velocity $\|v\|_E$ with respect to the time-space is

$$\tau=\gamma\tau_0.$$

Similarly if we consider a moving rod which points move constant velocity with respect to the time space such that it is always parallel to the velocity vector $\mathbf{K}(v,\tau)$. Then we have

$$\|v\|_E=\frac{L_0}{T}$$

where T is the time calculated from the length L_0 and the velocity vector v by such an observer which moves with the rod. Another observer can calculate the length L from the measured time T_0 and the velocity v by the formula

$$\|v\|_E=\frac{L}{T_0}.$$

Using the above formula of dilatation we get the known Fitzgerald contraction of the rod:

$$L=L_0\sqrt{1-\beta^2}=\frac{L_0}{\gamma}.$$

2.1. **Lorentz transformation.**

Lorentz transformation in time space also based on the usual experiment in which we send a ray of light to a mirror in direction of the unit vector e with distance d from me.

2.1.1. *Deduction of Lorentz transformation in time-space.*

If we at rest we can determine in time space the points A, C and B of departure, turn and arrival of the ray of light, respectively. A and B are on the absolute time-axis at heights τ_A, and τ_B, respectively. The position of C is

$$(\tau_C-\tau_A)\mathbf{K}(ce,\tau_C-\tau_A)+\tau_C e_4=\frac{\tau_B-\tau_A}{2}\mathbf{K}\left(ce,\frac{\tau_B-\tau_A}{2}\right)+\frac{\tau_B+\tau_A}{2}e_4,$$

since we know that the light take the road back and forth over the same time. We observe that the norm of the space-like component s_C is

$$\|s_C\|^{\tau_C}=c\frac{\tau_B-\tau_A}{2}$$

as in the usual case of space-time.

The moving observer synchronized its clock with the observer at rest in the origin, and moves in the direction v with velocity $\|v\|_E$. We assume that the moving observer also sees the experiment thus its time-axis corresponding to the vector v meats the world-line of the light in two points A' and B' positioning on the respective curves AC and CB. This implies that the respective space-like components of the world-line of the light and the world-line of the axis are parallels to each other in every minutes. By formula we have:

$$\|v\|_E\mathbf{K}(e,\tau)=\mathbf{K}(v,\tau).$$

From this we get the equality

$$\tau_{A'}\mathbf{K}(v,\tau_{A'})+\tau_{A'}e_4=(\tau_{A'}-\tau_A)\mathbf{K}(ce,\tau_{A'}-\tau_A)+\tau_{A'}e_4.$$

This implies that

$${\tau_{A'}}^2{\|v\|_E}^2-c^2{\tau_{A'}}^2=(\tau_{A'}-\tau_A)^2c^2-c^2{\tau_{A'}}^2$$

and thus

$$\tau_{A'}=\frac{c}{c-\|v\|_E}\tau_A.$$

The proper time $(\tau_{A'})_0$ is

$$(\tau_{A'})_0=\sqrt{1-\beta^2}\frac{c}{c-\|v\|_E}\tau_A=\tau_A\sqrt{\frac{1+\beta}{1-\beta}}.$$

Similarly we also get that

$$(\tau_{B'})_0=\tau_B\sqrt{\frac{1-\beta}{1+\beta}},$$

and we determine the new time coordinate of the point C with respect to the new coordinate system:

$$(\tau_C)_0=\frac{(\tau_{A'})_0+(\tau_{B'})_0}{2}=\frac{1}{2}\left(\tau_A\sqrt{\frac{1+\beta}{1-\beta}}+\tau_B\sqrt{\frac{1-\beta}{1+\beta}}\right).$$

Since we have that the norm of the space-like component is

$$\|s_C\|_E=c\frac{\tau_B-\tau_A}{2},$$

we get that

$$\tau_A=\tau_C-\frac{\|s_C\|_E}{c}\text{ and }\tau_B=\tau_C+\frac{\|s_C\|_E}{c}$$

and thus

$$(\tau_C)_0=\frac{1}{2}\left(\left(\tau_C-\frac{\|s_C\|_E}{c}\right)\sqrt{\frac{1+\beta}{1-\beta}}+\left(\tau_C+\frac{\|s_C\|_E}{c}\right)\sqrt{\frac{1-\beta}{1+\beta}}\right)=$$

$$=\frac{\tau_C-\frac{\beta\|s_C\|_E}{c}}{\sqrt{1-\beta^2}}=\frac{\tau_C-\frac{\|v\|_E\|s_C\|_E}{c^2}}{\sqrt{1-\frac{\|v\|_E^2}{c^2}}}=\frac{\tau_C-\frac{[\mathbf{K}(s_C,\tau_C),\mathbf{K}(v,\tau_C)]^{\tau_C}}{c^2}}{\sqrt{1-\frac{\|v\|_E^2}{c^2}}}.$$

On the other hand we also have that the space-like component $((s_C)_0)_S$ of the transformed space-like vector $(s_C)_0$ arise also from a vector parallel to e thus it is of the form

$$\mathbf{K}(((s_C)_0)_S,\tau)=\|((s_C)_0)_S\|_E\mathbf{K}(e,\tau).$$

For the norm of $(s_C)_0$ we know that

$$\|(s_C)_0\|^{+,T}=c\frac{(\tau_{B'})_0-(\tau_{A'})_0}{2},$$

hence

$$\|(s_C)_0\|^{+,T}=\frac{\|s_C\|_E-\|v\|_E\tau_C}{\sqrt{1-\frac{\|v\|_E^2}{c^2}}}.$$

If we consider the vector

$$\widehat{(s_C)_0}=\gamma\left(\mathbf{K}(s_C,\tau_C)-\mathbf{K}(v,\tau_C)\tau_C\right)\in S,$$

we get a norm-preserving, bijective mapping $\widehat{L}$ from the world-line of the light into S with the definition

$$\widehat{L}:\mathbf{K}((s_C)_0,(\tau_C)_0)\mapsto\gamma\left(\mathbf{K}(s_C,\tau_C)-\mathbf{K}(v,\tau_C)\tau_C\right).$$

The connection between the space-like coordinates of the point with respect to the two frames now has a more familiar form. Henceforth the Lorentz transformation means for us the correspondence:

$$\begin{array}{rcl} s & \mapsto & \widehat{\mathbf{K}(s',\tau')}=\gamma\left(\mathbf{K}(s,\tau)-\mathbf{K}(v,\tau)\tau\right)\\ \tau & \mapsto & \tau'=\gamma\left(\tau-\dfrac{[\mathbf{K}(s,\tau),\mathbf{K}(v,\tau)]^{\tau}}{c^2}\right), \end{array}$$

and the inverse Lorentz transformation the another one

$$\begin{array}{rcl} \widehat{\mathbf{K}(s',\tau')} & \mapsto & \mathbf{K}(s,\tau)=\gamma\left(\mathbf{K}(s',\tau')+\mathbf{K}(v,\tau')\tau'\right)\\ \tau' & \mapsto & \tau=\gamma\left(\tau'+\dfrac{[\mathbf{K}(s',\tau'),\mathbf{K}(v,\tau')]^{\tau'}}{c^2}\right). \end{array}$$

2.1.2. *Consequences of Lorentz transformation.* First note that we can determine the components of $(s_C)_0$ with respect to the absolute coordinate system, too. Since $(s_C)_0$ and $\tau\mathbf{K}(v,\tau)+\tau e_4$ are orthogonal to each other we get that

$$[\mathbf{K}(((s_C)_0)_S,\tau_C),\mathbf{K}(v,\tau_C)]^{\tau_C}=c^2((s_C)_0)_T,$$

implying that

$$((s_C)_0)_T=\frac{\|((s_C)_0)_S\|_E\|v\|_E}{c^2}.$$

Thus we get the equality

$$\|((s_C)_0)_S\|_E^2\left(1-c^2\left(\frac{\|v\|_E}{c^2}\right)^2\right)=\left(\frac{\|s_C\|_E-\|v\|_E\tau_C}{\sqrt{1-\frac{\|v\|_E^2}{c^2}}}\right)^2,$$

implying that

$$\|((s_C)_0)_S\|_E=\frac{\|s_C\|_E-\|v\|_E\tau_C}{\left(1-\frac{\|v\|_E^2}{c^2}\right)}=\gamma^2\left(\|s_C\|_E-\|v\|_E\tau_C\right)$$

and

$$((s_C)_0)_T=\frac{\|((s_C)_0)_S\|_E\|v\|_E}{c^2}=\frac{\|v\|_E\|s_C\|_E-\|v\|_E^2\tau_C}{c^2-\|v\|_E^2}.$$

We get that

$$(s_C)_0=\gamma^2\left(\|s_C\|_E-\|v\|_E\tau_C\right)\left(\mathbf{K}(e,\tau_C)+\frac{\|v\|_E}{c^2}e_4\right)=$$

$$=\gamma^2\left(\mathbf{K}(s_C,\tau_C)-\mathbf{K}(v,\tau_C)\tau_C\right)+\left(\frac{\gamma}{1-\gamma}\right)^2\left(\|s_C\|_E-\|v\|_E\tau_C\right)e_4.$$

We can determine also the length of this vector in the new coordinate system, too. Since

$$[(s_C)_0,(s_C)_0]^{+,T} = \left(\|(s_C)_0\|^{+,T}\right)^2 = \frac{(\|s_C\|^{\tau_C} - \|v\|_E\tau_C)^2}{1-\frac{\|v\|_E^2}{c^2}} =$$

$$= \frac{[s_C,s_C]^{\tau_C} - 2\|s_C\|^{\tau_C}\|v\|_E\tau_C + (\|v\|_E\tau_C)^2}{1-\frac{\|v\|_E^2}{c^2}}$$

and

$$\left((\tau_C)_0\right)^2 = \frac{(\tau_C)^2 - 2\tau_C\frac{\|v\|_E\|s_C\|^{\tau_C}}{c^2} + \frac{(\|v\|_E\|s_C\|^{\tau_C})^2}{c^4}}{1-\frac{\|v\|_E^2}{c^2}},$$

hence the equality

$$[(s_C)_0,(s_C)_0]^{+,T} - c^2\left((\tau_C)_0\right)^2 = [s_C,s_C]^{\tau_C} - c^2\left(\tau_C\right)^2$$

shows that under the action of the Lorentz transformation the "norm-squares" of the vectors of the time-space are invariant as in the case of the usual space-time.

Finally we determine those points of the space which new time-coordinates are zero and thus we get a mapping from the subspace S into the time-space. Let $s \in S$ arbitrary and consider the corresponding point $\mathbf{K}(s,\tau) + \tau e_4$ and assume that

$$0 = \tau_0 = \gamma\tau - \gamma\frac{\|v\|_E}{c^2}\|\mathbf{K}(s,\tau)\|^{\tau},$$

hence

$$\tau = \frac{\|v\|_E\|s\|_E}{c^2}.$$

Then we get the mapping of the coordinate subspace S under the action of the isometry corresponding to that Lorentz transformation which sends the absolute time-axis into the time-axis $\tau\mathbf{K}(v,\tau) + \tau e_4$ in question. This is the set

$$S_0 = \left\{\mathbf{K}\left(s, \frac{\|v\|_E\|s\|_E}{c^2}\right) + \frac{\|v\|_E\|s\|_E}{c^2}e_4 \quad | \quad s \in S\right\}.$$

For a boost in an arbitrary direction with velocity v, it is convenient to decompose the spatial vector s into components perpendicular and parallel to v:

$$s = s_1 + s_2$$

so that

$$[\mathbf{K}(s,\tau),\mathbf{K}(v,\tau)]^{\tau} = [\mathbf{K}(s_1,\tau),\mathbf{K}(v,\tau)]^{\tau} + [\mathbf{K}(s_2,\tau),\mathbf{K}(v,\tau)]^{\tau} = [\mathbf{K}(s_2,\tau),\mathbf{K}(v,\tau)]^{\tau}.$$

Then, only time and the component $\mathbf{K}(s_2,\tau)$ in the direction of $\mathbf{K}(v,\tau)$;

$$\begin{aligned} \tau' &= \gamma\left(\tau - \frac{[\mathbf{K}(s,\tau),\mathbf{K}(v,\tau)]^{\tau}}{c^2}\right) \\ \widehat{\mathbf{K}(s',\tau')} &= \mathbf{K}(s_1,\tau) + \gamma(\mathbf{K}(s_2,\tau) - \mathbf{K}(v,\tau)\tau) \end{aligned}$$

are "distorted" by the Lorentz factor γ. The second equality can be written also in the form:

$$\widehat{s'} = \mathbf{K}(s,\tau) + \left(\frac{\gamma-1}{\|v\|_E^2}[\mathbf{K}(s,\tau),\mathbf{K}(v,\tau)]^{\tau} - \gamma\tau\right)\mathbf{K}(v,\tau).$$

Remark. If we have two time-axes $\tau\mathbf{K}(v',\tau) + \tau e_4$ and $\tau\mathbf{K}(v'',\tau) + \tau e_4$ then there are two subgroups of the corresponding Lorentz transformations mapping the absolute time-axis onto another time-axes, respectively. These two subgroups are also subgroups of G_c. Their elements can be paired on the base of their action on S. The pairs of these isometries define a new isometry of the space (and its inverse) on a natural way, with the composition one of them and the inverse of the other. Omitting the absolute time-axis from the space (as we suggest earlier) the invariance of the product on the remaining space and also the physical axioms of special relativity can remain in effect.

2.1.3. *Addition of velocities.* If $\mathbf{K}(u,\tau)$ and $\mathbf{K}(v,\tau')$ are two velocity vectors then using the formula for inverse Lorentz transformation of the corresponding differentials we get that

$$\mathrm{d}\tau=\gamma\left(\mathrm{d}\tau'+\frac{[\mathbf{K}(\mathrm{d}\widehat{s'},\mathrm{d}\tau'),\mathbf{K}(v,\tau')]^{\tau'}}{c^2}\right)$$

and

$$\mathbf{K}(\mathrm{d}s,\mathrm{d}\tau)=\mathbf{K}(\mathrm{d}\widehat{s'},\mathrm{d}\tau')+\left(\frac{1-\gamma}{\|v\|_E^2}[\mathbf{K}(\mathrm{d}\widehat{s'},\mathrm{d}\tau'),\mathbf{K}(v,\tau')]^{\tau'}+\gamma\mathrm{d}\tau'\right)\mathbf{K}(v,\tau').$$

Thus

$$\mathbf{K}(u,\tau)=\frac{\mathbf{K}(\mathrm{d}s,\mathrm{d}\tau)}{\mathrm{d}\tau}=\frac{\mathbf{K}(\mathrm{d}\widehat{s'},\mathrm{d}\tau')+\left(\frac{1-\gamma}{\|v\|_E^2}[\mathbf{K}(\mathrm{d}\widehat{s'},\mathrm{d}\tau'),\mathbf{K}(v,\tau')]^{\tau'}+\gamma\mathrm{d}\tau'\right)\mathbf{K}(v,\tau')}{\gamma\left(\mathrm{d}\tau'+\frac{[\mathbf{K}(\mathrm{d}\widehat{s'},\mathrm{d}\tau'),\mathbf{K}(v,\tau')]^{\tau'}}{c^2}\right)}=$$

$$=\frac{\left(\mathbf{K}(v,\tau')+\frac{1}{\gamma}\frac{\mathbf{K}(\mathrm{d}\widehat{s'},\mathrm{d}\tau')}{\mathrm{d}\tau'}+\frac{1+\gamma}{\gamma c^2}\left[\frac{\mathbf{K}(\mathrm{d}\widehat{s'},\mathrm{d}\tau')}{\mathrm{d}\tau'},\mathbf{K}(v,\tau')\right]^{\tau'}\mathbf{K}(v,\tau')\right)}{1+\frac{\left[\frac{\mathbf{K}(\mathrm{d}\widehat{s'},\mathrm{d}\tau')}{\mathrm{d}\tau'},\mathbf{K}(v,\tau')\right]^{\tau'}}{c^2}}$$

$$=\frac{\left(\mathbf{K}(v,\tau')+\frac{1}{\gamma}\mathbf{K}(u',\mathrm{d}\tau')+\frac{1+\gamma}{\gamma c^2}[\mathbf{K}(u',\mathrm{d}\tau'),\mathbf{K}(v,\tau')]^{\tau'}\mathbf{K}(v,\tau')\right)}{1+\frac{[\mathbf{K}(u',\mathrm{d}\tau'),\mathbf{K}(v,\tau')]^{\tau'}}{c^2}}.$$

2.2. **Acceleration, momentum and energy.** Our starting point is *the velocity vector (or four-velocity).* The absolute time coordinate is τ, this defines a world line of form $S(\tau)=\mathbf{K}(s(\tau),\tau)+\tau e_4$. Its proper time is $\tau_0=\frac{\tau}{\gamma}=\tau\sqrt{1-\frac{\|v\|_E^2}{c^2}}$, where v is the velocity vector of the moving frame. By definition

$$V(\tau):=\frac{\mathrm{d}S(\tau)}{\mathrm{d}\tau_0}=\gamma\left(\frac{\mathrm{d}(\mathbf{K}(s(\tau),\tau))}{\mathrm{d}\tau}+e_4\right).$$

If the shape-function is a linear mapping then $\frac{\mathrm{d}(\mathbf{K}(s(\tau),\tau))}{\mathrm{d}\tau}=\mathbf{K}(\dot{s}(\tau),1):=\mathbf{K}(v(\tau),1)$ and we also have

$$[V(\tau),V(\tau)]^{+,T}=\gamma^2\left([\mathbf{K}(v(\tau),1),\mathbf{K}(v(\tau),1)]^1-c^2\right)=-c^2.$$

The *acceleration* is defined as the change in four-velocity over the particle's proper time. Hence now the velocity of the particle is also a function of τ as without γ we have the function $\gamma(\tau)$. The definition is:

$$A(\tau):=\frac{\mathrm{d}V}{\mathrm{d}\tau_0}=\gamma(\tau)\frac{\mathrm{d}V}{\mathrm{d}\tau}=\gamma^2(\tau)\frac{\mathrm{d}^2\mathbf{K}(s(\tau),\tau)}{\mathrm{d}\tau^2}+\gamma(\tau)\gamma'(\tau)\frac{\mathrm{d}(\mathbf{K}(s(\tau),\tau))}{\mathrm{d}\tau}+\gamma(\tau)\gamma'(\tau)e_4,$$

where with notation $a(\tau)=v'(\tau)=s''(\tau)$,

$$\gamma'(\tau)=\left(\frac{1}{\sqrt{1-\frac{\|v(\tau)\|_E^2}{c^2}}}\right)'=\left(\frac{1}{\sqrt{1-\frac{[\mathbf{K}(v(\tau),1),\mathbf{K}(v(\tau),1)]^1}{c^2}}}\right)'=$$

$$= \frac{\left[\frac{\mathrm{d}(\mathbf{K}(v(\tau),1)}{\mathrm{d}\tau}, \mathbf{K}(v(\tau),1)\right]^1}{c^2\left(1-\frac{[\mathbf{K}(v(\tau),1),\mathbf{K}(v(\tau),1)]^1}{c^2}\right)^{\frac{3}{2}}} = \frac{\left[\frac{\mathrm{d}(\mathbf{K}(v(\tau),1)}{\mathrm{d}\tau}, \mathbf{K}(v(\tau),1)\right]^1}{c^2}\gamma^3(\tau),$$

In the case of linear shape-function it has the form

$$A(\tau) = \gamma^2(\tau)\mathbf{K}(a(\tau),0) + \gamma(\tau)\gamma'(\tau)\mathbf{K}(v(\tau),1)) + \gamma(\tau)\gamma'(\tau)e_4,$$

Since in this case $[V(\tau),V(\tau)]^{+,T} = -c^2$, we have

$$[A(\tau),V(\tau)]^{T,+} = \gamma^3(\tau)\left([\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)]^1 + \right.$$

$$\left. +\gamma^2(\tau)\frac{[\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)]^1}{c^2}\|v(\tau)\|_E^2 - \gamma^2(\tau)\left[\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)\right]^1\right) =$$

$$= \gamma^3(\tau)\left([\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)]^1 - \frac{c^2-\|v(\tau)\|_E^2}{c^2-\|v(\tau)\|_E^2}\left[\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)\right]^1\right) = 0.$$

By Theorem 2 on the derivative of the product (corresponding to smooth and strictly convex norms) we also get this result, in fact we have

$$0 = \frac{\mathrm{d}[V(\tau),V(\tau)]^{+,T}}{\mathrm{d}\tau} = 2\left[\frac{\mathrm{d}V}{\mathrm{d}\tau},V\right]^{+,T} + \frac{\partial[V(\tau),V(\tau)]^{\tau}}{\partial\tau}(1)\cdot 0 = \frac{2}{\gamma}[A(\tau),V(\tau)]^{+,T}.$$

Also in the case of linear shape-function the *momentum* is

$$P = m_0V = \gamma m_0\left(\mathbf{K}(v(\tau),\tau) + e_4\right)$$

where m_0 is the invariant mass. We also have that

$$[P,P]^{+,T} = \gamma^2 m_0^2(\|v\|_E^2 - c^2) = (m_0c)^2.$$

Similarly the *force* is

$$F = \frac{\mathrm{d}P}{\mathrm{d}\tau} = m_0\gamma^2(\tau)\mathbf{K}(a(\tau),\tau) + \gamma(\tau)\gamma'(\tau)\mathbf{K}(v(\tau),\tau)) + \gamma(\tau)\gamma'(\tau)e_4,$$

and thus holds

$$[F,V]^{+,T} = 0.$$

3. General relativity theory

In time-space there is a way to describe and visualize certain spaces which are solutions of Einstein's equation. The first method is when we embed into an at least four-dimensional time-space a four-dimensional manifold which inner metric is a solution of the Einstein equation. Our basic references here are the books [1] and [8].

3.1. **Metrics embedded into a time-space.**

3.1.1. *Minkowski-Lorentz metric.* The simplest example of a Lorentz manifold is the *flat-space metric* which can be given as $\mathbb{R}^4$ with coordinates (t,x,y,z) and the metric function:

$$\mathrm{d}s^2=-c^2\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2.$$

In the above coordinates, the matrix representation is

$$\eta=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

In spherical coordinates (t,r,θ,ϕ), the flat space metric takes the form

$$\mathrm{d}s^2=-c^2\mathrm{d}t^2+\mathrm{d}r^2+r^2\mathrm{d}\Omega^2.$$

Here $f(r)\equiv 0$, $g=\mathrm{id}$ and $\tau=t$ implying that $\mathbf{K}\left(v,\tau\right)=v$ and the hypersurface is the light-cone defined by $\tau=\|v\|_E$. It can be considered also in a 5-dimensional time-space with shape-function $\mathbf{K}\left(v,\tau\right)=v$ as the metric of a 4-dimensional subspace through the absolute time-axis. By the equivalence of time axes in a usually space-time it also can be considered as arbitrary subspace distinct to the 4-dimensional subspace of space-like vectors, too.

3.1.2. *The de Sitter and the anti-de Sitter metrics.* The *de Sitter space* is the space defined on the de Sitter sphere of a Minkowski space of one higher dimension. Usually the metric can be considered as the restriction of the Minkowski metric

$$\mathrm{d}s^2=-c^2\mathrm{d}t^2+\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}x_3^2+\mathrm{d}x_4^2$$

to the sphere $-x_0^2+x_1^2+x_2^2+x_3^2+x_4^2=\alpha^2=\frac{3}{\Lambda}$, where Λ is the cosmological constant (see e.g. in [8]). Using also our constant c this latter equation can be rewrite as

$$-ct^2+(x_1')^2+(x_2')^2+(x_3')^2+(x_4')^2=1 \text{ where } x_0=t\,,\ \frac{1}{\alpha}=c \text{ and } x_i'=\frac{1}{\alpha}x_i.$$

This shows that in the 5-dimensional time space with shape-function $\mathbf{K}\left(v,\tau\right)=v$ it is the hyperboloid with one sheet with circular symmetry about the absolute time-axis.

The *anti-de Sitter space* is the hyperbolic analogue of the elliptic de Sitter space. The Minkowski space of one higher dimension can be restricted to the so called *anti-de Sitter sphere* (also called by in our terminology as imaginary sphere) defined by the equality $-x_0^2+x_1^2+x_2^2+x_3^2=-\alpha^2$. The shape function again is $\mathbf{K}\left(v,\tau\right)=v$ and the corresponding 4-submanifold is the hyperboloid of two sheets with hyperplane symmetry as the 4-subspace S of space-time vectors.

3.1.3. *Friedmann-Lemaître-Robertson-Walker metrics.* A standard metric forms of the Friedmann-Lemaître-Robertson-Walker metrics (F-L-R-W) family of space-times can be obtained by using suitable coordinate parameterizations of the 3-spaces of constant curvature. One of its forms is

$$\mathrm{d}s^2=-\mathrm{d}t^2+\frac{R^2(t)}{1+\frac{1}{4}k(x^2+y^2+z^2)}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right)$$

where $k\in\{-1,0,1\}$ is fixed. By the parametrization $\tau=t$ this metric is the metric of a time-space with shape-function $\mathbf{K}\left(v,\tau\right)$. Observe that

$$\|v\|_E^2=\left[\mathbf{K}\left(v,\tau\right),\mathbf{K}\left(v,\tau\right)\right]^\tau=\frac{R^2(\tau)}{1+\frac{1}{4}k\|v\|_E^2}\|\mathbf{K}\left(v,\tau\right)\|_E^2.$$

Note that we can choose the constant k also as a function of the absolute time τ giving a deterministic time-space with more generality. Hence the shape-function is

$$\mathbf{K}\left(v,\tau\right)=\frac{\sqrt{1+\frac{1}{4}k(\tau)\|v\|_E^2}}{R(\tau)}v.$$

3.2. **Three-dimensional visualization of a metric in a four-time-space.**

The second method is when we consider a four-dimensional time-space and a three-dimensional sub-manifold in it with the property that the metric of the time-space at the points of the sub-manifold can be corresponded to the given one. This method gives a good visualization of the solution in a case when the examined metric has some speciality e.g. there is no dependence on time or (and) the metric has a spherical symmetry. The examples of this section are also semi-Riemannian manifolds. We consider now such solutions which have the form:

$$\mathrm{d}s^2=-(1-f(r))c^2\mathrm{d}t^2+\frac{1}{1-f(r)}\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)$$

where

$$\mathrm{d}\Omega^2:=\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2$$

is the standard metric on the 2-sphere. Thus we have to search a shape function $\mathbf{K}\left(v,\tau\right)$ of the embedding space and a sub-manifold of it on which the Minkowski-metric gives the required one. If the metric isotropic we have a chance to give it by isotropic coordinates. To this we substitute the parameter r by the function $r=g(r^\star)$, and solve the differential equation:

$$f(g(r^\star))=1-\left(\frac{r^\star g'(r^\star)}{g(r^\star)}\right)^2$$

for the unknown function $g(r^\star)$. Then we get the metric in the isotropic form

$$\mathrm{d}s^2=-\left(\frac{r^\star g'(r^\star)}{g(r^\star)}\right)^2c^2\mathrm{d}t^2+\frac{g^2(r^\star)}{r^{\star 2}}\left(\mathrm{d}r^{\star 2}+r^{\star 2}(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)\right).$$

For isotropic rectangular coordinates $x=r^\star\sin\theta\cos\phi$, $y=r^\star\sin\theta\sin\phi$ and $z=r^\star\cos\theta$ the metric becomes

$$\mathrm{d}s^2=-\left(\frac{r^\star g'(r^\star)}{g(r^\star)}\right)^2c^2\mathrm{d}t^2+\frac{g^2(r^\star)}{r^{\star 2}}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right),$$

where $r^\star=\sqrt{x^2+y^2+z^2}$. From this substituting $ds^2=0$ and rearranging the equality, we get that the velocity of the light is

$$\sqrt{\frac{\mathrm{d}x^2}{\mathrm{d}t^2}+\frac{\mathrm{d}y^2}{\mathrm{d}t^2}+\frac{\mathrm{d}z^2}{\mathrm{d}t^2}}=\frac{r^{\star 2}g'(r^\star)}{g^2(r^\star)}c,$$

independent from its direction and varies with only the radial distance $r^\star$ (from the point mass at the origin of the coordinates). In the points of the hypersurface $t=r^\star=\sqrt{x^2+y^2+z^2}$ the metric can be parameterized by the time:

$$\mathrm{d}s^2=-\left(\frac{tg'(t)}{g(t)}\right)^2c^2\mathrm{d}t^2+\frac{g^2(t)}{t^2}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right),$$

and from the equation

$$\frac{tg'(t)}{g(t)}\mathrm{d}t=\mathrm{d}\tau$$

we can give a re-scale of the time by the parametrization

$$\tau := \int t\frac{g'(t)}{g(t)}\mathrm{d}t = t\ln(g(t)) - \int \ln(g(t))\mathrm{d}t.$$

From this equation we determine the inverse function $\hat{g}$ for which $t = \hat{g}(\tau)$. Since $\hat{g}(\tau) = t = r^\star = \sqrt{x^2+y^2+z^2}$ we also have that the examined set of points of the space-time is a hypersurface defined by the equality:

$$\tau = \left(t\ln(g(t)) - \int \ln(g(t))dt\right)\sqrt{x^2+y^2+z^2}.$$

This implies a new form of the metric at the points of this hypersurface:

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 + \frac{g^2(\hat{g}(\tau))}{\hat{g}(\tau)^2}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right).$$

The corresponding inner product has the matrix form:

$$\begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{g^2(\hat{g}(\tau))}{\hat{g}(\tau)^2} & 0 & 0 \\ 0 & 0 & \frac{g^2(\hat{g}(\tau))}{\hat{g}(\tau)^2} & 0 \\ 0 & 0 & 0 & \frac{g^2(\hat{g}(\tau))}{\hat{g}(\tau)^2} \end{pmatrix}$$

and hence the Euclidean lengthes of the vectors of the space depend only on the absolute moment τ in which we would like to measure it. Thus we can visualize the examined metric as a metric at the points of the hypersurface

$$\tau = \left(t\ln(g(t)) - \int \ln(g(t))\mathrm{d}t\right)\|v\|_E$$

of certain time-space. We note that this is not the inner metric of the examined surface of dimension 3 which can be considered as metric of a three-dimensional space-time. To determine the shape-function observe that

$$\|v\|_E^2 = [\mathbf{K}(v,\tau), \mathbf{K}(v,\tau)]^\tau = \frac{g^2(\hat{g}(\tau))}{\hat{g}(\tau)^2}\|\mathbf{K}(v,\tau)\|_E^2$$

from which we get that

$$\mathbf{K}(v,\tau) = \frac{\hat{g}(\tau)}{g(\hat{g}(\tau))}v.$$

We now give some examples.

3.2.1. *Schwarzschild metric.* Besides the flat space metric the most important metric in general relativity is the *Schwarzschild metric* which can be given in the set of local polar-coordinates (t, r, φ, θ) by

$$\mathrm{d}s^2 = -\left(1-\frac{2GM}{c^2r}\right)c^2\mathrm{d}t^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

where, again, $\mathrm{d}\Omega^2$ is the standard metric on the 2-sphere. Here G is the *gravitation constant* and M is a constant with the dimensions of mass. The function f is

$$f(r) = \frac{2GM}{c^2r} := \frac{r_s}{r} \text{ with constant } r_s = \frac{2GM}{c^2}.$$

The differential equation on g is

$$\frac{r_s}{g(r^\star)}=1-\left(\frac{r^\star g'(r^\star)}{g(r^\star)}\right)^2$$

with the solution

$$g(r^\star)=\frac{r_s}{4}c_1r^\star\left(1+\frac{1}{c_1r^\star}\right)^2,$$

and if we choose $\frac{4}{r_s}$ the parameter c_1 we get the known (see in [1]) solution

$$g(r^\star)=r^\star\left(1+\frac{r_s}{4r^\star}\right)^2.$$

For isotropic rectangular coordinates the metric becomes

$$\mathrm{d}s^2=-\frac{(1-\frac{r_s}{4r^\star})^2}{(1+\frac{r_s}{4r^\star})^2}\,c^2\mathrm{d}t^2+\left(1+\frac{r_s}{4r^\star}\right)^4(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2).$$

The equation between τ and t is

$$\tau=\int\frac{(1-\frac{r_s}{4t})}{(1+\frac{r_s}{4t})}\mathrm{d}t=\int\frac{4t-r_s}{4t+r_s}\mathrm{d}t=t-2r_s\int\frac{1}{4t+r_s}\mathrm{d}t=t-\frac{r_s}{2}\ln\left(t+\frac{r_s}{4}\right)+C.$$

Of course we can choose $C=0$. Similarly to the known tortoise-coordinates there is no explicite inverse function of this parametrization which we denote by $\hat{g}(\tau)=t$. The shape-function of the corresponding time-space is

$$\mathbf{K}\left(v,\tau\right)=\frac{\hat{g}(\tau)}{g(\hat{g}(\tau))}v=\left(1+\frac{r_s}{4\hat{g}(\tau)}\right)^{-2}v.$$

3.2.2. *Reissner-Nordström metric.* In spherical coordinates (t,r,θ,ϕ), the line element for the Reissner-Nordström metric is

$$\mathrm{d}s^2=-\left(1-\frac{r_\mathrm{S}}{r}+\frac{r_Q^2}{r^2}\right)c^2\,\mathrm{d}t^2+\frac{1}{1-\frac{r_\mathrm{S}}{r}+\frac{r_Q^2}{r^2}}\,\mathrm{d}r^2+r^2\,\mathrm{d}\theta^2+r^2\sin^2\theta\mathrm{d}\phi^2,$$

here again t is the time coordinate (measured by a stationary clock at infinity), r is the radial coordinate, $r_S=2GM/c^2$ is the Schwarzschild radius of the body, and r_Q is a characteristic length scale given by

$$r_Q^2=\frac{Q^2G}{4\pi\varepsilon_0c^4}.$$

Here $1/4\pi\varepsilon_0$ is the Coulomb force constant. The function f is

$$f(r)=\frac{r_s}{r}-\frac{r_Q^2}{r^2}$$

The differential equation on g is

$$\frac{r_s}{g(r^\star)}-\frac{r_Q^2}{g^2(r^\star)}=1-\left(\frac{r^\star g'(r^\star)}{g(r^\star)}\right)^2$$

with the solution

$$g(r^\star)=\sqrt{\frac{r_s^2}{4}-r_Q^2}\frac{c_1}{2}r^\star\left(1+\frac{1}{c_1r^\star}\right)^2-\sqrt{\frac{r_s^2}{4}-r_Q^2}+\frac{r_s}{2},$$

if we choose $c_1 := \frac{2}{\sqrt{\frac{r_s^2}{4}-r_Q^2}}$ we get a more simple form:

$$g(r^\star) = r^\star\left(1+\frac{\sqrt{\frac{r_s^2}{4}-r_Q^2}}{2r^\star}\right)^2 - \sqrt{\frac{r_s^2}{4}-r_Q^2}+\frac{r_s}{2} = r^\star\left(1+\frac{\frac{r_s^2}{4}-r_Q^2}{4r^{\star 2}}\right)+\frac{r_s}{2}.$$

For the isotropic rectangular coordinates we have:

$$\mathrm{d}s^2 = -\left(\frac{r^\star\left(1-\frac{\frac{r_s^2}{4}-r_Q^2}{4r^{\star 2}}\right)}{r^\star\left(1+\frac{\frac{r_s^2}{4}-r_Q^2}{4r^{\star 2}}\right)+\frac{r_s}{2}}\right)^2 c^2\mathrm{d}t^2 + \left(\frac{r^\star\left(1+\frac{\frac{r_s^2}{4}-r_Q^2}{4r^{\star 2}}\right)+\frac{r_s}{2}}{r^\star}\right)^2 (\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2).$$

Our process now leads to the new time parameter

$$\tau = t - \left(\frac{r_s}{4}-\frac{r_Q}{2}\right)\ln\left(\left(t+\frac{r_s}{4}\right)^2-\frac{r_Q^2}{4}\right) - r_Q\ln\left(t+\frac{r_s}{4}+\frac{r_Q}{2}\right)+C,$$

which in the case of $C = r_Q = 0$ gives back the parametrization of Schwarzschild solution. The shape-function of the searched time-space can be determined by the corresponding inverse $t = \hat{g}(\tau)$, it is

$$\mathbf{K}(v,\tau) = \frac{\hat{g}(\tau)}{g(\hat{g}(\tau))}v = \frac{\hat{g}(\tau)}{\hat{g}(\tau)\left(1+\frac{\frac{r_s^2}{4}-r_Q^2}{4\hat{g}(\tau)^2}\right)+\frac{r_s}{2}}v.$$

Analogously can be computed the time-space visualization of the Schwarzschild-de Sitter solution which we now omit.

3.2.3. *Bertotti-Robinson metric.* The Bertotti-Robinson space-time is the only conformally flat solution of the Einstein-Maxwell equalities for a non-null source-free electromagnetic field. The metric is:

$$\mathrm{d}s^2 = \frac{Q^2}{r^2}\left(-\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right),$$

and on the light-cone $t = r$ it has the form

$$\mathrm{d}s^2 = -\frac{Q^2}{t^2}\mathrm{d}t^2 + \frac{e^2}{t^2}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right).$$

By the new time coordinate

$$\tau = Q\ln t \text{ or } t = e^{\frac{\tau}{Q}}$$

using orthogonal space coordinates we get the form

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \frac{Q^2}{e^{\frac{2\tau}{Q}}}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right).$$

Thus it can be visualize on the hypersurface $\tau = e\ln r$ of the time-space with shape-function:

$$\mathbf{K}(v,\tau) := \frac{e^{\frac{\tau}{Q}}}{Q}v.$$

3.3. **Einstein's equation.** As we saw in the previous section the direct embedding of a solution of Einstein's equation into a time-space requires non-linear and very complicated shape-functions. It can be seen also that there are such solutions which there are no natural embedding into a time-space. This motivates the investigations of the present section. Our building up follows the one of the clear paper of Prof. Alan Heavens [9], we would like to thank to him for his downloadable PDF.

3.3.1. *Homogeneous time-space-manifolds and the Equivalence Principle.* We consider now such manifolds which tangent spaces are four-dimensional time-spaces with given shape-functions. More precisely:

Definition 10. *Let* $\mathcal{S}$ *be the set of linear mappings* $\mathbf{K}(v,\tau):\mathbb{E}^3\times\mathbb{R}\longrightarrow\mathbb{E}^3$ *holding the properties of a linear shape-function given in Definition 7. Giving for it the natural topology we say that* K *is* the space of shape-functions. *If we have a pair a four-dimensional topological manifold* M *and a smooth* (C^∞) *mapping* $\mathcal{K}:M\longrightarrow\mathcal{S}$ *with the property that at the point* $P\in M$ *the tangent space is the time-space defined by* $\mathbf{K}^P(s,\tau)\in\mathcal{S}$ *we say that it is a* time-space-manifold. *The time-space manifold is* homogeneous *if the mapping* $\mathcal{K}$ *is a constant function.*

Note that a Lorentzian manifold is such a homogeneous time-space manifold which shape-function is independent from the time and it is the identity mapping on its space-like components, namely $\mathbf{K}^P(s,\tau)=s$ for all P and for all τ. Its matrix-form (using the column representation of vectors in time-space) is:

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}$$

Our purpose to build up the theory of global relativity in a homogeneous time-space-manifolds. We accept the so-called *Strong Equivalence Principle* of Einstein in the following form:

Axiom 3. *(Equivalence Principle) At any point in a homogeneous time-space manifold it is possible to choose a* locally-inertial frame *in which the laws of physics are the same as the special relativity of the corresponding time-space.*

According to this principle, there is a coordinate-system in which a freely-moving particle moves with constant velocity with respect to the time-space $\mathcal{K}(P)=\mathbf{K}^P(s,\tau)=\mathbf{K}(s,\tau)$. It is convenient to write the world line

$$S(\tau)=\mathbf{K}(s(\tau),\tau)+\tau e_4$$

parametrically, as a function of the proper time $\tau_0=\frac{\tau}{\gamma(\tau)}$. In subsection 2.2 we determined the velocity using the time-space parameter τ:

$$V(\tau)=\gamma(\tau)\left(\frac{\mathrm{d}(\mathbf{K}(s(\tau),\tau))}{\mathrm{d}\tau}+e_4\right)=\gamma(\tau)\left(\mathbf{K}(v(\tau),1)+e_4\right).$$

Taking into consideration again that the shape-function is linear, the acceleration is :

$$A(\tau)=\gamma^2(\tau)\mathbf{K}(a(\tau),0)+\gamma^4(\tau)\frac{[\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)]^\tau}{c^2}\mathbf{K}(v(\tau),1)+$$
$$+\gamma^4(\tau)\frac{[\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)]^\tau}{c^2}e_4,$$

giving the differential equation $A(\tau)=0$ for such particle which moves linearly with respect to this frame.

3.3.2. *Affine connection and the metric on a homogeneous time-space-manifold.* Consider any other coordinate system in which the particle coordinates are $S'(\tau_0)$. Using the chain rule, the defining equation

$$0 = A(\tau_0) = \frac{\mathrm{d}V(\tau_0)}{\mathrm{d}\tau_0} = \frac{\mathrm{d}^2 S(\tau_0)}{\mathrm{d}\tau_0^2}$$

becomes

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}\tau_0}\left(\frac{\mathrm{d}S}{\mathrm{d}S'}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}\right) &= \frac{\mathrm{d}S}{\mathrm{d}S'}\frac{\mathrm{d}^2 S'(\tau_0)}{\mathrm{d}\tau_0^2} + \frac{\mathrm{d}}{\mathrm{d}\tau_0}\left(\frac{\mathrm{d}S}{\mathrm{d}S'}\right)\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0} = \\ &= \frac{\mathrm{d}S}{\mathrm{d}S'}\frac{\mathrm{d}^2 S'(\tau_0)}{\mathrm{d}\tau_0^2} + \frac{\mathrm{d}^2 S}{\mathrm{d}S'\mathrm{d}S'}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}, \end{aligned}$$

where $\frac{\mathrm{d}S}{\mathrm{d}S'}$ means the total derivatives of the mapping of the time-space sending the path $S'(\tau_0)$ into the specific path $S(\tau_0)$, and the trilinear function $\frac{\mathrm{d}^2 S}{\mathrm{d}S'\mathrm{d}S'}$ is the second total derivatives of the same mapping. (If there is a general smooth transformation between the coordinate-frames, the corresponding derivatives are exist.) From this equality we get the tensor form of the so called *geodesic equation* of homogeneous time-space manifold, it is:

$$\frac{\mathrm{d}^2 S'(\tau_0)}{\mathrm{d}\tau_0^2} + \left(\frac{\mathrm{d}S'}{\mathrm{d}S}\frac{\mathrm{d}^2 S}{\mathrm{d}S'\mathrm{d}S'}\right)\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0} = \frac{\mathrm{d}^2 S'(\tau_0)}{\mathrm{d}\tau_0^2} + \Gamma(S',S)\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0} = 0.$$

Here we denote the inverse of the total derivatives $\frac{\mathrm{d}S}{\mathrm{d}S'}$ by $\frac{\mathrm{d}S'}{\mathrm{d}S}$. The name of $\Gamma(S',S)$ is the *affine connection*.

For the uniform labelling we denote by x^4 the identity function. Since the shape function is a linear mapping we can represent it as the multiplication on left by the 3×4 matrix $K = [k_{ij}] = k^i{}_j$. In the rest of this paragraph we apply all conventions of general relativity. The Greek alphabet is used for space and time components, where indices take values 1,2,3,4 (frequently used letters are $\mu, \nu, \cdots$) and the Latin alphabet is used for spatial components only, where indices take values 1,2,3 (frequently used letters are $i, j, ...$) and according to the Einstein's convention, when an index variable appears twice in a single term it implies summation of that term over all the values of the index. The upper indices are indices of coordinates, coefficients or basis vectors.

The mapping $\mathcal{S} : S'(\tau_0) \longrightarrow S(\tau_0)$ sends $K(x'^1, x'^2, x'^3, x'^4)^T + x'^4 e_4$ into the vector $K(x^1, x^2, x^3, x^4)^T + x^4 e_4$. Denote by $\widetilde{K}$ the 4×4 matrix with coefficients:

$$\begin{pmatrix} k^1{}_1 & k^1{}_2 & k^1{}_3 & k^1{}_4 \\ k^2{}_1 & k^2{}_2 & k^2{}_3 & k^2{}_4 \\ k^3{}_1 & k^3{}_2 & k^3{}_3 & k^3{}_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then we get $\mathcal{S} : \widetilde{K}(x'^1, x'^2, x'^3, x'^4)^T \mapsto \widetilde{K}(x^1, x^2, x^3, x^4)^T$. If the shape-function $\mathbf{K}$ restricted to the subspace S is a regular linear mapping than we also have

$$\widetilde{K}^{-1}\mathcal{S}\widetilde{K}(x'^1, x'^2, x'^3, x'^4)^T = (x^1, x^2, x^3, x^4)^T$$

and we have that

$$\left[\frac{\partial x^\alpha}{\partial x'^\mu}\right] = \frac{\mathrm{d}\widetilde{K}^{-1}\mathcal{S}\widetilde{K}}{\mathrm{d}S'} = \widetilde{K}^{-1}\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}S'}\widetilde{K} \text{ and so } \frac{\mathrm{d}\mathcal{S}}{\mathrm{d}S'} = \widetilde{K}\left[\frac{\partial x^\alpha}{\partial x'^\mu}\right]\widetilde{K}^{-1}.$$

Hence

$$\frac{\mathrm{d}S'}{\mathrm{d}S} = \widetilde{K}\left[\frac{\partial x^\alpha}{\partial x'^\mu}\right]^{-1}\widetilde{K}^{-1} = \widetilde{K}\left[\frac{\partial x'^\mu}{\partial x^\alpha}\right]\widetilde{K}^{-1} \text{ and } \left[\frac{\mathrm{d}^2 S}{\mathrm{d}S'\mathrm{d}S'}\right]^\alpha = \widetilde{K}\left[\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu}\right]\widetilde{K}^{-1}$$

implying that the affine connection is:

$$\Gamma(S',S)^{\lambda}{}_{\mu\nu}=\widetilde{K}\frac{\partial x'^{\lambda}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}}\widetilde{K}^{-1}=\widetilde{K}\Gamma^{\lambda}{}_{\mu\nu}\widetilde{K}^{-1}=\widetilde{K}\left\{\begin{array}{c}\lambda\\ \mu\nu\end{array}\right\}\widetilde{K}^{-1}.$$

Since $S'(\tau_0)=\widetilde{K}(x'^1,x'^2,x'^3,x'^4)^T$ thus we also get three equalities, the first one is:

$$\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}=\widetilde{K}\left(\frac{\mathrm{d}x'^1}{\mathrm{d}\tau_0},\frac{\mathrm{d}x'^2}{\mathrm{d}\tau_0},\frac{\mathrm{d}x'^3}{\mathrm{d}\tau_0},\frac{\mathrm{d}x'^4}{\mathrm{d}\tau_0}\right)^T=\left(k^1{}_{\alpha}\frac{\mathrm{d}x'^{\alpha}}{\mathrm{d}\tau_0},k^2{}_{\alpha}\frac{\mathrm{d}x'^{\alpha}}{\mathrm{d}\tau_0},k^3{}_{\alpha}\frac{\mathrm{d}x'^{\alpha}}{\mathrm{d}\tau_0},k^4{}_{\alpha}\frac{\mathrm{d}x'^{\alpha}}{\mathrm{d}\tau_0}\right)^T=$$

$$=\left[k^{\lambda}{}_{\alpha}\frac{\mathrm{d}x'^{\alpha}}{\mathrm{d}\tau_0}\right].$$

The second equality is:

$$\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}=\widetilde{K}\left(\frac{\mathrm{d}x'^1}{\mathrm{d}\tau_0},\frac{\mathrm{d}x'^2}{\mathrm{d}\tau_0},\frac{\mathrm{d}x'^3}{\mathrm{d}\tau_0},\frac{\mathrm{d}x'^4}{\mathrm{d}\tau_0}\right)^T\left(\frac{\mathrm{d}x'^1}{\mathrm{d}\tau_0},\frac{\mathrm{d}x'^2}{\mathrm{d}\tau_0},\frac{\mathrm{d}x'^3}{\mathrm{d}\tau_0},\frac{\mathrm{d}x'^4}{\mathrm{d}\tau_0}\right)\widetilde{K}^T=$$

$$=\widetilde{K}\left[\frac{\mathrm{d}x'^{\mu}}{\mathrm{d}\tau_0}\frac{\mathrm{d}x'^{\nu}}{\mathrm{d}\tau_0}\right]\widetilde{K}^T,$$

and the third one is:

$$\frac{\mathrm{d}^2S'(\tau_0)}{\mathrm{d}\tau_0^2}=\widetilde{K}\left(\frac{\mathrm{d}^2x'^1}{\mathrm{d}\tau_0^2},\frac{\mathrm{d}^2x'^2}{\mathrm{d}\tau_0^2},\frac{\mathrm{d}^2x'^3}{\mathrm{d}\tau_0^2},\frac{\mathrm{d}^2x'^4}{\mathrm{d}\tau_0^2}\right)^T=\left[k^{\lambda}{}_{\alpha}\frac{\mathrm{d}^2x'^{\alpha}}{\mathrm{d}\tau_0^2}\right].$$

The geodesic equation now:

$$0=\widetilde{K}\left(\frac{\mathrm{d}^2x'^1}{\mathrm{d}\tau_0^2},\frac{\mathrm{d}^2x'^2}{\mathrm{d}\tau_0^2},\frac{\mathrm{d}^2x'^3}{\mathrm{d}\tau_0^2},\frac{\mathrm{d}^2x'^4}{\mathrm{d}\tau_0^2}\right)^T+\widetilde{K}\Gamma^{\lambda}{}_{\mu\nu}\widetilde{K}^{-1}\widetilde{K}\left[\frac{\mathrm{d}x'^{\mu}}{\mathrm{d}\tau_0}\frac{\mathrm{d}x'^{\nu}}{\mathrm{d}\tau_0}\right]\widetilde{K}^T,$$

or equivalently

$$0=\left(\frac{\mathrm{d}^2x'^1}{\mathrm{d}\tau_0^2},\frac{\mathrm{d}^2x'^2}{\mathrm{d}\tau_0^2},\frac{\mathrm{d}^2x'^3}{\mathrm{d}\tau_0^2},\frac{\mathrm{d}^2x'^4}{\mathrm{d}\tau_0^2}\right)^T+\Gamma^{\lambda}{}_{\mu\nu}\left[\frac{\mathrm{d}x'^{\mu}}{\mathrm{d}\tau_0}\frac{\mathrm{d}x'^{\nu}}{\mathrm{d}\tau_0}\right]\widetilde{K}^T,$$

implying that

$$0=\frac{\mathrm{d}^2x'^{\lambda}}{\mathrm{d}\tau_0^2}+\Gamma^{\lambda}{}_{\mu\nu}\frac{\mathrm{d}x'^{\mu}}{\mathrm{d}\tau_0}k^{\nu}{}_{\zeta}\frac{\mathrm{d}x'^{\zeta}}{\mathrm{d}\tau_0}.$$

Since for the proper time we have the equality

$$-c^2\mathrm{d}\tau_0^2=\mathrm{d}S^T\left(\begin{array}{cc}1&0\\0&-c^2\end{array}\right)\mathrm{d}S=\left(\frac{\mathrm{d}S}{\mathrm{d}S'}\mathrm{d}S'\right)^T\eta\frac{\mathrm{d}S}{\mathrm{d}S'}\mathrm{d}S'=\mathrm{d}S'^Tg\mathrm{d}S'$$

hence

$$g(S',S)=\left(\frac{\mathrm{d}S}{\mathrm{d}S'}\right)^T\eta\frac{\mathrm{d}S}{\mathrm{d}S'}.$$

Let denote by $[{}_j{}^ik]$ the transpose of the matrix $[k^i{}_j]$ and $K^i{}_j$ the elements of the inverse of $\widetilde{K}$. Then since

$$g(S',S)=\left(\widetilde{K}^{-1}\right)^T\left[\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\right]^T\widetilde{K}^T\eta\widetilde{K}\left[\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\right]\widetilde{K}^{-1}$$

thus

$$g(S',S)_{\varphi\psi}={}_{\varphi}{}^{\mu}K\frac{\partial x^{\alpha}}{\partial x'^{\mu}}{}_{\alpha}{}^{\delta}k\eta_{\delta,\varepsilon}k^{\varepsilon}{}_{\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}K^{\nu}{}_{\psi}.$$

This matrix is the *metric tensor* of the homogeneous time-space manifold in question. If $\widetilde{K}$ is the unit matrix, then $\mu=\varphi$, $\nu=\psi$, $\alpha=\delta$ and $\beta=\varepsilon$ implying the known formula

$$g_{\mu\nu}=\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\eta_{\alpha\beta}.$$

Also note that if $\widetilde{K}$ is an orthogonal transformation then we get a more simple form of the metric:

$$g(S',S)=\widetilde{K}\left[\frac{\partial x^{l}}{\partial x'^{i}}\right]^{T}\eta\left[\frac{\partial x^{l}}{\partial x'^{i}}\right]\widetilde{K}^{T}.$$

To determine the connection between the metric and the affine connection we determine the partial derivative of the metric.

$$\frac{\partial g(S',S)}{\partial x'^{\lambda}}=\left(\widetilde{K}^{-1}\right)^{T}\left[\frac{\partial^{2}x^{\alpha}}{\partial x'^{\mu}\partial x'^{\lambda}}\right]^{T}\widetilde{K}^{T}\eta\widetilde{K}\left[\frac{\partial x^{\beta}}{\partial x'^{\nu}}\right]\widetilde{K}^{-1}+$$

$$+\left(\widetilde{K}^{-1}\right)^{T}\left[\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\right]^{T}\widetilde{K}^{T}\eta\widetilde{K}\left[\frac{\partial^{2}x^{\beta}}{\partial x'^{\nu}\partial x'^{\lambda}}\right]\widetilde{K}^{-1},$$

and since

$$\frac{\partial^{2}x^{\alpha}}{\partial x'^{\mu}\partial x'^{\lambda}}=\frac{\partial x^{\alpha}}{\partial x'^{\rho}}\widetilde{K}^{-1}\Gamma(S',S)^{\rho}{}_{\mu\lambda}\widetilde{K}=$$

we have

$$\frac{\partial g(S',S)_{\varphi\psi}}{\partial x'^{\lambda}}=\Gamma(S',S)^{\rho}{}_{\varphi\lambda}g(S',S)_{\rho\psi}+g(S',S)_{\varphi\rho}\Gamma(S',S)^{\rho}{}_{\lambda\psi}$$

as in the classical case. Denote by $g(S,S')^{\varphi\rho}$ the inverse of the metric tensor then we get the connection:

$$\Gamma(S',S)^{\sigma}{}_{\lambda\mu}=\frac{1}{2}g(S,S')^{\nu\sigma}\left\{\frac{\partial g(S',S)_{\mu,\nu}}{\partial x'^{\lambda}}+\frac{\partial g(S',S)_{\lambda,\nu}}{\partial x'^{\mu}}-\frac{\partial g(S',S)_{\mu,\lambda}}{\partial x'^{\nu}}\right\}.$$

3.3.3. *Covariant derivative, parallel transport and the curvature tensor.* Since we determined the affine connection we can define the *covariant derivative* of a vectors fields on the way:

$$V^{\mu}{}_{;\lambda}=\frac{\partial V^{\mu}}{\partial x'^{\lambda}}+\Gamma(S',S)^{\mu}{}_{\lambda\rho}V^{\rho}=\frac{\partial V^{\mu}}{\partial x'^{\lambda}}+\widetilde{K}\Gamma^{\mu}{}_{\lambda\delta}\widetilde{K}^{-1}V^{\delta}.$$

In fact, it converts vectors into tensor on the basis of the following calculation:

$$\widetilde{K}\left[\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right]\left[\frac{\partial x^{\rho}}{\partial x'^{\lambda}}\right]\widetilde{K}^{-1}V^{\nu}{}_{;\rho}=\widetilde{K}\left[\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right]\left[\frac{\partial x^{\rho}}{\partial x'^{\lambda}}\right]\widetilde{K}^{-1}\left(\frac{\partial V^{\nu}}{\partial x^{\rho}}+\widetilde{K}\Gamma^{\nu}{}_{\rho\delta}\widetilde{K}^{-1}V^{\delta}\right)=$$

$$=\widetilde{K}\left[\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right]\left[\frac{\partial x^{\rho}}{\partial x'^{\lambda}}\right]\widetilde{K}^{-1}\left(\frac{\partial V^{\nu}}{\partial x^{\rho}}+\widetilde{K}\frac{\partial x'^{\nu}}{\partial x^{\alpha}}\frac{\partial^{2}x^{\alpha}}{\partial x'^{\rho}\partial x'^{\delta}}\widetilde{K}^{-1}V^{\delta}\right)=$$

$$=\frac{\partial V'^{\mu}}{\partial x'^{\lambda}}+\widetilde{K}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial^{2}x^{\alpha}}{\partial x'^{\lambda}\partial x'^{\delta}}\widetilde{K}^{-1}V'^{\delta}=\frac{\partial V'^{\mu}}{\partial x'^{\lambda}}+\widetilde{K}\Gamma^{\mu}{}_{\lambda\delta}\widetilde{K}^{-1}V'^{\delta}=V'^{\mu}{}_{;\lambda}.$$

Note that the covariant derivative of a co-vector is

$$V_{\mu;\lambda}=\frac{\partial V_{\mu}}{\partial x'^{\lambda}}-\Gamma(S',S)^{\mu}{}_{\lambda\rho}V^{\rho},$$

and the covariant derivative of a tensor has the rule, each upper index adds a Γ term and each lower index subtracts one. For this reason the covariant derivative of the metric tensor (by our calculation above) vanishes.

Again from the definition of the covariant derivative we get that the *equation of parallel transport* is now:

$$\frac{\mathrm{d}V^{\mu}}{\mathrm{d}\tau_0} = -\Gamma(S',S)^{\mu}{}_{\lambda\nu}\frac{\mathrm{d}x'^{\lambda}}{\mathrm{d}\tau_0}V^{\nu}.$$

From this it follows that the parallel-transport along a side $\delta x'^{\beta}$ of a small closed parallelogram is

$$\delta V^{\alpha} = -\Gamma^{\alpha}{}_{\beta\nu}(S',S)V^{\nu}\delta x'^{\beta}$$

and thus the total change around a small closed parallelogram with sides δa^{μ}, δb^{ν} is

$$\delta V^{\alpha} = (\Gamma^{\alpha}{}_{\beta\nu;\rho}(S',S)V^{\nu} + \Gamma^{\alpha}{}_{\beta\nu}(S',S)V^{\nu}{}_{;\rho} - \Gamma^{\alpha}{}_{\rho\nu;\beta}(S',S)V^{\nu} - \Gamma^{\alpha}{}_{\rho\nu}(S',S)V^{\nu}{}_{;\beta})\,\delta a^{\beta}\delta b^{\rho}$$

implying that

$$\delta V^{\alpha} = R(S',S)^{\alpha}{}_{\sigma\rho\beta}V^{\sigma}\delta a^{\beta}\delta b^{\rho}.$$

Here $R(S',S)^{\alpha}{}_{\sigma\rho\beta}$ is the *Riemann curvature tensor* defined by

$$R(S',S)^{\alpha}{}_{\sigma\rho\beta} := \Gamma(S',S)^{\alpha}{}_{\beta\sigma;\rho} - \Gamma(S',S)^{\alpha}{}_{\rho\sigma;\beta} + \Gamma(S',S)^{\alpha}_{\rho\nu}\Gamma(S',S)^{\nu}_{\sigma\beta} - \Gamma(S',S)^{\alpha}_{\beta\nu}\Gamma(S',S)^{\nu}_{\sigma\rho}.$$

The Ricci Tensor and the scalar curvature defined by

$$R(S',S)_{\sigma\beta} := R(S',S)^{\alpha}{}_{\sigma\alpha\beta} \text{ and } R(S',S) := R(S',S)^{\sigma}{}_{\sigma},$$

respectively.

3.3.4. *Einstein's equation.* As we can saw in the previous paragraph all of the notion of global relativity can be defined in a time-space-manifold thus all of the equations between them is a well-defined equation. On the other hand Einstein's equation take into consideration the facts of physic; hence contains parameters which can not be changed. Fortunately we noted earlier that the covariant derivative of our metric tensor vanishes, too. Thus also vanishes the covariant derivative its inverse and hence we can write the Einstein's equation with *cosmological constant* Λ, too. The equation is formally the same that the original one, but contains a new (undetermined) parameter which is the matrix $\widetilde{K}$ of the shape-function. It is:

$$R(S',S)^{\mu\nu} - \frac{1}{2}g(S',S)^{\mu\nu}R(S',S) - \Lambda g(S',S)^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu},$$

where the parameter G can be adjusted so that the active and gravitational masses are equal and $T^{\mu\nu}$ is the *energy-momentum tensor.*

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