IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 235305 (10pp) doi:10.1088/1751-8113/41/23/235305

Simplified solutions of the Cox–Thompson inverse scattering method at fixed energy

Tamás Pálmai[1], **Miklós Horváth**[2] **and Barnabás Apagyi**[1]

[1] Department of Theoretical Physics, Institute of Physics, Budapest University of Technology and Economics, H-1111 Budapest, Hungary
[2] Department of Mathematical Analysis, Institute of Mathematics, Budapest University of Technology and Economics, H-1111 Budapest, Hungary

E-mail: apagyi@phy.bme.hu

Received 13 February 2008, in final form 10 April 2008
Published 20 May 2008
Online at stacks.iop.org/JPhysA/41/235305

Abstract
Simplified solutions of the Cox–Thompson inverse quantum scattering method at fixed energy are derived if a finite number of partial waves with only even or odd angular momenta contribute to the scattering process. Based on new formulae various approximate methods are introduced which also prove applicable to the generic scattering events.

PACS numbers: 02.30.Zz, 02.60.Cb, 02.60.Pn, 03.65.Nk, 24.10.−i, 25.40.Fq, 25.70.Bc

1. Introduction

The Cox–Thompson (CT) inversion method [1] has a number of useful properties. As an input it requires a finite set consisting of N phase shifts. As an output it produces potentials which possess non-vanishing first momentum, $\int rV(r)\,\mathrm{d}r \neq 0$, and finite values at the origin, $|V(0)| < \infty$ [2]. These properties make the CT method an attractive constructive procedure which can be applied to experimental phase shifts in order to determine interaction potentials between composite quantum systems. However, the CT method leads to the solution of equations which are of nonlinear character, involving the inverse of a matrix containing the unknown quantities themselves. The solution of the CT equations is therefore numerically difficult to perform and, sometimes, only successful with deployment of sophisticated nonlinear solvers such as the simulated annealing method [5].

In this paper we investigate the possibility of making the CT equations simpler and introduce some alternative forms which follow from the analytical structure of the method. We shall derive equations for calculating the unknown generalized angular momenta, L's, from the finite input set of phase shifts, δ_l's. The characteristic property of these equations is that they do not involve any complicated matrix inversion. The simplification is only possible

if the even angular momenta can be handled separately from the odd ones. Physically such a situation occurs in a certain case of identical bosonic (fermionic) collision when only even (odd) partial waves contribute to the differential cross section and, thus, only phase shifts belonging to even (odd) angular momenta can be derived from experiments. We note that according to [3, 4], the even (odd) partial wave phase shifts alone would result in the same (unique) potential if an infinite subset of them were used as input and a proper constructive procedure were known. In the case of a finite set of phase shifts, however, when also even and odd partial waves contribute generally to the scattering, we can still use our simplified forms as approximations, and we shall provide various approximate examples for such usages. Finally, we also investigate the possibility of using the so-called one-term solution (or total decoupling case) approximation $L = l - 2\delta_\ell/\pi$, which can be obtained from the CT equations with $N = 1$.

In the next section we collect all the equations necessary to outline the CT method. The simplified equations valid in the case of even or odd angular momenta are derived in section 3, and the examples for the various applications with and without approximations are presented in section 4. Section 5 is devoted to the conclusion.

2. The Cox–Thompson method

Let us introduce first the dimensionless quantities, $x = kr$ for the distance, and $q(x) = \frac{V(x/k)}{E}$ for the potential where $E = \hbar^2 k^2/2m$ denotes the scattering energy, m the reduced mass, and k the wave number.

The inverse potential corresponding to the radial scattering Schrödinger equation reads as follows:

$$q(x) = -\frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{K(x,x)}{x}, \tag{1}$$

where the quantity $K(x, x)$ is the diagonal part of the transformation kernel defined by a Gel'fand–Levitan–Marchenko (GLM) type integral equation,

$$K(x,y) = g(x,y) - \int_0^x \mathrm{d}t\, t^{-2} K(x,t) g(t,y), \qquad x \geqslant y. \tag{2}$$

In the CT method we take the following separable expansion for the input kernel:

$$g(x,y) = \sum_{l\in S} \gamma_l u_l(x_<) v_l(x_>), \qquad \begin{Bmatrix} x_< \\ x_> \end{Bmatrix} = \begin{Bmatrix} \min \\ \max \end{Bmatrix} (x,y) \tag{3}$$

where γ_l are expansion coefficients (not required to be determined), $u_l(v_l)$ denote the regular (irregular) solutions of the free radial Schrödinger equation and the summation set S consists of the physical angular momenta $\{l\}$.

To solve the GLM equation (2), Cox and Thompson [1] introduced another separable ansatz for the transformation kernel, in the form of a sum over an artificial angular momentum space $L \in T$,

$$K(x,y) = \sum_{L\in T} A_L(x) u_L(y) \tag{4}$$

where A_L's are unknown expansion functions and T is interpreted as a set of unknown shifted angular momenta to be determined under the constraint that it has the same number N of *different* elements as S does, i.e., $N = |S| = |T|$ and $S \cap T = \emptyset$.

By inserting equations (3) and (4) into the GLM equation (2) and using the linear independence of the regular and irregular free solutions, one obtains equations for the determination of the expansion functions $A_L(x)$ as follows:

$$\sum_{L\in T} A_L(x)\frac{W[u_L(x), v_l(x)]}{l(l+1)-L(L+1)} = v_l(x), \qquad l \in S. \tag{5}$$

Here, the only unknowns are the set T with elements L, and W denotes the Wronskian defined by $W[a, b] \equiv ab' - a'b$.

In order to determine the set T one makes use of the Povzner–Levitan representation of the radial scattering wavefunction which reads as

$$\psi_l(x) = u_l(x) - \int_0^x \mathrm{d}t\, t^{-2} K(x, t) u_l(t), \qquad l \in S, \tag{6}$$

whose asymptotic form containing the input phase shift data takes the form

$$B_l \sin\left(x - l\frac{\pi}{2} + \delta_l\right) = \sin\left(x - l\frac{\pi}{2}\right) - \sum_{L\in T} A_L^{\mathrm{a}}(x)\frac{\sin\left((l-L)\frac{\pi}{2}\right)}{l(l+1)-L(L+1)}, \quad l \in S. \tag{7}$$

Here, B_l is a normalization constant and we have defined the asymptotic expansion functions $A_L^{\mathrm{a}}(x) \equiv A_L(x \to \infty)$ which can be calculated from the asymptotic version of equation (5) which is given by

$$\sum_{L\in T} A_L^{\mathrm{a}}(x)\frac{\cos\left((l-L)\frac{\pi}{2}\right)}{l(l+1)-L(L+1)} = -\cos\left(x - l\frac{\pi}{2}\right), \qquad l \in S. \tag{8}$$

Using the last two equations, (7) and (8), one can easily derive the following equations for the determination of the unknown L's from the input phase shifts, δ_l's [2, 6]:

$$S_l = \frac{1 + \mathrm{i}\mathcal{K}_l^+}{1 - \mathrm{i}\mathcal{K}_l^-} \qquad \text{or} \qquad \tan(\delta_l) = \frac{\mathcal{K}_l^+ + \mathcal{K}_l^-}{2 + \mathrm{i}(\mathcal{K}_l^+ - \mathcal{K}_l^-)}, \qquad l \in S \tag{9}$$

where $S_l = \mathrm{e}^{2\mathrm{i}\delta_l}$ and the 'shifted' reactance matrix elements are defined as

$$\mathcal{K}_l^{\pm} = \sum_{L\in T, l'\in S} [M_{\sin}]_{lL}[M_{\cos}^{-1}]_{Ll'}\, \mathrm{e}^{\pm\mathrm{i}(l-l')\pi/2}, \qquad l \in S, \tag{10}$$

with

$$\begin{Bmatrix} M_{\sin} \\ M_{\cos} \end{Bmatrix}_{lL} = \frac{1}{L(L+1)-l(l+1)} \begin{Bmatrix} \sin\left((l-L)\frac{\pi}{2}\right) \\ \cos\left((l-L)\frac{\pi}{2}\right) \end{Bmatrix}, \qquad l \in S, \quad L \in T. \tag{11}$$

The appearance of the inverse of the matrix $M_{\cos}$ containing the unknown L's is the problem that makes the solution of the nonlinear equations (9) especially hard.

If one succeeds in solving either one or both of the highly nonlinear equations (9) and thus finds the set T, the CT inverse potential $q_{\mathrm{CT}}(x)$ can easily be obtained by employing equations (5), (4) and (1).

3. Simplified solutions and approximations

In this section, we present simplifications to equations (9) which can be used if only even (odd) partial waves arise during the collision. Otherwise, the simplified equations can be employed to construct different approximations which will be discussed in separate subsections.

3.1. Even (odd) angular momentum treatment

Let us differentiate equation (8) twice with respect to the variable x. Then we arrive at the following equation:

$$\frac{\mathrm{d}^2 A_L^{\mathrm{a}}(x)}{\mathrm{d}x^2} = -A_L^{\mathrm{a}}(x), \tag{12}$$

which has a periodic solution as

$$A_L^{\mathrm{a}}(x) = a_L \cos(x) + b_L \sin(x). \tag{13}$$

Now, by inserting this solution (13) into equation (8) and taking into account the independence of the sine and cosine functions, one gets the following two equations:

$$\sum_{L\in T} \begin{Bmatrix} a_L \\ b_L \end{Bmatrix} \frac{\cos\left((l-L)\frac{\pi}{2}\right)}{L(L+1)-l(l+1)} = \begin{Bmatrix} \cos\left(l\frac{\pi}{2}\right) \\ \sin\left(l\frac{\pi}{2}\right) \end{Bmatrix}, \qquad l \in S. \tag{14}$$

Consider the decomposition

$$S = S_{\mathrm{e}} \cup S_{\mathrm{o}}$$

where S_{e} and S_{o} contains, respectively, the even and odd elements of S. Instead of (14) we consider two systems:

$$\sum_{L\in T_{\mathrm{e}}} \begin{Bmatrix} a_L \\ b_L \end{Bmatrix} \frac{\cos\left((l-L)\frac{\pi}{2}\right)}{L(L+1)-l(l+1)} = \begin{Bmatrix} \cos\left(l\frac{\pi}{2}\right) \\ \sin\left(l\frac{\pi}{2}\right) \end{Bmatrix}, \qquad l \in S_{\mathrm{e}}$$

and

$$\sum_{L\in T_{\mathrm{o}}} \begin{Bmatrix} a_L \\ b_L \end{Bmatrix} \frac{\cos\left((l-L)\frac{\pi}{2}\right)}{L(L+1)-l(l+1)} = \begin{Bmatrix} \cos\left(l\frac{\pi}{2}\right) \\ \sin\left(l\frac{\pi}{2}\right) \end{Bmatrix}, \qquad l \in S_{\mathrm{o}},$$

where $|T_{\mathrm{e}}| = |S_{\mathrm{e}}|$, $|T_{\mathrm{o}}| = |S_{\mathrm{o}}|$ and $T_{\mathrm{e}} \cap S_{\mathrm{e}} = \emptyset$, $T_{\mathrm{o}} \cap S_{\mathrm{o}} = \emptyset$. These systems have the solutions

$$a_L = \frac{\prod_{l\in S_{\mathrm{e}}}(L(L+1)-l(l+1))}{\prod_{L'\in T_{\mathrm{e}}\setminus\{L\}}(L(L+1)-L'(L'+1))} \frac{1}{\cos\left(L\frac{\pi}{2}\right)}, \qquad b_L = 0, \qquad L \in T_{\mathrm{e}}, \tag{15}$$

and

$$a_L = 0, \qquad b_L = \frac{\prod_{l\in S_{\mathrm{o}}}(L(L+1)-l(l+1))}{\prod_{L'\in T_{\mathrm{o}}\setminus\{L\}}(L(L+1)-L'(L'+1))} \frac{1}{\sin\left(L\frac{\pi}{2}\right)}, \qquad L \in T_{\mathrm{o}}, \tag{16}$$

respectively. In the case of $T_{\mathrm{e}} \cap T_{\mathrm{o}} \neq \emptyset$ the formulae (15) and (16) may assign different values to the same a_L and b_L but this is not a real ambiguity because we always use separately the solution vectors (15) and (16) in our calculations.

Now, by using the explicit expressions (15) in equations (13) and (7), one obtains the final solution to the CT method as

$$\tan(\delta_l) = -\sum_{L\in T_{\mathrm{e}}} \frac{\prod_{l'\in S_{\mathrm{e}}\setminus\{l\}}(L(L+1)-l'(l'+1))}{\prod_{L'\in T_{\mathrm{e}}\setminus\{L\}}(L(L+1)-L'(L'+1))} \tan\left(L\frac{\pi}{2}\right), \qquad l \in S_{\mathrm{e}}, \tag{17}$$

valid for the case of even l's. Similarly, using equations (16) we get the solution to the CT method as

$$\tan(\delta_l) = \sum_{L\in T_{\mathrm{o}}} \frac{\prod_{l'\in S_{\mathrm{o}}\setminus\{l\}}(L(L+1)-l'(l'+1))}{\prod_{L'\in T_{\mathrm{o}}\setminus\{L\}}(L(L+1)-L'(L'+1))} \cot\left(L\frac{\pi}{2}\right), \qquad l \in S_{\mathrm{o}} \tag{18}$$

which are valid in the case of odd l's. These equations determine the unknown sets T_{e} or T_{o} of shifted angular momenta L and, thus, replace either of the corresponding generic solutions (9).

Note the simplified structure of the nonlinear equations (17) and (18), compared to equations (9). While equations (9) contain an explicit matrix inversion of a matrix involving the unknowns of shifted angular momenta, L's, formulae (17) and (18) do not require such a nonlinear operation. They 'only' contain products and the tangent (cotangent) operations and are thus presumably easier to be solved for the sets T_e or T_o, if the respective input phase shifts are given.

Finding the sets T_e or T_o, the corresponding potentials $q_e(x)$ or $q_o(x)$ can be obtained similarly as in the general case, by employing equations (5), (4) and (1).

3.2. Equivalence of solutions (9) and (17) or (18)

By an explicit calculation one can check the equivalence of equations (9) and (17) or (18) for the special cases of even or odd l's. For the case of either even or odd l's, the relation $\mathcal{K}_l^+ = \mathcal{K}_l^-$ holds. Now, specifying ourselves to the even l case only, $l \in S_e$, the second of the general solutions (9) can be written as

$$\tan(\delta_l) = \sum_{L\in T_e, l'\in S_e} [M_{\sin}]_{lL}\left[M_{\cos}^{-1}\right]_{Ll'}(-)^{(l-l')/2}, \qquad l \in S_e. \tag{19}$$

Using equations (8) and (13), we get the expression

$$a_L \cos(x) = \sum_{l'\in S_e}\left[M_{\cos}^{-1}\right]_{Ll'} \cos\left(x - l'\frac{\pi}{2}\right), \qquad L \in T_e \tag{20}$$

which simplifies to

$$a_L = \sum_{l'\in S_e}\left[M_{\cos}^{-1}\right]_{Ll'}(-)^{l'/2}, \qquad L \in T_e. \tag{21}$$

By multiplying both sides of this equation by $(-)^{l/2}[M_{\sin}]_{lL}$, and performing the sum over L's, one may write

$$\sum_{L\in T_e} a_L[M_{\sin}]_{lL}(-)^{l/2} = \sum_{L\in T_e, l'\in S_e}[M_{\sin}]_{lL}\left[M_{\cos}^{-1}\right]_{Ll'}(-)^{(l-l')/2}, \qquad l \in S_e. \tag{22}$$

According to equation (19), the right-hand side is already equal to $\tan(\delta_l)$, and, by noting that the matrix $M_{\sin}$ on the left-hand side can be written, on account of (11), as $[M_{\sin}]_{lL} = -(-)^{l/2}\sin\left(L\frac{\pi}{2}\right)/(L(L+1) - l(l+1))$, one gets the formula

$$\tan(\delta_l) = -\sum_{L\in T_e} a_L \frac{\sin\left(L\frac{\pi}{2}\right)}{L(L+1) - l(l+1)}, \qquad l \in S_e, \tag{23}$$

which is the same as equation (17) if one takes into consideration the solution (15) for the coefficient a_L, $L \in T_e$.

A similar procedure can be applied to proving equivalence of equations (9) and (18) for odd l's.

3.3. Approximations

Equations (17) and (18) can be used to derive inverse potentials only in the case when either even or odd partial waves arise during the collision process. In other words, only identical bosonic or fermionic scattering can be treated by the simplified equations (17) and (18). However, the simplified equations offer several possibilities of introducing various approximate treatments of the general scattering case. The type of approximations will be classified according to the level it is applied to.

3.3.1. Potential approximation, A. If the general equations (9) can be solved by neither of the nonlinear solvers at hand, one may try to assess the inverse potential by solving the simplified equations (17) and (18) for the sets T_e and T_o. Then, separately constructing the corresponding potentials $q_e(x)$ and $q_o(x)$, one simply adds them together to get an approximation of the interaction potential, $q_A(x) = q_e(x) + q_o(x)$.

3.3.2. T-set approximation, T. One may try to approximate the set of the shifted angular momenta themselves. By unifying the two sets obtained by solving equations (17) and (18), one gets the T-set approximation $T_a = T_e \cup T_o$. Using this approximate set, T_a, in conjunction with equations (5), (4) and (1), one gets the approximate inverse potential $q_T(x)$.

3.3.3. One-term approximation, L. If the collision is dominated overwhelmingly by a single partial wave (as in the case of resonance scattering) then equations (9) are to be solved at $N = 1$, and this results in the simple expression $L = l - 2\delta_l/\pi$ for the shifted angular momentum, assuming that the lth partial wave is dominating. If however this is not the case, one still may try to use the approximate expressions

$$L_a = l - 2\delta_l/\pi \tag{24}$$

to form an approximate set T_L. Using this approximate set T_L in conjunction with equations (5), (4) and (1), one gets the approximate inverse potential denoted by $q_L(x)$.

4. Examples

In this section, we first apply the simplified solution (17) to calculate an effective potential related to phase shifts derived from bosonic collisions. Then, two exploratory calculations follow demonstrating applicability of different approximations introduced in the preceding section.

4.1. Effective ^{87}Rb+^{87}Rb potential at $E = 303\ \mu K$

In this subsection we recalculate one result of [7] where effective Rb–Rb inter-atomic potentials have been derived from ultracold Bose-gas collision data.

The inverse calculation is performed by using equation (17) at $E = 303\ \mu$K. The corresponding phase shifts δ_l^{orig} measured (in rad) are as follows: -1.287, 1.635, 0.005 related to partial waves with $l = 0, 2, 4$, respectively.

The resulting inverse potential $V_{CT}(r)$ is plotted in figure 1. To control the procedure we recalculated the phase shifts from the inverse potential and the difference $\Delta^{CT} = \left|\delta_l^{\text{orig}} - \delta_l^{CT}\right|$ between the phase shifts is as follows: 0.014, 0.042, 0.004 for $l = 0, 2, 4$, respectively.

The result shown in figure 1 is identical to that obtained by [7] using the generic CT procedure. Indeed we also calculated the inverse potential by using equations (9), and obtained the same result as depicted in figure 1. In other words, the equivalence of equations (17) and (9) in the case of even l's has also been numerically verified.

4.2. Gauss potential

As a next example, we explore the applicability of different approximations introduced in subsection 3.2. For the analysis we use the prescribed potential of Gauss form:

$$V_G(r) = -2\exp(-5r^2) \tag{25}$$

where both distance r and energy are measured in atomic units (au).

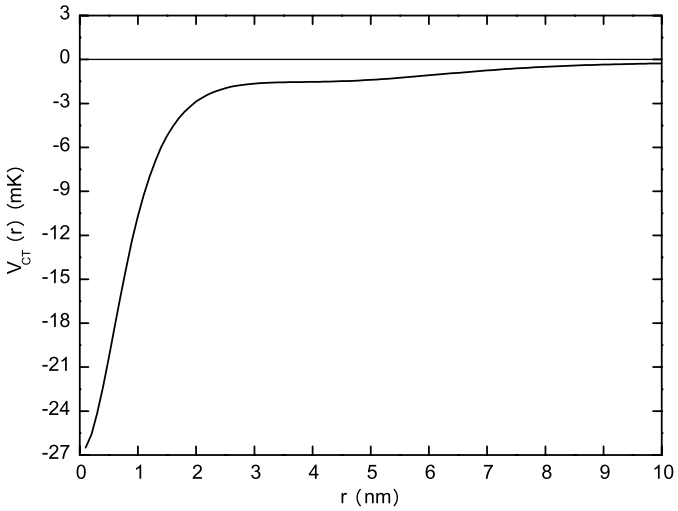

Figure 1. Effective ^{87}Rb+^{87}Rb inverse potential V_{CT} (in mK) as a function of the radial distance r (in nm) at cm energy $E = 303$ μK.

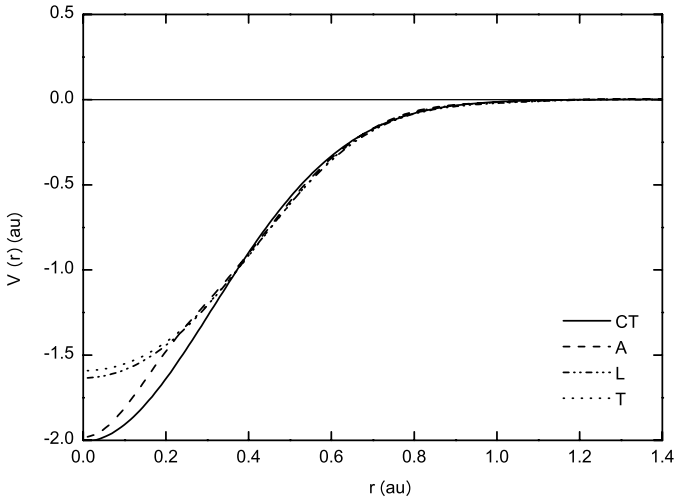

Figure 2. Inverse potentials $V(r)$ obtained from input phase shifts δ_l^{orig} (see table 1) as a function of the radial distance r at energy $E = 18$ ($k = 6$) au. Curves obtained by the CT and approximate methods are labeled according to the procedures discussed in the text.

Potential (25) provides a set of input phase shifts δ_l^{orig} listed in table 1 at scattering energy of $E = 18$ au ($k = 6$ au). Using this input set one calculates the CT inverse potential $V_{\mathrm{CT}}(r) = Eq_{\mathrm{CT}}(kr)$ which is depicted in figure 2. Note that V_{CT} is the same as V_{G}, within the width of line, therefore the plot of V_{G} is omitted in figure 2.

The different approximations V_{A}, V_{L}, and V_{T} can thus be compared to V_{CT} which may be taken to be exact. In figure 2 we see that, for this particular example, the potential $V_{\mathrm{A}}(r) = Eq_{\mathrm{A}}(kr)$ is a good approximation to the original one in view of its initial behavior

Table 1. Original phase shifts δ_l^{orig} produced by the Gauss potential (25) at scattering energy $E = 18$ au. Shifted angular momenta L corresponding to solution of equations (9), (17) or (18), and (24).

l	δ_l^{orig}	(9)	(17) or (18)	(24)
0	0.1294	−0.0893	−0.0809	−0.0824
1	0.0964	0.9392	0.9391	0.9386
2	0.0535	1.9676	1.9666	1.9659
3	0.0232	2.9865	2.9861	2.9852
4	0.0082	3.9955	3.9954	3.9948
5	0.0025	4.9989	4.9989	4.9984
6	0.0006	5.9999	5.9999	5.9996
7	0.0001	7.0001	7.0001	6.9999
8	0.0000	8.0002	8.0002	8.0000
9	0.0000	9.0001	9.0001	9.0000
10	0.0000	10.0001	10.0001	10.0000

(depth) and asymptotical property (range). Recall that approximation V_{A} is obtained by simply adding inversion approximations V_{e} and V_{o} derived by inverting the separate sets S_{e} and S_{o} of phase shifts belonging, respectively, to the sets of even and odd angular momenta l's. Approximation V_{T} is obtained by unifying the calculated sets T_{e} and T_{o} of shifted angular momenta, L's, listed in table 1 under heading (17) or (18). Finally, the approximation V_{L} has been obtained by simply using the one-term approximate values L_{a}, equation (24), which are also listed in table 1 under heading (24). It is interesting to see in figure 2 that this (totally analytical) version, the approximation V_{L}, provides a somewhat better result than that of the more involved approximation V_{T}.

4.3. $n + {}^{12}C$ scattering at $E = 10$ MeV

Our last example is taken from nuclear physics. Chen and Thornow [8] have derived 88 sets of complex-valued phase shifts from a comprehensive analysis of n scattering by ^{12}C target nucleus in the scattering energy region 7 MeV $\leqslant E_{\text{n}}^{\text{lab}} \leqslant$ 24 MeV. Because of the spin–orbit coupling, each partial wave provides two phase shifts, δ_l^+ and δ_l^-. In case of weak spin–orbit coupling the combined phase shifts $\delta_l = [(l+1)\delta_l^+ + l\delta_l^-]/(2l+1)$ are characteristic of the underlying central potential [9].

One set of such combined phase shifts, valid to the neutron scattering by ^{12}C at the energy of $E_{\text{n}}^{\text{lab}} = 10$ MeV is listed in table 2. Here, δ_l^{orig} denotes the real part of the combined phase shifts, Re δ_l, and η_l^{orig} stands for the elasticity, $\eta_l^{\text{orig}} = |\exp(2\text{i}\delta_l)|$. The results of the various inversion procedures are shown in figure 3.

In order to draw the conclusion about the applicability of the different approximate inverse procedures A, T and L, one can compare the corresponding potential curves to that obtained by the CT method which is assumed to be the best. (There is no model potential in this case.) Indeed, if one calculates back the phase shifts and elasticities that the different inverse potentials V_{CT}, V_{A}, V_{T} and V_{L} provide, we see that the reproduction is the best for the CT potential. All these facts can be studied in table 3 where we list the differences Δ_l and Ξ_l between the original data δ_l^{orig} and η_l^{orig} and the recalculated ones provided by the inverse potentials of the respective methods. The potential curves V_{CT}, V_{A}, V_{T} and V_{L} are drawn in figure 3 and we see that the curves obtained by the methods A, T and L approximate well the inverse potential V_{CT} considered to be the reference. Because the numerical reproduction is

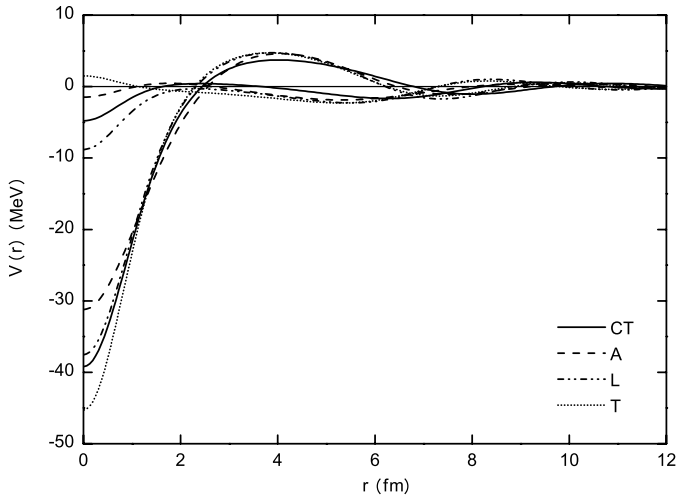

Figure 3. Inverse potentials $V(r)$ obtained from input phase shifts δ_l^{orig} and elasticities η_l^{orig} (see table 2) as a function of the radial distance r at energy $E_{\text{n}}^{\text{lab}} = 10$ MeV ($E_{\text{n}}^{\text{c.m}} = 9.23$ MeV, $k = 0.638$ fm^{-1}). Curves obtained by the CT and approximate methods are labeled according to the procedures discussed in the text. The imaginary part of the potentials concentrates around the abscissa, meaning a weak absorption.

Table 2. Original phase shifts δ_l^{orig} and elasticities η_l^{orig} taken from the phase shift data of [8] for n scattering by ^{12}C nucleus at the energy $E_{\text{n}}^{\text{lab}} = 10$ MeV. Real (left column) and imaginary (right column) parts of the shifted angular momenta L listed under headings CT, A and L, respectively, correspond to solutions of equations (9), (17) or (18), and (24).

l	δ_l^{orig}	η_l^{orig}	CT	CT	A	A	L	L
0	0.827	0.580	−0.581	−0.085	−0.583	−0.076	−0.527	−0.173
1	−0.562	0.723	1.226	0.001	1.360	−0.122	1.358	−0.103
2	−0.365	0.526	2.349	−0.192	2.259	−0.203	2.232	−0.205
3	0.057	0.846	2.981	−0.073	3.011	−0.084	2.964	−0.053
4	0.021	0.959	4.010	−0.062	4.001	−0.050	3.987	−0.013

Table 3. Differences Δ_l and Ξ_l between the original data and the calculated ones provided by the inverse potentials (shown in figure 3) obtained by the different inverse methods CT, A, T and L.

l	Δ_l^{CT}	Ξ_l^{CT}	Δ_l^{A}	Ξ_l^{A}	Δ_l^{L}	Ξ_l^{L}	Δ_l^{T}	Ξ_l^{T}
0	0.014	0.004	0.027	0.054	0.027	0.111	0.057	0.023
1	0.015	0.022	0.085	0.085	0.120	0.276	0.121	0.315
2	0.040	0.054	0.122	0.093	0.158	0.099	0.127	0.103
3	0.029	0.071	0.025	0.060	0.070	0.146	0.009	0.119
4	0.036	0.056	0.036	0.013	0.000	0.001	0.018	0.060

also surprisingly good we may conclude that the proposed approximations to the CT method can be used for a global orientation about the nature of the underlying interaction, if input phase shifts are known from another source and the solution of the generic equation (9) is not possible.

5. Conclusion

By observing the simple analytical property (13) of the asymptotic expansion functions $A_L^a(x)$ of the transformation kernel $K(x, y)$ involved in the Cox–Thompson (CT) inverse scattering method at fixed energy, simplified solutions of the CT method have been derived (see equations (17) and (18)) applicable to certain special scattering of identical bosons (fermions) when only even (odd) partial waves are arising (and thus measurable).

The new formulae (17) and (18) obtained are easier to solve by usual nonlinear solvers (such as that based, e.g., on the Newton–Raphson procedure [5]). This is because the new formulae do not involve the inverse of a matrix containing the unknowns themselves; they 'only' contain product and tangent (cotangent) operations. Therefore the range of applicability of the new formulae is wider than that of the generic equation (9). However, identical bosonic (fermionic) collision experiments are rare in practice. Therefore we have also developed several approximations, based on the new formulae (17) and (18), classified according to the level in which they have been introduced. By taking examples from the atomic collision experiment, potential scattering and nuclear physics, we have demonstrated the wide applicability of the new equations which make the solution of the CT inverse scattering method at fixed energy easier.

Acknowledgments

One of us (TP) wishes to express his gratitude to Professor Werner Scheid for kind hospitality and reading the manuscript. This work was supported by the Hungarian Scientific Research Fund, under contracts OTKA-T47035, T61311, IN67371 and the MTA-DFG grant (436 UNG 113/158).

References

[1] Cox J R and Thompson K W 1970 *J. Math. Phys.* **3** 805, 815
[2] Apagyi B, Harman Z and Scheid W 2003 *J. Phys. A: Math. Gen.* **36** 4815
[3] Ramm A G 1999 *Commun. Math. Phys.* **207** 231
[4] Horvath M 2006 *Trans Am Math Soc* **358** 5161
[5] Press W H, Teukolsky S A, Vetterling W T and Flannery B P 1992 *Numerical Recipes* (Cambridge: Cambridge University Press)
[6] Melchert O, Scheid W and Apagyi B 2006 *J. Phys. G: Nucl. Part. Phys.* **32** 849
[7] Schumayer D, Melchert O, Scheid W and Apagyi B 2008 *J. Phys. B: At. Mol. Phys.* **41** 035302
[8] Chen Z P and Tornow W 2005 *J. Phys. G: Nucl. Part. Phys.* **31** 1249
[9] Leeb H, Huber H and Fiedeldey H 1995 *Phys. Lett.* B **344** 18