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The stability of inverse scattering with fixed energy*

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Abstract

We consider a three-dimensional inverse scattering problem with fixed energy and with a spherically symmetrical, compactly supported potential. The resulting one-dimensional radial Schrödinger operator defines the sequence of phase shifts. We give some estimates of the potential perturbation by the perturbation of the phase shifts. More precisely, an exact estimate is given for an integral norm of the potential perturbation by the forward differences of the normalized perturbation of phase shifts. Another upper bound is provided if only the first few phase shifts are available with some error.

1. Introduction

The inverse scattering problem investigated in this paper is defined in the following way. Consider the equation

$$\varphi_n''(r) - \frac{n(n+1)}{r^2}\varphi_n(r) - q(r)\varphi_n(r) + k^2\varphi_n(r) = 0 \qquad r \geqslant 0 \tag{1.1}$$

with fixed energy $k^2 = 1$. This comes from three-dimensional quantum inverse scattering with a spherically symmetrical potential $V(x) = q(r), r = |x|$. Let $0 < a < \infty, \delta > 0$ and suppose that the real-valued potential $q(r)$ vanishes for $r > a$ and $r^{1-\delta}q(r) \in L_1(0, a)$. It is known (see [3]) that there exists a unique solution of (1.1) with

$$\varphi_n(r) = \frac{r^{n+1}}{(2n+1)!!}(1 + \mathbf{o}(1)) \qquad r \to 0+ \tag{1.2}$$

and

$$\varphi_n(r) = |F_n| \sin(r - n\pi/2 + \delta_n) + \mathbf{o}(1) \qquad r \to +\infty. \tag{1.3}$$

The quantities δ_n are called *phase shifts*.

The inverse scattering problem investigated here consists of the recovery of the potential q from the phase shifts δ_n.

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Recall the notion of forward differences

$$\Delta\mu_n = \mu_{n+1} - \mu_n, \qquad \Delta^k \mu_n = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \mu_{n+j}. \tag{1.4}$$

Introduce the quantities

$$\mu_n = a^{-2n-1-\delta}[(2n+1)!!]^2 |F_n F_n^*| \sin(\delta_n^* - \delta_n). \tag{1.5}$$

The value μ_n can be considered as a kind of relative error of the nth phase shift; it is used in order to eliminate the extremely rapid decay of the δ_n as $n \to \infty$. This is justified in remark 1.

The following stability result gives the exact order of the potential perturbation in terms of the perturbation of phase shifts. The potential perturbation is measured in an integral norm, while the perturbation of δ_n appears in the quantity μ_n.

Theorem 1.1. *Let $\delta > 0$, $q(r) = 0$ for $r > a$, $r^{1-\delta}q(r) \in L_1(0, a)$ and analogously for q^*. Let δ_n, δ_n^* be the corresponding phase shifts. Suppose that $\|r^{1-\delta}q(r)\|_1 \leqslant D$, $\|r^{1-\delta}q^*(r)\|_1 \leqslant D$. Then there are positive constants $c_1(D,\delta)$, $c_2(D,\delta)$ such that*

$$\begin{aligned} c_1(D,\delta) \int_0^a |q^*(r) - q(r)| r^{1-\delta}\,\mathrm{d}r &\leqslant \sup_n \sum_{k=0}^{n} \sum_{j=0}^{\infty} \binom{j + (\delta-1)/2}{j} \binom{n}{k} |\Delta^{n-k+j}\mu_k| \\ &\leqslant c_2(D,\delta) \int_0^a |q^*(r) - q(r)| r^{1-\delta}\,\mathrm{d}r. \end{aligned} \tag{1.6}$$

Remark 1. We know from Ramm [17] that if q has a constant nonzero sign in a small segment $(a - \varepsilon, a)$, then

$$\lim_{n\to\infty} \frac{2n+1}{e} |\delta_n|^{1/(2n)} = a, \tag{1.7}$$

i.e., $\delta_n = \left(\frac{ae}{2n}\right)^{2n} \cdot \alpha_n$ with a factor α_n of less than exponential growth or decay at infinity. This formula helps us to estimate the growth order of μ_n from (1.5) as follows. We easily get from the Stirling formula that $(2n+1)!!$ has the order $\left(\frac{2n}{e}\right)^{n+1}$. Next consider $|F_n|$. From [3], (1.5.17) and (1.1.24) we know that

$$|F_n| = F_n\,\mathrm{e}^{\mathrm{i}\delta_n}, \qquad F_n = 1 + \int_0^a w_n(r) q(r) \varphi_n(r)\,\mathrm{d}r \tag{1.8}$$

with

$$w_n(r) = i(-1)^n \left(\frac{\pi r}{2}\right)^{1/2} H_{n+1/2}(r). \tag{1.9}$$

From (2.5), (2.6) given below it follows that

$$|\varphi_n(r)| \leqslant \frac{c}{(2n+1)!!} r^{n+1} \qquad r \leqslant a \tag{1.10}$$

with a constant $c = c(a)$ uniform in n. We can check that

$$|H_{n+1/2}(r)| \leqslant c r^{-n-1/2}(2n+1)!! \qquad r \leqslant a. \tag{1.11}$$

Indeed, the Taylor series expansion of $z^{-\nu}J_\nu(z)$ implies that

$$|J_\nu(z)| \leqslant \frac{c(a)}{\min_{n\geqslant 1}|\nu + n|} \frac{(|z|/2)^\nu}{|\Gamma(\nu+1)|} \qquad |z| \leqslant a \tag{1.12}$$

and then

$$|H_{n+1/2}(r)| = |J_{n+1/2}(r) + i(-1)^{n+1}J_{-n-1/2}(r)| \leqslant c\frac{(r/2)^{-n-1/2}}{|\Gamma(-n-1/2)|}$$
$$\leqslant c(r/2)^{-n-1/2}\frac{(2n+1)!!}{2^n} \leqslant cr^{-n-1/2}(2n+1)!!.$$

Thus $|w_n(r)| \leqslant cr^{-n}(2n+1)!!$ and then by (1.13) and (1.15), $|F_n| \leqslant 1 + c\int_0^a r|q(r)|\,\mathrm{d}r$ is bounded. This means that the quantity

$$a^{-2n}[(2n+1)!!]^2|F_nF_n^*|\sin\delta_n$$

can grow only subexponentially; it can be considered as a 'normalized' value of δ_n. Thus we have an interpretation of μ_n from (1.5) as a quantity close to the relative error of δ_n^* (with respect to δ_n); theorem 1.1 gives an exact estimate of an integral norm of $q^* - q$ in terms of the relative errors of the shifts δ_n.

Remark 2. If the shifts δ_n are taken from measurements, it may cause some uncontrolled small errors in the quantities μ_n. For example, if $|\mu_n| < \varepsilon\,\forall n$, then $|\Delta^n\mu_k| < 2^n\varepsilon$, hence substituting measured data into (1.6) can be highly unstable. On the other hand, all phase shifts δ_n with large index n are measured to be zero, i.e., to be the phase shifts of the zero potential. If n_0 is the largest index with $\mu_{n_0} \neq 0$ then the inner sum in (1.6) is infinite. Indeed, for large j and for $k \leqslant k_0$, $\Delta^{n-k+j}\mu_k$ is of order j^{k_0-k} and $\binom{j+(\delta-1)/2}{j}$ is of order $j^{(\delta-1)/2} \geqslant j^{-1/2}$. So both sides of (1.6) are infinite showing that the situation is impossible: if all but a finite number of phase shifts are identical, i.e. $\delta_n^* = \delta_n$ for all $n > N$, then in fact all shifts are the same (and $q^* = q$). This statement (and more) has been previously proven by Ramm [18].

To overcome the above-mentioned stability problems we apply further conditions on the potentials.

Definition. *Let $D > 0$. The set M_D consists of all real-valued functions $q(r)$ with support in* $[0, a]$ *and such that*

$$\int_0^a |\mathrm{d}(r^2q(r))| \leqslant D \tag{1.13}$$

and

$$r|q(r)| \leqslant D. \tag{1.14}$$

In other words, $r^2q(r)$ has a total variation bounded by D and $r|q(r)|$ is bounded by D. This is a rather wide class for large D; it includes, for example, the functions, having a bounded continuous derivative except for finitely many points where jumps are allowed.

Theorem 1.2. *Let $q, q^* \in M_D$ and suppose that*

$$|\mu_n| < \varepsilon(<1) \quad \forall n. \tag{1.15}$$

Then

$$\|r^2(q^*(r) - q(r))\|_{L_2(0,a)} \leqslant c\left(\log\frac{1}{\varepsilon}\right)^{-1/2} \tag{1.16}$$

with a constant $c = c(a, D)$ independent of q, q^* and ε.

A similar former result is due to Ramm [19].

Since in real experiments only the first few phase shifts can be measured as nonzero quantities, we formulate the following variant of the above theorem:

Theorem 1.3. *If* $q, q^* \in M_D$ *and*

$$|\mu_n| < \varepsilon \qquad \forall n \leqslant N, \tag{1.17}$$

then

$$\|r^2(q^*(r) - q(r))\|_{L_2(0,a)} \leqslant c\left(\frac{1}{\sqrt{N}} + \left(\log\frac{1}{\varepsilon}\right)^{-1/2}\right). \tag{1.18}$$

Remark 3. The considerations in remark 1 imply that

$$|\mu_n| \leqslant c(a)\left(\frac{2n}{ae}\right)^{2n+2}|\sin(\delta_n^* - \delta_n)|. \tag{1.19}$$

Thus in (1.15) and (1.17), $|\mu_n| < \varepsilon$ can be substituted by

$$\left(\frac{2n}{ae}\right)^{2n+2}|\sin(\delta_n^* - \delta_n)| < \varepsilon. \tag{1.20}$$

This is a bit stronger but explicit condition, not containing $|F_n F_n^*|$. To check it we only need an error estimate for δ_n, taken, e.g., from some measurements.

1.1. Historical remarks and perspectives

Although the reconstruction of a quantummechanical system from fixed-energy scattering data is an important and widely investigated area in mathematical physics, not very much is known concerning stability issues. The first difficulty is the lack of uniqueness: if we have no *a priori* information on the potential, the inverse problem has infinitely many solutions. For example, there are nontrivial potentials, oscillating and of order $r^{-3/2}$ at infinity for which all the phase shifts vanish; see, e.g., [12, chapter 20.4].

Several methods are known for the solution at fixed energy, such as the WKB method [9], direct inversion methods as the Bargmann inversion method [5], the finite-difference method [7], the Newton–Sabatier [3] and the Cox–Thompson method [4]. If we seek for a solution of the inverse scattering problem only by a fixed method using some ansatz (such as Newton–Sabatier or Cox–Thompson) or in a very restricted function class, then several uniqueness theorems can be proved; see, e.g., [3]. However, these considerations allow the existence of another potential, not accessible, e.g., by the Newton–Sabatier construction, producing the same phase shifts. Note that the N–S method itself can be used to construct a set of these 'equivalent potentials' if we allow an $r^{-3/2}$ tail [3, pp 202–8]: why this nonuniqueness appears was recently revisited in a paper by Sabatier [20]. A general uniqueness result is obtained by Ramm [15]: if $q \in L_2(\mathbf{R}^3)$ is of compact support (and not necessarily spherically symmetrical), then it can be uniquely identified by the scattering amplitude. As a matter of fact, in the spherically symmetric case, global theorems giving sets of potentials which are uniquely identified by the scattering amplitude were produced very early by Loeffel [10] (see also [3, section 8.3]); one of these sets is that of compactly supported potentials. A uniqueness result for bounded potentials having an arbitrarily small exponential decay at infinity is due to Novikov in 1988; see [13]. The proofs are based upon the Dirichlet-to-Neumann map, a key notion in many kinds of inverse problems. Later on, it turned out that in the symmetrical case a relatively small part of all phase shifts is still enough to ensure uniqueness; see [18]. As we have already mentioned, the vanishing or the strong decay of the potential at large distances is not a real restriction from a practical point of view since the measurement errors will necessarily destroy any information about the tail of the potential (unless we allow potentials that are singular, e.g. with strong repulsive singularities, and/or decrease more slowly than r^{-2} at infinity).

Many reconstruction procedures are based on a Gelfand–Levitan–Marchenko-type integral equation, whose input kernel is obtained by solving a linear or nonlinear system of equations; see, e.g., [3]. Since the stability of all these steps are well known, most of such procedures show quite good stability properties in the class of potentials, reachable by that method. On the other hand, we find that the inverse problem itself has very weak stability properties. For examples of very different (step function) potentials producing almost the same phase shifts, see [14]. The logarithmic bounds given above in theorems 1.2 and 1.3 also show very poor stability since $\log r$ tends to infinity extremely slowly. There is no evidence that these bounds are best possible; however, many independent investigations conclude in similar logarithmic estimates. Maybe, the common root is the reconstruction of the potential from the Dirichlet-to-Neumann map which requires logarithmic stability bounds. This idea appears in Alessandrini [1] in studying the stability of the inverse conductivity problem. Similar results are obtained by Stefanov [21] for the inverse scattering with fixed energy; roughly speaking, he proved that building up the Dirichlet-to-Neumann operator from the spectral amplitude is strongly stable, while reconstructing the potential from the D-to-N map has a weak (logarithmic) stability. All these results are local, i.e., are true only for small perturbations of a fixed potential. A global stability result is given in [16]: if we consider potentials in $L_2(\mathbf{R}^3) \cap L_\infty(\mathbf{R}^3)$ with compact support and if the perturbation of the scattering amplitude is bounded by δ, then the 3D Fourier transform of the potential perturbation is bounded by $\log|\log\delta|/|\log\delta|$. In theorem 1.3 a direct estimate of the potential perturbation is given, based on the first few phase shifts.

It would be interesting to obtain better 'method-independent' stability, e.g., by some smoothness of the potential and systematic study of the stability of the most popular methods.

2. Proof of theorem 1.1

We start with some results on moment problems.

Lemma 2.1. *Let* $\alpha > 0$, $g \in BV[0,1]$, $\mu_n = \int_0^1 t^{n+\alpha}\,\mathrm{d}g(t)$, $\nu_n = \int_0^1 t^n\,\mathrm{d}g(t)$. *Then*

$$\mu_n = \sum_{j=0}^{\infty} \binom{\alpha}{j} \Delta^j \nu_n \qquad \textit{and} \qquad \nu_n = \sum_{j=0}^{\infty} \binom{-\alpha}{j} \Delta^j \mu_n, \tag{2.1}$$

$$\int_0^1 |\mathrm{d}g| = \sup_n \sum_{k=0}^{n} \sum_{j=0}^{\infty} \binom{j+\alpha-1}{j} \binom{n}{k} \left|\Delta^{n-k+j} \mu_k\right|. \tag{2.2}$$

Proof. Recall first that

$$\Delta^n \nu_k = \int_0^1 t^k (t-1)^n\,\mathrm{d}g(t).$$

Since the series $t^\alpha = \sum_{j=0}^{\infty} \binom{\alpha}{j} (t-1)^j$ is uniformly convergent in [0, 1], we obtain

$$\mu_n = \sum_{j=0}^{\infty} \binom{\alpha}{j} \int_0^1 t^n (t-1)^j\,\mathrm{d}g(t) = \sum_{j=0}^{\infty} \binom{\alpha}{j} \Delta^j \nu_n.$$

Conversely, from the monotone and bounded convergence of $t^n = \sum_{j=0}^{\infty} \binom{-\alpha}{j} t^{n+\alpha}(t-1)^j$ it follows that

$$\nu_n = \int_0^1 t^{n+\alpha} t^{-\alpha}\, \mathrm{d}g(t) = \sum_{j=0}^{\infty} \binom{-\alpha}{j} \int_0^1 t^{n+\alpha}(t-1)^j\, \mathrm{d}g(t) = \sum_{j=0}^{\infty} \binom{-\alpha}{j} \Delta^j \mu_n.$$

Finally, recall a classical result of Hausdorff [6]:

$$\begin{aligned}\int_0^1 |\mathrm{d}g| &= \sup_n \sum_{k=0}^{n} \binom{n}{k} |\Delta^{n-k}\nu_k| \\ &\leqslant \sup_n \sum_{k=0}^{n}\sum_{j=0}^{\infty} \binom{j+\alpha-1}{j}\binom{n}{k} |\Delta^{n-k+j}\mu_k|\end{aligned}$$

since $\binom{-\alpha}{j} = (-1)^j \binom{\alpha+j-1}{j}$. On the other hand, from

$$\Delta^{n-k+j}\mu_k = \int_0^1 t^{k+\alpha}(t-1)^{n-k+j}\, \mathrm{d}g(t)$$

it follows that

$$\begin{aligned}\sum_{k=0}^{n}\sum_{j=0}^{\infty} \binom{j+\alpha-1}{j}\binom{n}{k} \left|\Delta^{n-k+j}\mu_k\right| &\leqslant \int_0^1 \sum_{k=0}^{n} \binom{n}{k} t^{k+\alpha}(1-t)^{n-k} \sum_{j=0}^{\infty} \binom{-\alpha}{j} (t-1)^j |\mathrm{d}g(t)| \\ &= \int_0^1 \sum_{k=0}^{n} \binom{n}{k} t^{k+\alpha}(1-t)^{n-k} t^{-\alpha} |\mathrm{d}g(t)| = \int_0^1 |\mathrm{d}g(t)|\end{aligned}$$

which proves (2.2).

□

Lemma 2.2. *Denote φ_n^* be the solution of (1.1), (1.2) corresponding to q^*. Then*

$$\varphi_n(r) = \sqrt{\frac{\pi r}{2}} |F_n| \cdot [\cos\delta_n \cdot J_{n+1/2}(r) - \sin\delta_n \cdot Y_{n+1/2}(r)] \qquad r \geqslant a, \tag{2.3}$$

$$\varphi_n'(a)\varphi_n^*(a) - \varphi_n(a)\varphi^{*\prime}(a) = |F_n F_n^*| \sin(\delta_n^* - \delta_n). \tag{2.4}$$

Here J_s and Y_s are the Bessel and Neumann functions; see [2].

Proof. For $r > a$, we have the free operator (1.1), i.e., $q = 0$. Consequently, it can be expressed linearly by the functions $r^{1/2} J_{n+1/2}(r)$ and $r^{1/2} Y_{n+1/2}(r)$, so

$$\varphi_n(r) = c_n r^{1/2} [\cos\alpha_n \cdot J_{n+1/2}(r) + \sin\alpha_n \cdot Y_{n+1/2}(r)].$$

Taking into account that

$$J_{n+1/2}(r) = \sqrt{\frac{2}{\pi r}} \sin(r - n\pi/2) + \mathbf{o}(1),$$

$$Y_{n+1/2}(r) = -\sqrt{\frac{2}{\pi r}} \cos(r - n\pi/2) + \mathbf{o}(1),$$

we obtain

$$\varphi_n(r) = c_n \sqrt{\frac{2}{\pi}} \sin(r - n\pi/2 - \alpha_n) + \mathbf{o}(1) \qquad r \to +\infty.$$

Comparing it with (1.3) gives (2.3). Now

$$\begin{aligned}\varphi_n'(a)\varphi_n^{*}(a)-\varphi_n(a)\varphi_n^{*\prime}(a)&=\frac{\pi}{2}|F_nF_n^{*}|\cdot\big[(\cos\delta_n\sqrt{r}J_{n+1/2}(r)-\sin\delta_n\sqrt{r}Y_{n+1/2}(r))'\\&\quad\cdot(\cos\delta_n^{*}\sqrt{r}J_{n+1/2}(r)-\sin\delta_n^{*}\sqrt{r}Y_{n+1/2}(r))\\&\quad-(\cos\delta_n\sqrt{r}J_{n+1/2}(r)-\sin\delta_n\sqrt{r}Y_{n+1/2}(r))\\&\quad\cdot(\cos\delta_n^{*}\sqrt{r}J_{n+1/2}(r)-\sin\delta_n^{*}\sqrt{r}Y_{n+1/2}(r))'\big]_{r=a}\\&=\frac{\pi}{2}|F_nF_n^{*}|(\cos\delta_n\sin\delta_n^{*}-\cos\delta_n^{*}\sin\delta_n)\\&\quad\cdot\big[\sqrt{r}J_{n+1/2}(r)(\sqrt{r}Y_{n+1/2}(r))'-(\sqrt{r}J_{n+1/2}(r))'\sqrt{r}Y_{n+1/2}(r)\big]_{r=a}\\&=\frac{\pi}{2}|F_nF_n^{*}|\sin(\delta_n^{*}-\delta_n)a[J_{n+1/2}Y_{n+1/2}'-J_{n+1/2}'Y_{n+1/2}]_{r=a}.\end{aligned}$$

The Wronskian here is known to be $\frac{2}{\pi a}$ (see [2]) and this proves (2.4). □

In the next step, we investigate the behaviour of φ_n for $r<a$. Formula (2.5) given below is essentially a transformed form of the Povzner–Levitan representation. Some smoothness properties of the kernel are shown in [19].

Lemma 2.3. *Suppose that $rq(r)\ln\frac{a}{r}\in L_1(0,a)$. Then*

$$(2n+1)!!\varphi_n(r)=r^{n+1}+\int_0^r H(r,t)r^{1/2}t^{n-1/2}\,\mathrm{d}t\qquad 0\leqslant r\leqslant a\tag{2.5}$$

where the continuous kernel H satisfies

$$|H(r,t)|\leqslant\frac{1}{2}\int_0^{\sqrt{rt}}\tau|q(\tau)-1|\,\mathrm{d}\tau\cdot\mathrm{e}^{\int_0^a\tau|q(\tau)-1|\ln\frac{a}{\tau}\mathrm{d}\tau}.\tag{2.6}$$

Proof. Introduce the function $y_n(x)$ by

$$(2n+1)!!\varphi_n(r)=a^{n+1/2}r^{1/2}y_n\left(\ln\frac{a}{r}\right).\tag{2.7}$$

As shown in Horváth [8],

$$-y_n''+Q(x)y_n=\lambda_n^2y_n\qquad 0\leqslant x<\infty,\quad \lambda_n=\mathrm{i}(n+1/2)\tag{2.8}$$

where $Q(x)$ is defined by

$$q(r)=\frac{1}{r^2}Q\left(\ln\frac{a}{r}\right)+1\qquad 0<r\leqslant a.\tag{2.9}$$

Now we have

$$\int_0^\infty x|Q(x)|\,\mathrm{d}x=\int_0^a\ln\frac{a}{r}\left|Q\left(\ln\frac{a}{r}\right)\right|\frac{\mathrm{d}r}{r}\tag{2.10}$$

$$=\int_0^a r|q(r)-1|\ln\frac{a}{r}\mathrm{d}r<\infty.\tag{2.11}$$

Thus from (2.7) and (1.2) it follows that $y_n(x)=\mathrm{e}^{-(n+1/2)x}(1+\mathbf{o}(1))x\to+\infty$. Consequently [8],

$$y_n(x)=\mathrm{e}^{-(n+1/2)x}+\int_x^\infty H_1(x,t)\,\mathrm{e}^{-(n+1/2)t}\,\mathrm{d}t\qquad 0<x$$

with a continuous kernel H_1 with

$$|H_1(x,t)|\leqslant 1/2\int_{\frac{x+t}{2}}^\infty|Q|\,\mathrm{e}^{\int_0^\infty s|Q(s)|\,\mathrm{d}s}.$$

Substituting here $x = \ln\frac{a}{r}$, $t = \ln\frac{a}{\varrho}$, we easily obtain the statement of lemma 2.3 if

$$H(r,t) = H_1\left(\ln\frac{a}{r}, \ln\frac{a}{t}\right).$$

□

Lemma 2.4. *Denote* $C_\delta = \{h : \|h\|_\delta = \int_0^a r^{1-\delta}|h(r)|\,\mathrm{d}r < \infty\}$. *If* $h \in C_\delta$ *then*

$$[(2n+1)!!]^2 \int_0^a h\varphi_n\varphi_n^* = \int_0^a r^{2n+2}(B_{q^*}h)(r)\,\mathrm{d}r \tag{2.12}$$

where

$$B_{q^*}h(r) = h(r) + \frac{1}{r^2}\int_r^a 2\varrho h(\varrho)\cdot\left[(H+H^*)\left(\varrho, \frac{r^2}{\varrho}\right) + \int_{\frac{r^2}{\varrho}}^{\varrho} H(\varrho,t)H^*\left(\varrho, \frac{r^2}{t}\right)\mathrm{d}t\right]\mathrm{d}\varrho. \tag{2.13}$$

There exist positive constants $C_1(D,\delta)$, $C_2(D,\delta)$ *such that*

$$C_1(D,\delta)\|h\|_\delta \leqslant \|B_{q^*}h\|_\delta \leqslant C_2(D,\delta)\|h\|_\delta \qquad \forall\, h \in C_\delta, \quad \|q\|_\delta, \|q^*\|_\delta \leqslant D. \tag{2.14}$$

Proof.

$$\begin{aligned}[(2n+1)!!]^2 \int_0^a h\varphi_n\varphi_n^* &= \int_0^a h(r)r^{2n+2}\,\mathrm{d}r + \int_0^a h(r)r^{n+3/2}\int_0^r (H+H^*)(r,t)t^{n-1/2}\,\mathrm{d}t\,\mathrm{d}r \\ &\quad + \int_0^a h(r)r\int_0^r H(r,t)t^{n-1/2}\,\mathrm{d}t\int_0^r H^*(r,\tau)\tau^{n-1/2}\,\mathrm{d}\tau\,\mathrm{d}r \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Here

$$\begin{aligned} I_2 &= \int_0^a h(r)r^2\int_0^r (H+H^*)(r,t)(rt)^{n-1/2}\,\mathrm{d}t\,\mathrm{d}r \\ &= \int_0^a h(r)r^2\int_0^r (H+H^*)(r,\tau^2/r)2\frac{\tau^{2n}}{r}\,\mathrm{d}\tau\,\mathrm{d}r \\ &= \int_0^a \tau^{2n}\int_\tau^a rh(r)2(H+H^*)(r,\tau^2/r)\,\mathrm{d}r\,\mathrm{d}\tau, \\ I_3 &= \int_0^a rh(r)\int_0^r H(r,t)\int_0^r H^*(r,\tau)(t\tau)^{n-1/2}\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}r \\ &= \int_0^a rh(r)\int_0^r H(r,t)\int_0^{\sqrt{rt}} H^*(r,\varrho^2/t)2\varrho^{2n}\,\mathrm{d}\varrho\,\mathrm{d}t\,\mathrm{d}r \\ &= \int_0^a rh(r)\int_0^r 2\varrho^{2n}\int_{\varrho^2/r}^r H(r,t)H^*(r,\varrho^2/t)\,\mathrm{d}t\,\mathrm{d}\varrho\,\mathrm{d}r \\ &= \int_0^a \varrho^{2n}\int_\varrho^a rh(r)2\int_{\varrho^2/r}^r H(r,t)H^*(r,\varrho^2/t)\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\varrho. \end{aligned}$$

This verifies (2.13). The estimates (2.14) can be proved analogously as in Horváth [8]; we do not give the details. □

Proof of theorem 1.1. First observe that

$$\varphi_n'(a)\varphi_n^*(a) - \varphi_n(a)\varphi_n^{*\prime}(a) = -\int_0^a (q^*(r) - q(r))\varphi_n(r)\varphi_n^*(r)\,\mathrm{d}r. \tag{2.15}$$

Indeed,

$$\frac{d}{dr}(\varphi_n'\varphi_n^* - \varphi_n\varphi_n^{*\prime}) = \varphi_n''\varphi_n^* - \varphi_n\varphi_n^{*\prime\prime} = (q - q^*)\varphi_n\varphi_n^*.$$

Since φ_n' is of order r^n at 0 (see [3]), $\varphi_n'\varphi_n^*$ and $\varphi_n\varphi_n^{*\prime}$ tend to zero as $r \to 0+$ and this proves (2.15). Now putting together lemmas 2.2 and 2.4 gives that

$$[(2n+1)!!]^2|F_nF_n^*|\sin(\delta_n^*-\delta_n) = -[(2n+1)!!]^2\int_0^a (q^*(r)-q(r))\varphi_n(r)\varphi_n^*(r)\,\mathrm{d}r$$
$$= -\int_0^a r^{2n+1+\delta\cdot r^{1-\delta}}B_{q^*}(q^*-q)(r)\,\mathrm{d}r. \tag{2.16}$$

Denote

$$f(r) = r^{1-\delta}B_{q^*}(q^*-q)(r) \in L_1(0,a).$$

Then

$$\int_0^a r^{2n+1+\delta}f(r)\,\mathrm{d}r = a^{2n+1+\delta}\int_0^1 t^{n+\frac{\delta+1}{2}}f(a\sqrt{t})\frac{a}{2\sqrt{t}}\,\mathrm{d}t,$$

hence by lemma 2.1

$$\|B_{q^*}(q^*-q)\|_\delta = \int_0^a |f(r)|\,\mathrm{d}r = \int_0^1 f(a\sqrt{t})\frac{a}{2\sqrt{t}}\,\mathrm{d}t$$
$$= \sup_n\sum_{k=0}^n\sum_{j=0}^\infty \binom{j+(\delta-1)/2}{j}\binom{n}{k}\left|\Delta^{n-k+j}\mu_k\right|$$

where

$$\mu_n = -a^{-2n-1-\delta}[(2n+1)!!]^2|F_nF_n^*|\sin(\delta_n^*-\delta_n).$$

Taking (2.14) into account this finishes the proof of theorem 1.1 □

3. The proof of theorems 1.2 and 1.3

Denote

$$b(r,t) = \frac{2}{t}\left[(H+H^*)(t,r^2/t) + \int_{r^2/t}^t H(t,\tau)H^*(t,r^2/\tau)\,\mathrm{d}\tau\right]. \tag{3.1}$$

Using (2.13), formula (2.16) can be rewritten in the form

$$\mu_n = -a^{-2n-1-\delta}\int_0^a r^{2n}A[r^2(q^*(r)-q(r))]\,\mathrm{d}r \tag{3.2}$$

where

$$Ah(r) = h(r) + \int_r^a b(r,t)h(t)\,\mathrm{d}t. \tag{3.3}$$

In what follows c always denotes positive constants, depending only on a and D, possibly different in each occurrences.

Lemma 3.1. *Let $q^*, q \in M_D$, then $A: L_2(0,a)\to L_2(0,a)$ is isomorphism and $\|A^{-1}\| \leqslant c$.*

Proof. Let $A = I+K$, $Kh(r) = \int_r^a b(r,t)h(t)\,\mathrm{d}t$. Since $r|q(r)| \leqslant D$, $r|q^*(r)| \leqslant D$, we get in (2.6) that

$$|H(r,t)| \leqslant c\sqrt{rt}$$

and then

$$|b(r,t)| \leqslant \frac{c}{t}\left[r + \int_{r^2/t}^t \sqrt{t\tau}\sqrt{tr^2/\tau}\,\mathrm{d}\tau\right] \leqslant \frac{c}{t}[r+t^2r] \leqslant c. \tag{3.4}$$

We see by induction that

$$K^n h(r) = \int_r^a b_n(r,t)h(t)\,\mathrm{d}t \tag{3.5}$$

where $b_1(r,t) = b(r,t)$ and

$$b_{n+1}(r,t) = \int_r^t b(r,\tau)b_n(\tau,t)\,\mathrm{d}\tau. \tag{3.6}$$

Indeed,

$$\begin{aligned} K^{n+1}h(r) = K[K^n h](r) &= \int_r^a b(r,\tau)\int_\tau^a b_n(\tau,t)h(t)\,\mathrm{d}t\,\mathrm{d}\tau \\ &= \int_r^a h(t)\int_r^t b(r,\tau)b_n(\tau,t)\,\mathrm{d}\tau\,\mathrm{d}t. \end{aligned}$$

Another induction on n gives

$$|b_{n+1}(r,t)| \leqslant c^{n+1}\frac{(t-r)^n}{n!} \qquad n \geqslant 0 \tag{3.7}$$

with a constant $c = c(a, D)$ independent also of n. Indeed, (3.4) being the case $n = 0$, we get from (3.6) that

$$|b_{n+1}(r,t)| \leqslant \int_r^t c\cdot c^n\frac{(t-\tau)^{n-1}}{(n-1)!}\,\mathrm{d}\tau = c^{n+1}\frac{(t-r)^n}{n!}.$$

Consequently,

$$\begin{aligned} \|K^{n+1}h\|^2 &= \int_0^a \left|\int_r^a b_{n+1}(r,t)h(t)\,\mathrm{d}t\right|^2\mathrm{d}r \\ &\leqslant \int_0^a\int_r^a |b_{n+1}(r,t)|^2\,\mathrm{d}t\cdot\int_r^a |h|^2\,\mathrm{d}r \\ &\leqslant \int_0^a |h|^2\cdot\int_0^a\int_r^a |b_{n+1}(r,t)|^2\,\mathrm{d}t\,\mathrm{d}r \\ &\leqslant \|h\|^2\int_0^a\int_r^a c^{2n+2}\frac{(t-r)^{2n}}{(n!)^2}\,\mathrm{d}t\,\mathrm{d}r \\ &= \|h\|^2\int_0^a c^{2n+2}\frac{(a-r)^{2n+1}}{(2n+1)(n!)^2}\,\mathrm{d}r \\ &= \|h\|^2\frac{(ca)^{2n+2}}{(2n+2)(2n+1)(n!)^2} \leqslant \|h\|^2\frac{(ca)^{2n+2}}{[(n+1)!]^2} \end{aligned}$$

i.e.

$$\sum_0^\infty \|K^n\| \leqslant 1 + \sum_0^\infty \frac{(ca)^{n+1}}{(n+1)!} = \mathrm{e}^{ca}.$$

This means that the Neumann series $A^{-1} = I - K + K^2 - K^3 + \cdots$ is convergent in operator norm and $\|A^{-1}\| \leqslant \mathrm{e}^{ca}$. □

Define the function

$$G(z) = \int_0^a \cos rz\cdot A[r^2(q^*(r) - q(r))]\,\mathrm{d}r. \tag{3.8}$$

It is an even entire function. In terms of $G(z)$ formula (3.2) means that

$$-a^{2n+1+\delta}\mu_n = (-1)^n G^{(2n)}(0). \tag{3.9}$$

Lemma 3.2. *Let $q^*, q \in M_D$. Then*

$$|G(z)| \leqslant \frac{c}{1+|z|} \qquad z \in \mathbf{R} \tag{3.10}$$

with $c = c(a, D)$ independent of q, q^* and z.

Proof. We know that $r^2(q^*(r) - q(r))$ is bounded and the kernel $b(r, t)$ of K is also bounded by (3.4). Hence $A[r^2(q^*(r) - q(r))]$ and then $G(z)$ are bounded as well for bounded z. Thus it is enough to prove (3.10) for large values of $|z|$. This will be done by integrating by parts. Introduce the notation

$$h(r) = r^2(q^*(r) - q(r)),$$

then

$$\begin{aligned} G(z) = {} & \left[\frac{\sin rz}{z} Ah(r)\right]_{r=0}^{a} - \int_0^a \frac{\sin rz}{z}\, \mathrm{d}(h(r)) + \int_0^a \frac{\sin rz}{z} b(r, r)h(r)\, \mathrm{d}r \\ & - \int_0^a \frac{\sin rz}{z} \int_r^a \frac{\partial}{\partial r} b(r, t)h(t)\, \mathrm{d}t\, \mathrm{d}r \\ = {} & I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.11}$$

From the boundedness of Ah(r) we get at once that

$$|I_1| \leqslant \frac{c}{|z|}.$$

From (1.13) it follows that

$$|I_2| \leqslant \frac{1}{|z|} \int_0^a |\mathrm{d}(h)| \leqslant \frac{c}{|z|}.$$

Since $b(r, t)$ and $h(t)$ are bounded,

$$|I_3| \leqslant \frac{c}{|z|}.$$

To estimate I_4 we return to the proof of lemma 2.3. Denote

$$\sigma(x) = \int_x^\infty |Q|, \qquad \sigma_1(x) = \int_x^\infty \sigma.$$

It is proven in [11, chapter 3] that

$$\left|\frac{\partial}{\partial t} H_1(x, t) + \frac{1}{4} Q\left(\frac{x+t}{2}\right)\right| \leqslant \frac{1}{2}\sigma(x)\sigma\left(\frac{x+t}{2}\right) \mathrm{e}^{\sigma_1(0)} \tag{3.12}$$

and analogously for $\frac{\partial}{\partial x} H_1$. Using the identity

$$\sigma(\ln(a/\tau)) = \int_{\ln(a/\tau)}^\infty |Q| = \int_0^\tau r|q(r) - 1|\, \mathrm{d}r$$

we get from (3.12) that

$$\begin{aligned} \left|\frac{\partial}{\partial t} H(r, t)\right| & = |\partial_2 H_1(\ln(a/r), \ln(a/\tau))t^{-1}| \\ & \leqslant \frac{1}{4t}\left|Q\left(\ln \frac{a}{\sqrt{rt}}\right)\right| + \frac{c}{t} \int_0^r \varrho|q(\varrho) - 1|\, \mathrm{d}\varrho \cdot \int_0^{\sqrt{rt}} \varrho|q(\varrho) - 1|\, \mathrm{d}\varrho \\ & \leqslant \frac{1}{4t} rt|q(\sqrt{rt}) - 1| + \frac{c}{t} r\sqrt{rt} \leqslant c\sqrt{r/t}. \end{aligned}$$

Computing $\frac{\partial}{\partial r}b(r,t)$ from (3.1) gives

$$\frac{2}{t}\left|\partial_2(H+H^*)(t,r^2/t)\cdot\frac{2r}{t}\right| \leqslant \frac{c}{t}\sqrt{\frac{t}{r^2/t}}\cdot\frac{r}{t}\leqslant\frac{c}{t},$$
$$\frac{2}{t}\frac{2r}{t}|H(t,r^2/t)H^*(t,t)| \leqslant c\frac{r}{t^2}rt = c\frac{r^2}{t}$$

and

$$\frac{2}{t}\int_{r^2/t}^{t}|H(t,\tau)|\cdot|\partial_2 H^*(t,r^2/\tau)|\cdot\frac{2r}{\tau}\,\mathrm{d}\tau \leqslant \frac{c}{t}\int_{r^2/t}^{t}\sqrt{t\tau}\cdot\frac{\sqrt{t\tau}}{r}\cdot\frac{r}{\tau}\,\mathrm{d}\tau\leqslant ct.$$

Summing up, we have shown that

$$\left|\frac{\partial}{\partial r}b(r,t)\right|\leqslant\frac{c}{t}$$

and then

$$|I_4|\leqslant\frac{c}{|z|}\int_0^a\int_r^a\frac{1}{t}|h(t)|\,\mathrm{d}t\,\mathrm{d}r=\frac{c}{|z|}\int_0^a|h|\leqslant\frac{c}{|z|}.$$

Taking (3.11) into account, this finishes the proof. □

Proof of theorem 1.2. Suppose $|\mu_n| < \varepsilon$, then by (3.9), $|G^{(2n)}(0)| \leqslant ca^{2n}\varepsilon$. On the other hand, $G^{(2n+1)}(0) = 0$ since $G(z)$ is even. Now we have

$$|G(z)| = \left|\sum_{n=0}^{\infty}\frac{G^{(2n)}(0)}{(2n)!}z^{2n}\right|\leqslant c\varepsilon\sum_{n=0}^{\infty}\frac{(a|z|)^{2n}}{(2n)!} < c\varepsilon\,\mathrm{e}^{a|z|}.$$

As we have seen, $|G(z)| \leqslant c/|z|$ for large real z. Denote z_0 the unique positive solution of the equation

$$\varepsilon\,\mathrm{e}^{az_0} = 1/z_0.$$

If $\varepsilon > 0$ is small then for $z = 1/a\ln(1/\varepsilon)$ we have $z\,\mathrm{e}^{az} = 1/a\ln(1/\varepsilon)/\varepsilon > 1/\varepsilon$ and for $z = 1/(2a)\ln(1/\varepsilon)$, $z\,\mathrm{e}^{az} = 1/(2a)\ln(1/\varepsilon)/\sqrt{\varepsilon} < 1/\varepsilon$. Thus z_0 is between $1/(2a)\ln(1/\varepsilon)$ and $1/a\ln(1/\varepsilon)$. Now

$$\begin{aligned}\int_{-\infty}^{\infty}|G(z)|^2\,\mathrm{d}z &= \int_{-z_0}^{z_0}|G|^2+\int_{\mathbf{R}\setminus[-z_0,z_0]}|G|^2\\ &\leqslant c^2\varepsilon^2\int_{-z_0}^{z_0}\mathrm{e}^{2a|z|}\,\mathrm{d}z+c^2\int_{\mathbf{R}\setminus[-z_0,z_0]}z^{-2}\,\mathrm{d}z\\ &\leqslant c(\varepsilon^2\,\mathrm{e}^{2az_0}+1/z_0)\leqslant c\big(1\big/z_0^2+1/z_0\big)\leqslant\frac{c}{z_0}\leqslant\frac{c}{\ln(1/\varepsilon)}.\end{aligned}$$

Since the cosine Fourier transform is isomorphic, we get from (3.8) that

$$\|A[r^2(q^*(r)-q(r))]\|_{L_2(0,a)}\leqslant\frac{c}{\sqrt{\ln(1/\varepsilon)}}$$

and by lemma 3.1 that

$$\|r^2(q^*(r)-q(r))\|_{L_2(0,a)}\leqslant\frac{c}{\sqrt{\ln(1/\varepsilon)}},$$

which completes the proof. □

We know that the first few quantities μ_n are small, but we need an estimate for the other μ_n-s, too.

Lemma 3.3. *If $q, q^* \in M_D$ and $\delta < 1$ then*

$$|\mu_n| \leqslant \frac{c}{n+1} \tag{3.13}$$

with $c = c(a, D)$ independent of δ, n, q, q^*.

Proof. We have seen that for $q \in M_D$

$$|H(r,t)| \leqslant c\sqrt{rt}.$$

Putting this into (2.13) gives

$$\begin{aligned}|B_{q^*}h(r)| &\leqslant |h(r)| + \frac{c}{r^2}\int_r^a \varrho|h(\varrho)|\left[r + \int_{r^2/\varrho}^{\varrho} \sqrt{\varrho t}\cdot r\sqrt{\frac{\varrho}{t}}\,\mathrm{d}t\right]\mathrm{d}\varrho \\ &\leqslant |h(r)| + \frac{c}{r}\int_r^a \varrho|h(\varrho)|\,\mathrm{d}\varrho.\end{aligned}$$

Hence

$$r|B_{q^*}(q^* - q)(r)| \leqslant c.$$

By (2.16) this finally yields

$$|\mu_n| \leqslant a^{-\delta}\int_0^a \left(\frac{r}{a}\right)^{2n+1} r|B_{q^*}(q^* - q)(r)|\,\mathrm{d}r \tag{3.14}$$

$$\leqslant ca^{-\delta}\int_0^a \left(\frac{r}{a}\right)^{2n+1}\mathrm{d}r \leqslant ca^{1-\delta}\frac{1}{n+1} \leqslant \frac{c}{n+1} \tag{3.15}$$

if $\delta < 1$. □

Proof of theorem 1.3. As in theorem 1.2, we have to verify that

$$\int_{-\infty}^{\infty} |G(z)|^2\,\mathrm{d}z \leqslant c\left(\frac{1}{\log 1/\varepsilon} + \frac{1}{N}\right). \tag{3.16}$$

Take the value z_0 from the proof of theorem 1.2 and let

$$z_0^* = \min\left(z_0, \frac{\gamma N}{a}\right)$$

where $0 < \gamma < 2/e$ is an arbitrary constant. Applying (3.9) and lemma 3.3 we get that for $|z| \leqslant z_0^*$

$$\begin{aligned}|G(z)| &\leqslant c\varepsilon\sum_{n=0}^{N}\frac{(a|z|)^{2n}}{(2n)!} + c\sum_{n=N+1}^{\infty}\frac{(a|z|)^{2n}}{(2n)!}\frac{1}{n+1} \\ &\leqslant c\varepsilon\,\mathrm{e}^{a|z|} + \frac{(a|z|)^{2N+2}}{(2N+2)!}\frac{1}{N} \leqslant c\varepsilon\,\mathrm{e}^{a|z|} + \frac{c}{N}\frac{(a|z|)^{2N+2}}{\left(\frac{2N+2}{e}\right)^{2N+2}\sqrt{N}} \\ &= c\varepsilon\,\mathrm{e}^{a|z|} + \frac{c}{N^{3/2}}\left(ae\frac{|z|}{2N+2}\right)^{2N+2}.\end{aligned}$$

Consequently,

$$\int_{\mathbf{R}} |G|^2 \leqslant \begin{cases}\dfrac{c}{\log(1/\varepsilon)} & \text{if} \quad z_0 \leqslant \frac{\gamma N}{a} \\ \dfrac{c}{N} & \text{if} \quad z_0 > \frac{\gamma N}{a}.\end{cases}$$

Indeed, $z_0 \leqslant \gamma N/a$ implies $z_0^* = z_0$ and then

$$\int_{|z|>z_0^*} |G|^2 \leqslant \int_{|z|>z_0^*} \frac{c}{z^2}\,\mathrm{d}z = \frac{c}{z_0} = \mathbf{O}\left(\frac{1}{\log(1/\varepsilon)}\right),$$

$$\int_{|z|\leqslant z_0^*} |G|^2 \leqslant c\left[\varepsilon^2\,\mathrm{e}^{2az_0} + \frac{z_0}{N^3}\left(ae\frac{z_0}{2N+2}\right)^{4N+4}\right]$$

$$\leqslant c\left[\frac{1}{z_0^2} + \frac{1}{N^2}\right] \leqslant \frac{c}{z_0^2} = \mathbf{O}\left(\frac{1}{\log(1/\varepsilon)}\right).$$

Finally, if $z_0 > \gamma N/a$ then $z_0^* = \gamma N/a$, so

$$\int_{|z|>z_0^*} |G|^2 \leqslant \frac{c}{z_0^*} = \mathbf{O}\left(\frac{1}{N}\right),$$

$$\int_{|z|\leqslant z_0^*} |G|^2 \leqslant c\left[\varepsilon^2\,\mathrm{e}^{2az_0^*} + \frac{z_0^*}{N^3}\left(ae\frac{z_0^*}{2N+2}\right)^{4N+4}\right]$$

$$\leqslant c\left[\frac{1}{z_0^2} + \frac{1}{N^2}\right] = \mathbf{O}\left(\frac{1}{N}\right).$$

Thus (3.16) is proved. Using (3.8) and the isometric property of A this yields theorem 1.3. □

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