Modern Physics Letters B

NOTES ON THE DISTRIBUTION OF PHASE SHIFTS

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Received (Day Month Year)
Revised (Day Month Year)

We consider three-dimensional inverse scattering with fixed energy for which the spherically symmetrical potential is nonvanishing only in a ball. We give exact upper and lower bounds for the phase shifts. We provide a variational formula for the Weyl-Titchmarsh m-function of the one-dimensional Schrödinger operator defined on the half-line.

Keywords: Inverse scattering, phase shifts, Bessel functions.

1. Introduction

Consider the inverse scattering problem for the Schrödinger equation

$$-\Delta u + V(x)u = k^2 u \quad \text{on } \mathbf{R}^3 \tag{1.1}$$

with fixed energy $k^2 = 1$. Suppose that the potential is spherically symmetrical i.e.

$$V(x) = q(r) \quad r = |x|, \tag{1.2}$$

belonging to the weighted L_1-space

$$\int_0^\infty r|q(r)|\, dr < \infty. \tag{1.3}$$

By a separation of variables in (1.1) we get the following system of equations:

$$\varphi_n"(r) - \frac{n(n+1)}{r^2}\varphi_n(r) + (1-q(r))\varphi_n(r) = 0 \quad n \geq 0 \tag{1.4}$$

$$\varphi_n(r) = \gamma_n r^{n+1}(1 + \mathbf{o}(1)) \quad r \to 0+ \tag{1.5}$$

$$\varphi_n(r) = \sin(r - n\pi/2 + \delta_n) + \mathbf{o}(1) \quad r \to +\infty. \tag{1.6}$$

The constants δ_n are called phase shifts. Their definition can be extended to non-integer (and even complex) indices:

$$\varphi"(r,\lambda) - \frac{\lambda^2 - 1/4}{r^2}\varphi(r,\lambda) + (1-q(r))\varphi(r,\lambda) = 0 \quad \Re\lambda > 0 \tag{1.7}$$

$$\varphi(r,\lambda) = \gamma(\lambda) r^{\lambda+1/2}(1+\mathbf{o}(1)) \quad r \to 0+ \tag{1.8}$$

$$\varphi(r,\lambda) = \sin(r - \pi/2(\lambda - 1/2) + \delta(\lambda)) + \mathbf{o}(1) \quad r \to +\infty. \tag{1.9}$$

This defines the phase shifts only mod π; to make it unique we require that $\delta(\lambda)$ should depend continuously on q in the L_1-space (1.3) and that the shifts be zero for the zero potential.

Clearly we have $\delta_n = \delta(n+1/2)$ for $n \geq 0$. The scattering amplitude can be given by the phase shifts in the form

$$F(x) = \sum_{n=0}^{\infty}(2n+1)F_n P_n(x), \quad F_n = e^{i\delta_n}\sin\delta_n = \frac{e^{2i\delta_n - 1}}{2i} \tag{1.10}$$

where $P_n(x)$ are the Legendre polynomials.

Suppose that the potential $q(r)$ depends on some parameter s and the dependence is continuously differentiable in the sense that (denoting by a dot the derivative with respect to s)

$$\lim_{\Delta s \to 0} \frac{q(\cdot, s+\Delta s) - q(\cdot, s)}{\Delta s} = \dot{q}(\cdot, s) \tag{1.11}$$

and

$$s \mapsto \dot{q}(\cdot, s) \text{ is continuous in } s \tag{1.12}$$

where the limit and continuity are considered in the weighted L_1 space (1.3). In this case the following variational formulae hold

Theorem 1.1. Suppose (1.3), (1.11) and (1.12). Then for $\varphi = \varphi(r,\lambda)$, $\delta = \delta(\lambda)$, $\Re\lambda > 0$ and $\dot{q} = \dot{q}(r)$ we have

$$\left(\frac{\varphi'}{\varphi}\right)^{\cdot} = \frac{\int_0^r \dot{q}\varphi^2}{\varphi^2} \quad \text{if } \varphi(r,\lambda) \neq 0, \tag{1.13}$$

$$\dot{\delta} = -\int_0^\infty \dot{q}\varphi^2, \tag{1.14}$$

$$\dot{F}(x) = -\sum_{n=0}^{\infty}(2n+1)e^{2i\delta_n}P_n(x)\int_0^\infty \dot{q}\varphi_n^2 \tag{1.15}$$

and in particular for $q = 0$

$$\dot{F}(\cos\theta) = -\int_0^\infty \dot{q}(r) r^2 \cdot \frac{\sin(2r\sin(\theta/2))}{2r\sin(\theta/2)} dr, \quad 0 \leq \theta \leq \pi. \tag{1.16}$$

Remark that for the physical phase shifts δ_n a formula analogous to (1.14) has been heuristically derived in [4], see also [5].

Suppose that the potential vanishes beyond a ball of radius a for some $0 < a < \infty$:

$$q(r) = 0 \quad \text{for } r > a, \quad rq(r) \in L_1(0,a). \tag{1.17}$$

This is a natural assumption if the phase shifts are only approximately known from experiments, since the tail of q has little influence on the phase shifts, and thus it can not be reconstructed from measurements. The following exact bounds can be proved:

Theorem 1.2. Suppose (1.17). Then

$$c(\lambda) < \delta(\lambda) < \infty \quad \text{for } \lambda > 0 \tag{1.18}$$

with

$$c(\lambda) = \arctan \frac{J_\lambda(a)}{Y_\lambda(a)} - k\pi \tag{1.19}$$

where J_λ and Y_λ are the Bessel functions and k is the number of zeros of Y_λ on $(0,a)$ (in case $Y_\lambda(a) = 0$ let $\arctan = -\pi/2$). Neither of the bounds can be improved for the potential class (1.17).

For the physically relevant phase shifts the Bessel functions can be expressed by trigonometric functions, so the lower bounds have the following form

$$-a < \delta_0 < \infty \tag{1.20}$$

$$-a + \arctan a < \delta_1 < \infty \tag{1.21}$$

$$\frac{\pi}{2} - a + \arctan \frac{a^2-3}{3a} < \delta_2 < \infty \tag{1.22}$$

$$c_3 < \delta_3 < \infty \tag{1.23}$$

where

$$c_3 = \begin{cases} -a + \arctan \frac{a(a^2-15)}{6a^2-15} & \text{if } a < \sqrt{5/2} \\ -a + \pi/2 & \text{if } a = \sqrt{5/2} \\ -a + \arctan \frac{a(a^2-15)}{6a^2-15} + \pi & \text{if } a > \sqrt{5/2} \end{cases}$$

and so on.

To comment the above result remark first that $\lambda < y_1(\lambda)$ where $y_1(\lambda)$ is the first positive zero of $Y_\lambda(r)$, see [1], 9.5.2. Consequently for $\lambda \geq a$ (in fact for $y_1(\lambda) > a$) we have $k = 0$ in (1.19). Secondly it is worth while mentioning that for $\lambda \to \infty$ the bound $c(\lambda)$ decays very fast: from [1], 9.3.1 we infer

$$\frac{J_\lambda(a)}{Y_\lambda(a)} = -\frac{1}{2}\left(\frac{ea}{2\lambda}\right)^{2\lambda}(1 + \mathbf{o}(1))$$

and then

$$\lim_{\lambda\to\infty}\lambda|c(\lambda)|^{\frac{1}{2\lambda}}=\frac{ea}{2}.$$

It is closely related to the following result of Ramm et al. [7]: if q does not change sign in a small left neighborhood of a (and vanishes beyond a), then

$$\lim_{n\to\infty}n|\delta_n|^{\frac{1}{2n}}=\frac{ea}{2}.$$

Our last result is connected with the inverse spectral theory of the one-dimensional Schrödinger equation

$$-y"+Q(x)y=zy \quad x\in[0,\infty). \tag{1.24}$$

We suppose that

$$Q\in L_1(0,\infty). \tag{1.25}$$

It is known that under this condition the operator is in the limit point case, that is, for every $z\in\mathbf{C}\setminus[0,\infty)$ there is a nontrivial solution $y\in L_2(0,\infty)$ of (1.24) and this solution (the Weyl solution) is unique up to a constant factor. Define the Weyl-Titchmarsh m-function by

$$m(z)=\frac{y'(0)}{y(0)} \quad z\in\mathbf{C}\setminus[0,\infty) \tag{1.26}$$

where y is the Weyl solution of (1.24). We can prove the following variational formula about the dependence of the m-function on the potential:

Theorem 1.3. Let (1.25) be satisfied and suppose that Q has a continuously differentiable dependence on a parameter s in a sense analogous to (1.11)-(1.12). Then

$$\dot m(z)=-\frac{\int_0^\infty \dot Q y^2}{y^2(0)} \quad z\in\mathbf{C}\setminus[0,\infty),\ y(0)\neq 0 \tag{1.27}$$

where y is the Weyl solution of (1.24).

In particular this implies that $m(-\lambda^2)$ is decreasing if Q increases, $\lambda>0$ is constant and if λ is not an eigenvalue for any of the potentials touched in the deformation of Q.

2. Proofs

We will establish variational formulae for $\varphi(r,\lambda)$ and $\delta(\lambda)$ as functions of the potential $q(r)$. We will assume throughout that

$$\Re\lambda>0.$$

Introduce the Jost solutions as the (complex-valued) solutions $f_\pm(r,\lambda)$ of (1.7) satisfying

$$f_\pm(r,\lambda) = e^{\mp ir}(1+\mathbf{o}(1)) \quad r\to\infty. \tag{2.1}$$

In Alfaro, Regge [2] it is proved that the integral equation

$$f_\pm(r,\lambda) = e^{\mp ir} + \int_r^\infty \sin(t-r)\left[q(t)+\frac{\lambda^2-1/4}{t^2}\right] f_\pm(t,\lambda)\,dt \tag{2.2}$$

holds and that

$$f_\pm(r,\lambda) = \sum_{n=0}^\infty g_{n,\pm}(r,\lambda) \tag{2.3}$$

with

$$g_{0,\pm}(r,\lambda) = e^{\mp ir}, \tag{2.4}$$

$$g_{n+1,\pm}(r,\lambda) = \int_r^\infty \sin(t-r)\left[q(t)+\frac{\lambda^2-1/4}{t^2}\right] g_{n,\pm}(t,\lambda)\,dt. \tag{2.5}$$

Using the estimate

$$|\sin(t-r)| \le \min(t,1) \le \frac{2t}{1+t} \quad t\ge r\ge 0 \tag{2.6}$$

we get by induction on n that

$$g_{n,\pm}(r,\lambda) \le \frac{[2Q(r)]^n}{n!}, \quad Q(r) = \int_r^\infty \left|q(t)+\frac{\lambda^2-1/4}{t^2}\right| \frac{t}{1+t}\,dt. \tag{2.7}$$

Consequently

$$|f_\pm(r,\lambda)| \le e^{2Q(r)}, \quad |f_\pm(r,\lambda) - e^{\mp ir}| \le e^{2Q(r)} - 1. \tag{2.8}$$

From (2.2) we see that

$$f'_\pm(r,\lambda) = \mp i e^{\mp ir}(1+\mathbf{o}(1)), \quad r\to\infty \tag{2.9}$$

$$\dot f_\pm(r,\lambda) = \mathbf{o}(1)), \quad r\to\infty. \tag{2.10}$$

The function φ can be expressed by Jost solutions; comparing (1.9) and (2.1) gives

$$\varphi = \frac{1}{2i}\left[f_-\cdot e^{i(\delta(\lambda)-\pi/2(\lambda-1/2))} - f_+\cdot e^{-i(\delta(\lambda)-\pi/2(\lambda-1/2))}\right]. \tag{2.11}$$

We also need another solution $\psi(r,\lambda)$ of (1.7) defined by

$$\psi(r,\lambda) = \cos(r-\pi/2(\lambda-1/2)+\delta(\lambda)) + \mathbf{o}(1) \quad r\to+\infty. \tag{2.12}$$

It has an order of $r^{-\lambda+1/2}$ at $r\to 0+$. From its behavior at infinity we get

$$\psi = \frac{1}{2}\left[f_-\cdot e^{i(\delta(\lambda)-\pi/2(\lambda-1/2))} + f_+\cdot e^{-i(\delta(\lambda)-\pi/2(\lambda-1/2))}\right]. \tag{2.13}$$

From (2.9) and (2.10) it follows that

$$\varphi' = \cos(r - \pi/2(\lambda - 1/2) + \delta(\lambda)) + \mathbf{o}(1), \quad r \to \infty \tag{2.14}$$

$$\psi' = -\sin(r - \pi/2(\lambda - 1/2) + \delta(\lambda)) + \mathbf{o}(1), \quad r \to \infty \tag{2.15}$$

$$\dot\varphi = \dot\delta(\lambda)\cos(r - \pi/2(\lambda - 1/2) + \delta(\lambda)) + \mathbf{o}(1)), \quad r \to \infty \tag{2.16}$$

$$\dot\psi = -\dot\delta(\lambda)\sin(r - \pi/2(\lambda - 1/2) + \delta(\lambda)) + \mathbf{o}(1), \quad r \to \infty. \tag{2.17}$$

As a consequence, the Wronskian of φ and ψ can be computed by taking the limit $r \to \infty$:

$$\varphi\psi' - \varphi'\psi = -\sin^2(r - \pi/2(\lambda - 1/2)) - \cos^2(r - \pi/2(\lambda - 1/2)) = -1. \tag{2.18}$$

Lemma 2.1. If $h(r)r^{-\lambda+1/2} \in L_1^{\mathrm{loc}}[0,\infty)$ then

$$u = \varphi\int_0^r h\psi - \psi\int_0^r h\varphi \tag{2.19}$$

is a solution of

$$u" - \frac{\lambda^2 - 1/4}{r^2}u + (1-q)u = h \tag{2.20}$$

Proof. By the assumption on h the integrals in (2.19) exist. Repeated differentiations give the formulae

$$u' = \varphi'\int_0^r h\psi - \psi'\int_0^r h\varphi,$$

$$u" = \varphi"\int_0^r h\psi - \psi"\int_0^r h\varphi + (\varphi'\psi - \psi'\varphi)h.$$

Expressing $\varphi"$ and $\psi"$ from the differential equation and using (2.18) we get (2.20). □

Lemma 2.2. Define $\tilde\varphi = \varphi/\gamma(\lambda)$. Under the assumption (1.11), (1.12) we have

$$\dot{\tilde\varphi} = \left(\varphi\int_0^r \dot q\varphi\psi - \psi\int_0^r \dot q\varphi^2\right)/\gamma(\lambda). \tag{2.21}$$

Proof. We differentiate (1.7) with respect to the parameter to obtain

$$\dot{\tilde\varphi}" - \frac{\lambda^2 - 1/4}{r^2}\dot{\tilde\varphi} + (1 - q(r))\dot{\tilde\varphi} = \dot q\tilde\varphi. \tag{2.22}$$

Denote by u the right hand side of (2.21) and let $h = \dot q\tilde\varphi$. Then

$$hr^{-\lambda+1/2} = \mathbf{O}(\dot q r^{\lambda+1/2}\cdot r^{-\lambda+1/2}) = \mathbf{O}(r\dot q) \in L_1^{\mathrm{loc}}[0,\infty)$$

so by Lemma 2.1 u is a solution of (2.22). Thus all solutions of (2.22) are $u+\alpha\varphi+\beta\psi$. By (1.8), $\tilde{\varphi}$ is asymptotically $r^{\lambda+1/2}$ at $r\to 0+$. Take the following integral representation of $\tilde{\varphi}$ from [2]

$$\tilde{\varphi}(r,\lambda)=r^{\lambda+1/2}+\frac{1}{2\lambda}\int_0^r\left[\left(\frac{t}{r}\right)^{\lambda}-\left(\frac{r}{t}\right)^{\lambda}\right]\sqrt{rt}[1-q(t)]\tilde{\varphi}(t,\lambda)\,dt. \tag{2.23}$$

It implies that

$$\dot{\tilde{\varphi}}=\mathbf{o}(r^{\lambda+1/2})\quad r\to 0+\,. \tag{2.24}$$

A similar statement holds for u since

$$\begin{aligned}u&=\mathbf{O}\left(\int_0^r r^{\lambda+1/2}t^{\lambda+1/2}t^{-\lambda+1/2}|\dot{q}(t)|\,dt+\int_0^r r^{-\lambda+1/2}t^{2\lambda+1}|\dot{q}(t)|\,dt\right)\\&=r^{\lambda+1/2}\mathbf{O}\left(\int_0^r t|\dot{q}(t)|\,dt\right)=\mathbf{o}(r^{\lambda+1/2}).\end{aligned}$$

Since neither of φ and ψ is $\mathbf{o}(r^{\lambda+1/2})$ near zero, this means that in $\dot{\tilde{\varphi}}=u+\alpha\varphi+\beta\psi$ both α and β must be zero. □

Proof of Theorem 1.1. Differentiating (2.21) in r gives

$$\dot{\tilde{\varphi}}'=\left(\varphi'\int_0^r\dot{q}\varphi\psi-\psi'\int_0^r\dot{q}\varphi^2\right)/\gamma(\lambda).$$

Hence

$$\begin{aligned}\dot{\tilde{\varphi}}'\tilde{\varphi}-\dot{\tilde{\varphi}}\tilde{\varphi}'&=\left(\varphi'\int_0^r\dot{q}\varphi\psi-\psi'\int_0^r\dot{q}\varphi^2\right)\varphi/\gamma(\lambda)^2-\left(\varphi\int_0^r\dot{q}\varphi\psi-\psi\int_0^r\dot{q}\varphi^2\right)\varphi'/\gamma(\lambda)^2\\&=(\varphi'\psi-\varphi\psi')\int_0^r\dot{q}\tilde{\varphi}^2=\int_0^r\dot{q}\tilde{\varphi}^2\end{aligned}$$

which proves (1.13) since

$$\left(\frac{\varphi'}{\varphi}\right)^{\cdot}=\left(\frac{\tilde{\varphi}'}{\tilde{\varphi}}\right)^{\cdot}=\frac{\dot{\tilde{\varphi}}'\tilde{\varphi}-\dot{\tilde{\varphi}}\tilde{\varphi}'}{\tilde{\varphi}^2}=\frac{\int_0^r\dot{q}\tilde{\varphi}^2}{\tilde{\varphi}^2}=\frac{\int_0^r\dot{q}\varphi^2}{\varphi^2}.$$

From (2.2) and (2.11) it follows that

$$\begin{aligned}\varphi(r,\lambda)&=\sin(r-\pi/2(\lambda-1/2)+\delta(\lambda))\\&\quad+\int_r^\infty\sin(t-r)\left[q(t)+\frac{\lambda^2-1/4}{t^2}\right]\varphi(t,\lambda)\,dt\end{aligned} \tag{2.25}$$

which implies

$$\dot\varphi'(r,\lambda) = -\dot\delta(\lambda)\sin(r-\pi/2(\lambda-1/2)+\delta(\lambda)) + \mathbf{o}(1) \quad r\to\infty. \tag{2.26}$$

Together with (1.9), (2.14) and (2.16) this gives

$$\dot\varphi'\varphi - \varphi'\dot\varphi = -\dot\delta + \mathbf{o}(1)$$

that is,

$$-\dot\delta + \mathbf{o}(1) = \frac{\dot\varphi'\varphi - \varphi'\dot\varphi}{\varphi^2}\varphi^2 = \int_0^r \dot q\varphi^2$$

and this proves (1.14). Clearly (1.15) follows from (1.14).

Now introduce the spherical Bessel functions

$$j_n(r) = \sqrt{\frac{\pi}{2r}}J_{n+1/2}(r) \quad n\ge 0.$$

It is known that for the zero potential $\varphi_n(r) = rj_n(r)$ (see e.g. [3]), hence

$$\dot\delta_n = -\int_0^\infty \dot q(r)r^2 j_n^2(r)\,dr \quad \text{ if } q=0. \tag{2.27}$$

As a special case of the formula [1], 10.1.45 we get

$$\sum_{n=0}^\infty (2n+1)j_n^2(r)P_n(\cos\theta) = \frac{\sin R}{R}, \quad R = 2r\sin\frac{\theta}{2}. \tag{2.28}$$

For the zero potential (1.15) gives

$$\dot F = -\int_0^\infty \dot q r^2\sum(2n+1)j_n^2P_n$$

which yields (1.16). Theorem 1.1 is completely proved. □

To verify Theorem 1.2 we need some preliminary statements. First we provide a uniform bound for φ. In the following statement $c(D,\varepsilon_0)$ denotes positive constants possibly different in every occurrences.

Lemma 2.3. Let $0<\varepsilon_0\le\lambda\le D$ and $\int_0^\infty r|q(r)|\,dr \le D$. Suppose that the singularity of q near $r=0$ is uniformly controlled in the sense that there exists $r_0\ge c(D,\varepsilon_0)>0$ such that

$$\int_0^{r_0} t|1-q(t)|\,dt < \varepsilon_0/2. \tag{2.29}$$

Then

$$|\varphi(r,\lambda)| \le c(D,\varepsilon_0) \tag{2.30}$$

with a constant $c(D, \varepsilon_0)$ independent of r, λ and q.

Proof. In the estimate (2.8) we see that

$$Q(r) \leq c(D)(1 + \log\left(\frac{1+r}{r}\right)).$$

From (2.11) this implies the uniform boundedness of φ if $1/r$ is bounded. For small r we will use the representation (2.23) for $\tilde{\varphi} \approx r^{\lambda+1/2}$. By successive approximation it is proved in [2] that

$$|\tilde{\varphi}(r,\lambda) - r^{\lambda+1/2}| \leq r^{\lambda+1/2}[e^{P(r)/\lambda} - 1], \quad P(r) = \int_0^r t|1 - q(t)|\, dt. \tag{2.31}$$

From (2.29) we obtain $P(r_0)/\lambda < 1/2$ hence

$$0.3r^{\lambda+1/2} \leq \tilde{\varphi}(r,\lambda) \leq 1.7r^{\lambda+1/2} \quad \text{for } r \leq r_0. \tag{2.32}$$

Consequently $|\tilde{\varphi}(r)| \leq 6|\tilde{\varphi}(r_0)|$ for $r \leq r_0$. The same is true for φ (as a constant multiple of $\tilde{\varphi}$), so φ is uniformly bounded also in $[0, r_0]$. □

In what follows we use the notation $\delta(\lambda) = \delta(\lambda, q)$ to indicate the dependence on q. Let $0 < a < \infty$ be fixed and denote $\delta(\lambda, n)$ the phase shift for the box potential which is n for $r \leq a$ and 0 for $r > a$.

Lemma 2.4. If $q(r) = 0$ for $r > a$ and $rq(r) \in L_1(0, a)$ then

$$\delta(\lambda, q) > \lim_{n\to\infty} \delta(\lambda, n). \tag{2.33}$$

Proof. It follows from (1.14) that if q strictly increases on a set of positive measure then δ strictly decreases. So for q bounded from above (2.33) is straightforward; suppose that q is not bounded. Let $q_n = \max(q, n)$. Then there is a positive integer n_0 such that $\delta(\lambda, q) > \delta(\lambda, q_{n_0})$. Let $q_{n,m} = \min(m, q_n)$. Using the linear transformation $s \mapsto sq_{n_0,m} + (1 - s)q_{n_0}$ we get

$$0 \leq \delta(\lambda, q_{n_0,m}) - \delta(\lambda, q_{n_0}) = \int_0^1 \dot{\delta}\, ds \leq c \int_0^a (q_{n_0} - q_{n_0,m})\, dr \to 0 \quad m \to \infty$$

since by the previous Lemma the functions φ for $sq_{n_0,m} + (1 - s)q_{n_0}$ are uniformly bounded in r and m. Summing up, we see that $\delta(\lambda, q) > \delta(\lambda, q_{n_0})$, $\delta(\lambda, q_{n_0,m})$ tends to $\delta(\lambda, q_{n_0})$ and $\delta(\lambda, q_{n_0,m}) \geq \delta(\lambda, m)$. So necessarily $\delta(\lambda, q) > \delta(\lambda, m)$ for large m. □

Now consider the box potentials

$$q(r) = \begin{cases} q_0 & \text{if } 0 \leq r \leq a \\ 0 & \text{if } r > a \end{cases} \tag{2.34}$$

with a constant $q_0 \in \mathbf{R}$ and let $\gamma = \sqrt{1-q_0}$; this is nonnegative or purely imaginary. It is known that for the box potentials the functions φ and the phase shifts can be explicitly expressed by Bessel functions:

$$\varphi(r,\lambda) = \begin{cases} c\sqrt{r}J_\lambda(\gamma r) \text{ (or } cr^{\lambda+1/2} \text{ for } \gamma = 0) & \text{if } r \le a \\ \sqrt{\frac{r\pi}{2}}[\cos\delta(\lambda)J_\lambda(r) - \sin\delta(\lambda)Y_\lambda(r)] & \text{if } r \ge a. \end{cases} \tag{2.35}$$

Since $\varphi \in C^1$ at $r = a$, we infer the fundamental equation

$$\frac{\gamma J'_\lambda(\gamma a)}{J_\lambda(\gamma a)} = \frac{J'_\lambda(a) - \tan(\delta(\lambda))Y'_\lambda(a)}{J_\lambda(a) - \tan(\delta(\lambda))Y_\lambda(a)} \tag{2.36}$$

from which the tangent of the phase shifts can be simply expressed.

We need the monotonicity of the right hand side of (2.36) in the following sense:

Lemma 2.5. a. The function

$$\gamma \mapsto \frac{\gamma J'_\lambda(\gamma a)}{J_\lambda(\gamma a)}$$

is strictly decreasing in $\gamma \ge 0$ between the consecutive zeros of $J_\lambda(\gamma a)$.
b. The function

$$\tau \mapsto \frac{i\tau J'_\lambda(i\tau a)}{J_\lambda(i\tau a)}$$

is strictly increasing in $\tau \ge 0$.

Proof. The mapping

$$t \mapsto \frac{J'_\lambda(a) - tY'_\lambda(a)}{J_\lambda(a) - tY_\lambda(a)} \tag{2.37}$$

is strictly decreasing in $(-\infty, J_\lambda(a)/Y_\lambda(a))$ and in $(J_\lambda(a)/Y_\lambda(a), \infty)$; in $t = J_\lambda(a)/Y_\lambda(a)$ it jumps from $-\infty$ to $+\infty$. Indeed,

$$\frac{d}{dt}\frac{J'_\lambda(a) - tY'_\lambda(a)}{J_\lambda(a) - tY_\lambda(a)} = \frac{J'_\lambda(a)Y_\lambda(a) - J_\lambda(a)Y'_\lambda(a)}{(J_\lambda(a) - tY_\lambda(a))^2} = \frac{-2/(\pi a)}{(J_\lambda(a) - tY_\lambda(a))^2} < 0.$$

Now if $\gamma = \sqrt{1-q_0}$ increases, then q_0 decreases, so δ strictly increases and then $\tan(\delta)$ will also increase apart from possible jumps from $+\infty$ to $-\infty$. If γ moves between consecutive zeros of $J_\lambda(\gamma a)$ then $t = \tan(\delta)$ can not take the value $J_\lambda(a)/Y_\lambda(a)$. Thus the monotonicity of (2.37) implies that the right hand side of (2.36) is decreasing (if $\tan(\delta)$ goes through infinity, the right of (2.36) keeps decreasing in a continuous way). This proves the first part of Lemma 2.5. The second part is proved analogously: if $\gamma = i\tau$ and $\tau = \sqrt{q_0 - 1} \ge 0$ is increasing then $q_0 \ge 1$ is increasing and then $\tan(\delta)$ decreases apart from possible infinite jumps. So we can finish the proof as above. From the expansion

$$J_\lambda(ix) = \left(\frac{ix}{2}\right)^\lambda \sum_{m=1}^{\infty} \frac{x^{2m}}{4^m m!\Gamma(m+\lambda+1)} \tag{2.38}$$

we see that the function $J_\lambda(i\tau a)$ has no positive zeros. □

Proof of Theorem 1.2. By Lemma 2.4 we have to show that

$$\lim_{n\to-\infty}\delta(\lambda,n)=+\infty,\quad \lim_{n\to\infty}\delta(\lambda,n)=c(\lambda). \tag{2.39}$$

From (2.36) we see that

$$\tan(\delta(\lambda))=\pm\infty\Leftrightarrow \frac{\gamma J'_\lambda(\gamma a)}{J_\lambda(\gamma a)}=\frac{Y'_\lambda(a)}{Y_\lambda(a)}. \tag{2.40}$$

If $q_0<1,\ q_0\to-\infty$, then $\gamma\to+\infty$ and $\delta(\lambda)$ is strictly increasing, thus between consecutive zeros of $J_\lambda(\gamma a)$ $\tan\delta$ becomes infinite exactly once. Since $J_\lambda(\gamma a)$ has infinitely many positive zeros, $\delta\to+\infty$.

Now let $q_0>1,\ q_0\to\infty$, then $\gamma=i\tau,\ \tau\to\infty$ and δ is strictly decreasing. Using the known asymptotics

$$J_\lambda(z)=\sqrt{\frac{2}{\pi z}}\sin(z-(\lambda-1/2)\pi/2)+e^{\Im z}\mathbf{O}(|z|^{-3/2}),$$
$$J'_\lambda(z)=\sqrt{\frac{2}{\pi z}}\cos(z-(\lambda-1/2)\pi/2)+e^{\Im z}\mathbf{O}(|z|^{-3/2})$$

valid for $|z|\to\infty,\ |\arg(z)|<\pi$ (see e.g. [1]), we obtain

$$i\tau\frac{J'_\lambda(i\tau a)}{J_\lambda(i\tau a)}=\tau(1+\mathbf{O}(1/\tau))\to+\infty$$

This implies by (2.36) that

$$\lim_{n\to+\infty}\tan(\delta(\lambda,n))=\frac{J_\lambda(a)}{Y_\lambda(a)}. \tag{2.41}$$

That is, the tangent of $c(\lambda)$ is found and we have to determine the number of shifts by π in $c(\lambda)$. More precisely, $c(\lambda)$ is $\arctan(J_\lambda(a)/Y_\lambda(a))$ minus π times the number of solutions of (2.40) while q_0 runs over $(0,\infty)$.

Suppose first that $J_\lambda(\gamma a)$ has no zeros on $(0,1]$. For $q_0=0$ we have $\delta=0$, so the left hand side of (2.40) starts from the value $J'_\lambda(a)/J_\lambda(a)$. While q_0 is growing to 1, γ decreases to 0, so the left of (2.40) increases. If q_0 grows further from 1 to ∞, the left of (2.40) continues growing until $+\infty$. Since the monotonicity is strict, (2.40) has a single solution if $J'_\lambda(a)/J_\lambda(a)<Y'_\lambda(a)/Y_\lambda(a)$ and has no solution otherwise. Since

$$\frac{Y'_\lambda(a)}{Y_\lambda(a)}-\frac{J'_\lambda(a)}{J_\lambda(a)}=\frac{2/(\pi a)}{J_\lambda(a)Y_\lambda(a)},$$

we can summarize the above considerations in the statement $c(\lambda)=\arctan(J_\lambda(a)/Y_\lambda(a))-\pi\max(0,\operatorname{sign}(J_\lambda(a)Y_\lambda(a)))$. If $J_\lambda(\gamma a)$ has k_1 zeros on $(0,1]$ then between $\gamma=1$ and the largest zero the left of (2.40) goes again from $J'_\lambda(a)/J_\lambda(a)$ to $+\infty$; between the consecutive zeros and after the last zero it goes from $-\infty$ to $+\infty$. Hence (2.40) has exactly $k_1 + \max(0, \operatorname{sign}(J_\lambda(a)Y_\lambda(a)))$ solutions. This proves that

$$c(\lambda) = \arctan \frac{J_\lambda(a)}{Y_\lambda(a)} - \pi(k_1 + \max(0, \operatorname{sign}(J_\lambda(a)Y_\lambda(a)))) \tag{2.42}$$

and it remains to check that this is equivalent to (1.19). In fact both of them are equivalent to a third statement: if $\lambda > 0$ is fixed then $c(\lambda) = \arctan[J_\lambda(a)/Y_\lambda(a)]$ for small $a > 0$ and $c(\lambda) = \arctan[J_\lambda(a)/Y_\lambda(a)]$ minus integer times π for larger a, defined continuously in a. This is easy to check for (1.19): the zeros of J_λ and Y_λ are interlacing, thus if a crosses a zero of Y_λ, $\arctan[J_\lambda(a)/Y_\lambda(a)]$ jumps from $-\infty$ to $+\infty$. For (2.42) similar considerations apply: for small a, $\operatorname{sign}(J_\lambda(a)Y_\lambda(a)) < 0$, when a crosses the first zero of Y_λ, the jumps of sign and arctan cancel each other; when a crosses the first zero of J_λ, the jumps of sign and k_1 cancel each other and so on.

Finally we verify (1.20)-(1.23). From $J_{1/2}(a)/Y_{1/2}(a) = -\tan a$ and from the continuity of c_0 in a (1.20) follows at once. Now

$$\begin{aligned}
\frac{J_{3/2}(a)}{Y_{3/2}(a)} &= \frac{a - \tan a}{1 + a \tan a} = \tan(\arctan a - a), \\
\frac{J_{5/2}(a)}{Y_{5/2}(a)} &= \frac{1 + \frac{a^2-3}{3a} \tan a}{\tan a - \frac{a^2-3}{3a}} = \cot\left(a - \arctan \frac{a^2-3}{3a}\right) \\
&= \tan\left(\pi/2 - a + \arctan \frac{a^2-3}{3a}\right), \\
\frac{J_{7/2}(a)}{Y_{7/2}(a)} &= \frac{\frac{a(a^2-15)}{6a^2-15} - \tan a}{1 + \frac{a(a^2-15)}{6a^2-15} \tan a} = \tan\left(\arctan \frac{a(a^2-15)}{6a^2-15} - a\right)
\end{aligned}$$

implies (1.21)-(1.23) respectively. This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Consider the equation

$$-y" + Q(x)y = -\lambda^2 y \quad x \in [0, \infty),\ \Re\lambda > 0 \tag{2.43}$$

with a real-valued potential

$$Q \in L_1(0, \infty). \tag{2.44}$$

Suppose that y is the Weyl solution:

$$y(x) = e^{-\lambda x}(1 + \mathbf{o}(1)) \quad x \to \infty. \tag{2.45}$$

Fix any number $0 < a < \infty$ and introduce the new variable

$$r = ae^{-x}, \quad r \in (0, a].$$

Then the new eigenfunctions

$$\tilde{\varphi}(r) = \sqrt{r} y(x) \tag{2.46}$$

and the new potential

$$q(r) = Q(x)/r^2 + 1, \quad 0 < r \le a \tag{2.47}$$

satisfies (1.7). The formula (2.45) is transformed into

$$\tilde{\varphi}(r) = r^{\lambda+1/2}(1 + \mathbf{o}(1)), \quad r \to 0+$$

justifying the notation. Moreover

$$rq(r) \in L_1(0, a).$$

All these statements are proved in Horvath [6]. From (2.46) we see that

$$\frac{\varphi'(a)}{\varphi(a)} = \frac{\tilde{\varphi}'(a)}{\tilde{\varphi}(a)} = \frac{1}{2a} - \frac{1}{a}\frac{y'(0)}{y(0)}.$$

where φ is the solution of (1.7)-(1.9). Thus it follows from (1.13) that

$$\dot{m}(-\lambda^2) = \left(\frac{y'(0)}{y(0)}\right)^{\cdot} = -a\left(\frac{\varphi'(a)}{\varphi(a)}\right)^{\cdot} = -a\frac{\int_0^a \dot{q}\varphi^2}{\varphi^2(a)} = -\frac{\int_0^a \dot{q}\varphi^2}{y^2(0)}$$

and here

$$\int_0^a \dot{q}\varphi^2 = \int_0^\infty \dot{q}(ae^{-x})\varphi^2(ae^{-x})ae^{-x}\,dx = \int_0^\infty \dot{Q}(x)y^2(x)\,dx$$

which proves Theorem 1.3 $\square$

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