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SEMIMODULAR LATTICES AND THE HALL–DILWORTH GLUING CONSTRUCTION

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Abstract. We present a new gluing construction for semimodular lattices, related to the Hall–Dilworth construction.

The gluing constructions in the lattice theory started with a paper of M. Hall and R. P. Dilworth [4] to prove that there exists a modular lattice that cannot be embedded in any complemented modular lattice. This construction is the following: let K and L be lattices, let F be a filter of K, and let I be an ideal of L such that F and I are isomorphic with $\varphi : F \to I$. Then we form the disjoint union $G = K \cup L$ and identify $a \in F$ with $a\varphi \in I$, for all $a \in F$. $a \leqq b$ in G iff one of the following cases is satisfied: (i) $a \leqq_K b$, $a, b \in K$, (ii) $a \leqq_L b$, $a, b \in L$, (iii) $a \leqq_K z$ and $z\varphi \leqq_L b$, $a \in K, b \in L$ for some $z \in F$.

I applied in my paper [5] the following special gluing construction (to give a very short proof for the theorem that every semimodular lattice of finite length has a cover-preserving embedding into a simple semimodular lattice), which is shown in Fig. 1. We define this gluing.

Let L and K be semimodular lattices of finite length. Take a maximal chain C of L. Assume that K contains a filter C' isomorphic to C under the isomorphism $\psi : C \to C'$. Consider the attachment G of the lattice K to the lattice L over C by identifying C with C' along ψ, in the sense of G. Grätzer

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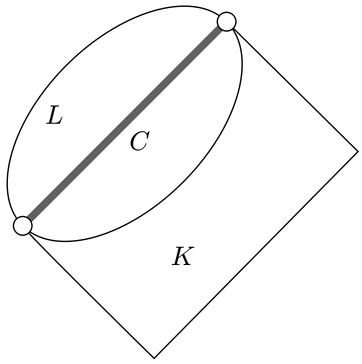

Fig. 1: The sketch of the gluing $G = L \cup_C K$

and D. Kelly [2]. So we have $G = L \cup K$ with $L \cap K = C$ and the order

$$a \leqq b = \begin{cases} a \leqq_K b & \text{if } \ a, b \in K; \\ a \leqq_L b & \text{if } \ a, b \in L; \\ a \leqq_K x\psi, \ x \leqq_L b & \text{if } \ a \in K,\, b \in L, \text{ for some } x \in C. \end{cases}$$

We denote the lattice thus obtained by $G = L \cup_C K$. This lattice is evidently not a Hall–Dilworth gluing. The maximal chain C is not an ideal of L, but this construction reminds of the Hall–Dilworth gluing.

Despite this we will prove that by the given construction the Hall–Dilworth gluing is in the 'background'. This makes it possible to introduce a new gluing construction for semimodular lattices.

A join-homomorphism $\varphi : L \to K$ is said to be *cover-preserving* iff it preserves the relation $\preceq$. Similarly, a join-congruence Φ of L is called cover-preserving if the natural join-homomorphism $L \to L/\Phi$, $x \mapsto [x]\Phi$ is cover-preserving. The image of a semimodular lattice by a cover-preserving join-homomorphism is a semimodular lattice (see [3]).

THEOREM. *$G = L \cup_C K$ is the cover-preserving join-homomorphic image of a semimodular lattice H, $\psi : H \to G$ such that:*

(1) *H is the Hall–Dilworth gluing of the semimodular lattices P and K over a chain isomorphic to C,*

(2) *P is the direct power of a chain isomorphic to C,*

(3) *L is the cover-preserving join-homomorphic image of P, $\varphi : P \to L$,*

(4) *the restriction of ψ to the filter P of H is φ and to K is the identity.*

PROOF. We have the semimodular lattices L and K of finite length. C is a maximal chain of L and K contains C' as filter. Further, $L \cap K = C$ and $G = L \cup_C K$ is the glued semimodular lattice.

In [1], we have proved (see Theorem 1): each finite semimodular lattice L is a cover-preserving join-homomorphic image of the direct product of $\mathrm{w}\bigl(J(L)\bigr)$ finite chains (the *width* $\mathrm{w}(P)$ of a (finite) poset P is defined to be $\max\{n : P$ has an n-element antichain$\}$). The outline the proof is: let $k = \mathrm{w}\bigl(J(L)\bigr)$. $J(L)$ is the union of k appropriate chains. Let us extend these chains of $J(L)$ to *maximal* chains $C_1, \dots, C_k$ of L. Then $J(L) \subseteqq C_1 \cup \dots \cup C_k$. We may assume that $k \geqq 2$. Denote $C_1 \times \dots \times C_k$ by P and define a join-homomorphism

$$\varphi : P \to L, \quad (x_1, \dots, x_k) \mapsto x_1 \vee \dots \vee x_k.$$

This proves (3). The maximal chains C_i are all isomorphic to C which means that (2) is satisfied, $P \cong C^k$.

Observe that to represent L as a cover-preserving join-homomorphic image of a direct product of chains we use only that $J(L) \subseteqq C_1 \cup \dots \cup C_k$ and therefore we may assume that $k \geqq \mathrm{w}\bigl(J(L)\bigr)$ and one of the maximal chains in the definition of P is the selected $C \subset L$, $C = C_k$. We will regard C_k as the ideal I of P in the usual manner. On the other hand, in K we have the filter C'. Take the Hall–Dilworth gluing of P and K by identifying I and C'. We get obviously a semimodular lattice $H = P \cup K$ and condition (1) is satisfied.

Finally, we define $\psi : H \to G$. Denote Φ the join congruence of P induced by φ. By the definition $\varphi : P \to L$ $(0, 0, \dots, 0, c)\varphi = c$ and therefore if $c, c' \in C$, $c \neq c'$ we have $(0, 0, \dots, 0, c) \neq (0, 0, \dots, 0, c')(\Phi)$. We define Ψ on H such that the restriction of Ψ to the filter P is Φ and the restriction to K is the trivial join congruence.

A sublattice $\{a_1 \wedge a_2, a_1, a_2, a_1 \vee a_2\}$ of a lattice is called a *covering square* if $a_1 \wedge a_2 \prec a_i \prec a_1 \vee a_2$ for $i = 1, 2$.

In [1] we proved the following characterization of cover-preserving join congruences: let Φ be a join-congruence of a finite semimodular lattice M. Then Φ is cover-preserving if and only if for any covering square $S = \{a \wedge b, a, b, a \vee b\}$ if $a \wedge b \not\equiv a$ (Φ) and $a \wedge b \not\equiv b$ (Φ) then $a \equiv a \vee b$ (Φ) implies $b \equiv a \vee b$ (Φ). Using this result we can see that Ψ is a cover-preserving join congruence of H. (4) is obviously satisfied.

Obviously, $(1, 1, \dots, 1, 1) = (0, 0, \dots, 0, 1)(\Phi)$, consequently, the image of $I \subset H$ is the maximal $C \subset G$. $\square$

We illustrate the proof with the following easy example. L is the well known seven element semimodular lattice N_7, and K is an arbitrary semimodular lattice with a filter which is a chain of length 3.

P is isomorphic to C^2 and φ is a cover-preserving join homomorphism. The image of all elements in A is the unit element.

From P and K we get the following lattice H with the Hall–Dilworth gluing:

In the class of all semimodular lattices we introduce a natural generalization of the Hall–Dilworth gluing:

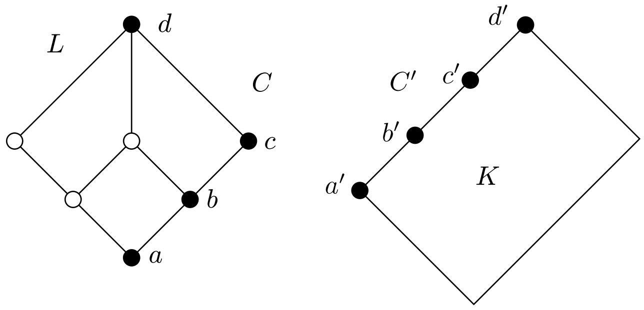

Fig. 2: The lattices L and K with the chains C, C'

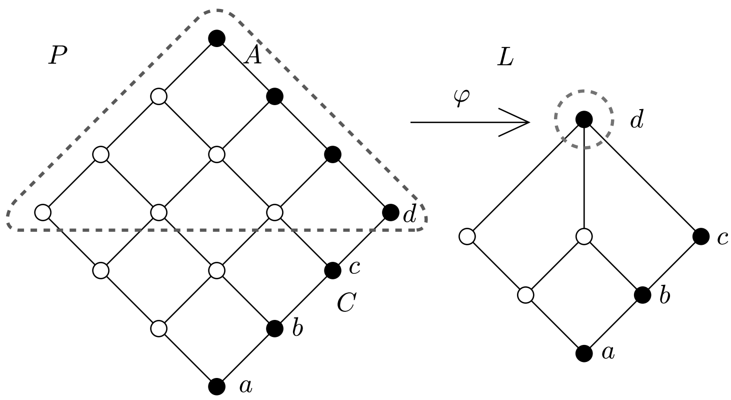

Fig. 3

DEFINITION. Let K and L be semimodular lattices, F a filter of K, and I an ideal of L. If F is isomorphic to I with φ the isomorphism, then we can form first the lattice G, the Hall–Dilworth gluing of L and K over F and I. Let Φ be a cover-preserving join-congruence of G. We call G/Φ the gluing of L/Φ and K/Φ.

L/Φ is in general only a cover-preserving sublattice of G/Φ. In our special case the restriction of Φ to K is the trivial join congruence ω.

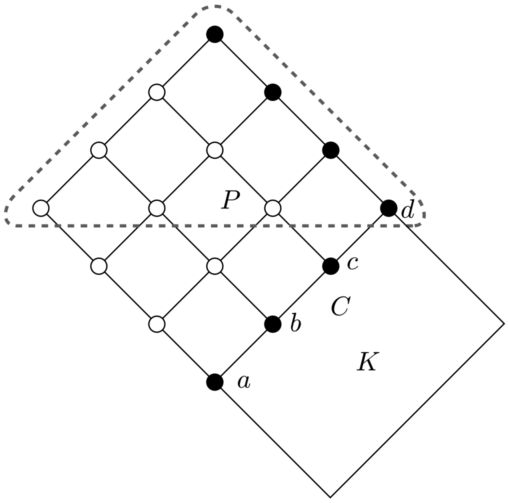

Fig. 4: $H = P \cup K$ with C

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