A SHORT PROOF
OF THE CONGRUENCE REPRESENTATION THEOREM
FOR SEMIMODULAR LATTICES

G. GRÄTZER AND E. T. SCHMIDT

Abstract. In a 1998 paper with H. Lakser, the authors proved that every
finite distributive lattice $D$ can be represented as the congruence lattice of
a finite semimodular lattice.
Some ten years later, the first author and E. Knapp proved a much stronger
result, proving the representation theorem for rectangular lattices.
In this note we present a short proof of these results.

1. Introduction

In [5], the authors with H. Lakser proved the following result:

Theorem 1. Let $D$ be a finite distributive lattice. Then there is a planar semi-
modular lattice $K$ such that

$$D \cong \text{Con} K.$$ 

A stronger result was proved some 10 years later. To state it, we need a few
concepts.
Let $A$ be a planar lattice. A left corner (resp., right corner) of the lattice $A$ is
a doubly-irreducible element in $A - \{0, 1\}$ on the left (resp., right) boundary of $A$.

We define a rectangular lattice $L$, as in G. Grätzer and E. Knapp [4], as a planar
semimodular lattice that has exactly one left corner, $u_l$, and exactly one right
corner, $u_r$, and they are complementary—that is, $u_l \lor u_r = 1$ and $u_l \land u_r = 0$.

The first author and E. Knapp [4] proved the following much stronger form of
Theorem 1:

Theorem 2. Let $D$ be a finite distributive lattice. Then there is a rectangular
lattice $K$ such that

$$D \cong \text{Con} K.$$ 

In this note we present a short proof of this result.

2. Notation

We use the standard notation, see [3].

For a rectangular lattice $L$, we use the notation $C_{ll} = \text{id}(u_l)$, $C_{ul} = \text{fil}(u_l)$,
$C_{lr} = \text{id}(u_r)$, $C_{ur} = \text{fil}(u_r)$ for the four boundary chains; if we have to specify the

\text{Date}: March 30, 2013.
\text{2010 Mathematics Subject Classification}. Primary: 06B10. Secondary: 06A06.
\text{Key words and phrases}. principal congruence, order, semimodular, rectangular.

The second author was supported by the Hungarian National Foundation for Scientific Research
(OTKA), grant no. K77432.

1
lattice $L$, we write $C_{il}(L)$, and so on. (See G. Czédli and G. Grätzer [1] for a survey of semimodular lattices, in general, and rectangular lattices, in particular.)

**Figure 1.** The $M_3$-grid for $n = 3$ and the lattice $S_8$

![Diagram of M_3-grid and S_8](image1)

**Figure 2.** A sketch of the lattice $K_i$ for $n \geq 3$ and $3 < i \leq e$

![Diagram of K_i](image2)

**3. Proof**

Let $D$ be the finite distributive lattice of Theorem 2. Let $P = Ji \ D$. Let $n$ be the number of elements in $P$ and $e$ the number of coverings in $P$.

We shall construct a rectangular lattice $K$ representing $D$ by induction on $e$. Let $m_i \prec n_i$, for $1 \leq i \leq e$, list all coverings of $P$. Let $P_j$, for $0 \leq j \leq e$, be the
order we get from $P$ by removing the coverings $m_i \prec n_i$ for $j < i \leq e$. Then $P_0$ is an antichain and $P_e = P$.

For all $0 \leq i \leq e$, we construct a rectangular lattice $K_i$ inductively. Let $K_0 = C_{n+1}^2$ be a grid, in which we replace the covering squares of the main diagonal by covering $M_3$-s; see Figure 1 for $n = 3$. Clearly, this lattice is rectangular and $\text{Con} K_0$ is the boolean lattice with $n$ atoms.

Now assume that $K_{i-1}$ has been constructed. Let the three-element chain $0 \prec m_i \prec n_i$ be represented by the lattice $S_8$, see Figure 1.

Take the four lattices $S_8$, $K_{i-1}$, $C_3 \times C_{ui}(K_{i-1}), C_{ur}(K_{i-1}) \times C_3$ and put them together as in Figure 2, where we sketch $K_{i-1}$ for $n \geq 3$ and $3 < i \leq e$. We add two more elements to turn two covering squares into covering $M_3$-s, see Figure 2, so that the prime interval of $S_8$ marked by $m$ defines the same congruence as the prime interval of $K_{i-1}$ marked by $m$; and the same for $n$. Let $K_i$ be the lattice we obtain. The reader should have no trouble to directly verify that $K_i$ is a rectangular lattice. (See G. Czédli and G. Grätzer [1] for general techniques that could be employed.)

The lattice $K$ for Theorem 2 is the lattice $K_e$.


References


Department of Mathematics, University of Manitoba, Winnipeg, MB R3T 2N2, Canada

E-mail address, G. Grätzer: gratzer@me.com
URL, G. Grätzer: http://server.maths.umanitoba.ca/homepages/gratzer/

Mathematical Institute of the Budapest University of Technology and Economics, H-1521 Budapest, Hungary

E-mail address, E. T. Schmidt: schmidt@math.bme.hu