Exam, 2017 05 10 – Solutions

1. In a box there are 2 red and 3 blue balls. You pick balls, one after the other with replacement. You stop picking when the first time a blue is drawn. Let X mean the number of draws you make.

   Question (a): Set up the formula of the weight function of the distribution of X.

   Answer:

   X follows the optimistic geometric distribution with parameter $p = \frac{3}{5}$:

$$p(x) = \text{P}(X = x) = \text{P}(\text{ first } (x-1) \text{ red and then 1 blue }) =$$

$$= \text{P}(\ (x-1) \text{ red } \ ) \cdot \text{P}(\text{ 1 blue }) = \left(\frac{2}{5}\right)^{x-1} \cdot \frac{3}{5} \qquad (x = 1, 2, \ldots)$$

   This is the optimistic geometric distribution with parameter $p = \frac{3}{5}$.

   Question (b): If you made 1000 experiments, approximately how much would be the average of the number of draws?

   Answer:

   The average of the number of draws is approximately equal to the expected value, which – for the optimistic geometric distribution – is equal to the reciprocal of the parameter:

$$\text{E}(X) = \frac{1}{p} = \frac{5}{3}$$

2. The number of mobile phones ringing during a theatre performance is a random variable.

   Question (a): Explain why this random variable follows Poisson distribution.

   Answer:

   The number of people with mobile phones is large. Each mobile phone may ring during a theatre performance with a small probability. They ring or do not independently of each other.

   Question (b): Assume that the probability that no mobile phones ring during a theatre performance is 0.6. What is the average number of mobile phones ringing during a theatre performance?

   Answer:

   The weight function of the Poisson distribution is

$$p(x) = \frac{\lambda^x}{x!} e^{-\lambda} \qquad (x = 0, 1, 2, \ldots)$$

   Since the probability that no mobile phones ring during a theatre performance is 0.6, we get that

$$p(0) = e^{-\lambda} = 0.6$$

   Solving this equation for λ, we get:

$$\lambda = -\ln(0.6) = 0.51$$

   The average number – in other words – the expected value of the Poisson distribution is equal to the parameter $\lambda = 0.51 \approx 0.5$.

3. Assume that the amount of milk in a bottle sold in a supermarket has a normal distribution with expectation 1 liter and standard deviation 0.01 liter.

   Question (a): Out of 1000 bottles approximately how many contain more than 1.02 liter of milk?

   Answer:

$$\text{P(a bottle contains more than 1.02 liter of milk)} = 1 - \text{NORM.DIST(1.02 ; 1 ; 0.01 ; TRUE)} = 0.023$$

   Out of 1000 bottles approximately 23 contain more than 1.02 liter of milk.

   Question (b): Determine the probability that out of 5 such bottles more than 2 contain less than 0.99 liter of milk.

Answer:

$$\text{P(a bottle contains less than 0.99 liter of milk)} = \text{NORM.DIST(0.99 ; 1 ; 0.01 ; TRUE)} = 0.16$$

$$\text{P(out of 5 bottles more than 2 contain less than 0.99 liter of milk)} =$$
$$= 1 - \text{BINOM.DIST(2 ; 5 ; 0.16 ; TRUE)} = 0.032$$

4. X is a random variable with values between $-\infty$ and ∞. The density function of X is $f(x) = \mathrm{e}^{-2|x|}$.

   Question (a): What is the probability that $-1 < X < 1$?

   Answer:

$$\mathrm{P}(-1 < X < 1) = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \mathrm{e}^{-2|x|}\,dx = 2 \cdot \int_{0}^{1} \mathrm{e}^{-2x}\,dx \qquad (= 0.86)$$

   Question (b): Determine the expected value of of X^2.

   Answer:

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot \mathrm{e}^{-2|x|}\,dx = 2\int_{0}^{\infty} x^2 \cdot \mathrm{e}^{-2x}\,dx \qquad \left(= \ldots \textit{integration by parts} \ldots = \frac{1}{2}\right)$$

5. The weight and the height of a randomly chosen woman – as a two-dimensional random variable – follows a normal distribution. The standard deviation of the weight of women with a height of 175 centimeters is 4 kgs. The correlation coefficient is 0.8.

   Question (a): How much is the standard deviation of the weight of women?

   Answer:

   Using the rule

$$\text{conditional standard deviation} = \text{unconditional standard deviation} \cdot \sqrt{1 - r^2}$$

   we get that

$$\text{unconditional standard deviation} = \frac{4}{\sqrt{1 - 0.8^2}} = \frac{20}{3} = 6.67$$

   Question (b): How much is the standard deviation of the weight of women who are 165 centimeters tall?

   Answer:

   Since the conditional standard deviation does not depend on the condition, the answer is simply 4 kgs.

6. **Question (a):**

   Give the meaning of the standard deviation of the data set $\{1; 3; 7; 8; 11\}$ by making simple calculations (without using calculator). *(Show the details of your calculations.)*

   Answer:

   The average of the numbers $1,\ 3,\ 7,\ 8,\ 11$ is obviously 6 . Taking the average of the squared differences $(1-6)^2,\ (3-6)^2,\ (7-6)^2,\ (8-6)^2,\ (11-6)^2$, we get the variance. Then, taking the square root, we get the standard deviation:

$$\sqrt{\frac{(1-6)^2 + (3-6)^2 + (7-6)^2 + (8-6)^2 + (11-6)^2}{5}} \qquad (= 3.58)$$

   Question (b):

   Give the meaning of the standard deviation of a continuous random variable by a correct mathematical formula.

   Answer:

   Expected value:

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

   Variance:

$$\mathrm{VAR}(X) = \int_{-\infty}^{\infty} (x - \mathrm{E}(X))^2 \cdot f(x)\,dx$$

   Standard deviation:

$$\mathrm{SD}(X) = \sqrt{\mathrm{VAR}(X)}$$