An interior-point solver for convex separable block-angular problems

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Outline

Block-angular problems

Modelling tool

- Multiperiod, multicommodity problems.
- Stochastic problems (two-stage, multi-stage optimization).
- Linking constraints.

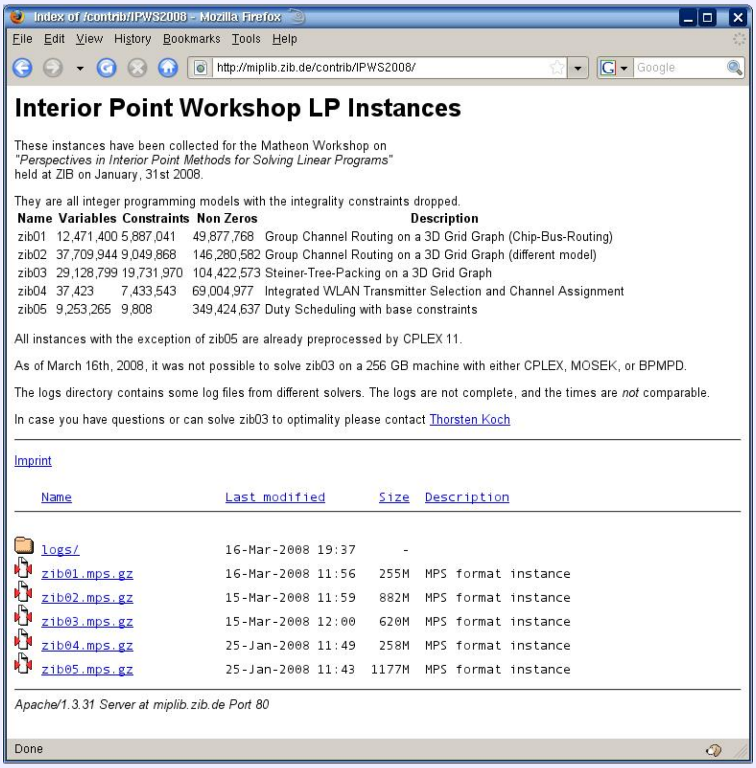
Applications

- Energy
- Logistics
- Telecommunications
- Big-data.

Size

- Very large-scale problems

IPMs successful for very large-scale problems...



Index of /contrib/IPWS2008 - Mozilla Firefox

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http://miplib.zib.de/contrib/IPWS2008/

Interior Point Workshop LP Instances

These instances have been collected for the Matheon Workshop on
"Perspectives in Interior Point Methods for Solving Linear Programs"
held at ZIB on January, 31st 2008.

They are all integer programming models with the integrality constraints dropped.

Name	Variables	Constraints	Non Zeros	Description
zib01	12,471,400	5,887,041	49,877,768	Group Channel Routing on a 3D Grid Graph (Chip-Bus-Routing)
zib02	37,709,944	9,049,868	146,280,582	Group Channel Routing on a 3D Grid Graph (different model)
zib03	29,128,799	19,731,970	104,422,573	Steiner-Tree-Packing on a 3D Grid Graph
zib04	37,423	7,433,543	69,004,977	Integrated WLAN Transmitter Selection and Channel Assignment
zib05	9,253,265	9,808	349,424,637	Duty Scheduling with base constraints

All instances with the exception of zib05 are already preprocessed by CPLEX 11.

As of March 16th, 2008, it was not possible to solve zib03 on a 256 GB machine with either CPLEX, MOSEK, or BPMPD.

The logs directory contains some log files from different solvers. The logs are not complete, and the times are *not* comparable.

In case you have questions or can solve zib03 to optimality please contact Thorsten Koch

Imprint

Name	Last modified	Size	Description
logs/	16-Mar-2008 19:37	-	
zib01.mps.gz	16-Mar-2008 11:56	255M	MPS format instance
zib02.mps.gz	15-Mar-2008 11:59	882M	MPS format instance
zib03.mps.gz	15-Mar-2008 12:00	620M	MPS format instance
zib04.mps.gz	25-Jan-2008 11:49	258M	MPS format instance
zib05.mps.gz	25-Jan-2008 11:43	1177M	MPS format instance

Apache/1.3.31 Server at miplib.zib.de Port 80

Done

... but some problems too-large for standard IPMs

Specialized vs standard IPMs

- Standard IPMs (CPLEX, XPRESS, MOSEK...) rely on Cholesky
- Specialized IPMs use PCG for systems of equations.
- Preconditioners are instrumental for efficiency.

Some preconditioners in IPMs

- Splitting preconditioners (Oliveira, Sorensen, 2005; Frangioni, Gentile 2004)
- Constraints preconditioners (Keller, Gould, Wathen 2000; Gondzio et al. 2007; Gondzio 2012)
- Partial Cholesky (Bellavia et al. 2013)
- IPM converge even if systems solved approximately (Gondzio 2013)

Formulation of block-angular problems

For convex separable problems (f_i convex separable)

$$
\begin{array}{ll}
\min & \sum_{i=0}^{k} f_i(x^i) \\
\text{subject to} & \begin{bmatrix} N_1 & & & \\ & \ddots & & \\ & & N_k & \\ L_1 & \dots & L_k & I \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^k \\ x^0 \end{bmatrix} = \begin{bmatrix} b^1 \\ \vdots \\ b^k \\ b^0 \end{bmatrix} \\
& 0 \le x^i \le u^i \quad i = 0, \dots, k.
\end{array}
$$

Particular cases

- Linear: $f_i(x^i) = {c^i}^\top x^i$
- Quadratic: $f_i(x^i) = {c^i}^\top x^i + \frac{1}{2}{x^i}^\top Q_i x^i$, Q_i diagonal

Approaches

- Dantzig-Wolfe, cutting planes
- But IPMs can also be used...

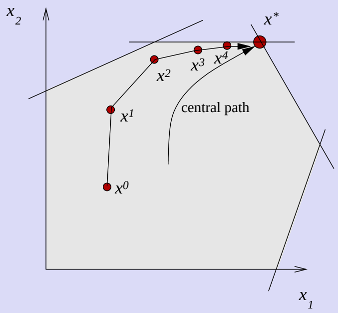
A path-following method

Convex optimization problem

$$(P)\quad \begin{array}{ll} \min & f(x) \\ \text{s.to} & Ax = b \qquad [\lambda] \\ & 0 \le x \le u \qquad [z,w] \end{array}$$

Central path defined by perturbed KKT-μ system

$$\begin{array}{rcl} A^\top\lambda + z - w - \nabla f(x) & = & 0 \\ Ax & = & b \\ (XZe, SWe) & = & (\mu e, \mu e) \quad \mu \in \mathbb{R}^+ \\ (z,w) > 0 & & (x,s) > 0 \quad s = u - x \end{array}$$



The linear algebra of IPMs

Augmented system

PCG-based IPMs usually solve the augmented system:

$$\begin{bmatrix} -\Theta^{-1} & A^\top \\ A & 0 \end{bmatrix}$$

Normal equations

BlockIP solves normal equations

$$(A\Theta A^\top)\Delta\lambda = g$$

where

$$\Theta = (ZX^{-1} + WS^{-1} + \nabla^2 f(x))^{-1}$$

is a diagonal matrix if problem is separable.

Solving normal equations

Exploiting structure of A and Θ

$$A = \begin{bmatrix} N_1 & & & \\ & \ddots & & \\ & & N_k & \\ L_1 & \dots & L_k & I \end{bmatrix} \qquad \Theta = \begin{bmatrix} \Theta_1 & & & \\ & \ddots & & \\ & & \Theta_k & \\ & & & \Theta_0 \end{bmatrix}$$

$$A\Theta A^\top = \left[\begin{array}{ccc|c} N_1\Theta_1 N_1^\top & & & N_1\Theta_1 L_1^\top \\ & \ddots & & \vdots \\ & & N_k\Theta_k N_k^\top & N_k\Theta_k L_k^\top \\ \hline L_1\Theta_1 N_1^\top & \dots & L_k\Theta_k N_k^\top & \Theta_0 + \sum_{i=1}^k L_i\Theta_i L_i^\top \end{array}\right] = \begin{bmatrix} B & C \\ C^\top & D \end{bmatrix}$$

The Schur complement

$$\begin{bmatrix} B & C \\ C^\top & D \end{bmatrix} \begin{bmatrix} \Delta\lambda_1 \\ \Delta\lambda_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \Longleftrightarrow \begin{array}{rl} (D - C^\top B^{-1} C)\Delta\lambda_2 = & (g_2 - C^\top B^{-1} g_1) \\ B\Delta\lambda_1 = & (g_1 - C\Delta y_2) \end{array}$$

- System with B solved by k Cholesky factorizations.
- Schur complement $S = D - C^\top B^{-1} C$ with large fill-in: system solved by PCG.

The preconditioner

Based on P-regular splitting $S = D - (C^\top B^{-1} C)$ (SIOPT00,COAP07)

Spectral radius of $D^{-1}(C^\top B^{-1} C))$ satisfies $\rho(D^{-1}(C^\top B^{-1} C))) < 1$ and then

$$(D - C^\top B^{-1} C)^{-1} = \left(\sum_{i=0}^{\infty} (D^{-1}(C^\top B^{-1} C))^i \right) D^{-1}$$

Preconditioner M^{-1} obtained truncating the power series at term h

$$\begin{array}{rcll} M^{-1} & = & D^{-1} & \text{if } h = 0, \\ M^{-1} & = & (I + D^{-1}(C^\top B^{-1} C))D^{-1} & \text{if } h = 1. \end{array}$$

Quality of preconditioner depends on

- $\rho < 1$: the farther from 1, the better the preconditioner.

Non-zero Hessians improve the preconditioner I

Proposition. Upper bound for ρ (MP11)

The spectral radius ρ of $D^{-1}(C^\top B^{-1} C)$ is bounded by

$$\rho \leq \max_{j\in\{1,\dots,l\}} \frac{\gamma_j}{\left(\frac{r_j}{v_j}\right)^2 \Theta_{0j} + \gamma_j} < 1,$$

where r is the eigenvector of $D^{-1}(C^\top B^{-1} C)$ associated to ρ; $\gamma_j, j = 1,\dots,l$, and $V = [V_1,\dots,V_l]$, are the eigenvalues and matrix of eigenvectors of $\sum_{i=1}^k L_i \Theta_i {L_i}^\top$, and $v = V^\top r$.

If $L_i = I$ the bound has the simple and computable form:

$$\rho \leq \max_{j\in\{1,\dots,l\}} \frac{\sum_{i=1}^k \Theta_{ij}}{\Theta_{0j} + \sum_{i=1}^k \Theta_{ij}} < 1.$$

Non-zero Hessians improve the preconditioner II

Proposition. PCG more efficient for quadratic or nonlinear problems

Under some mild conditions, the upper bound of ρ decreases for $\nabla^2 f(x) \succ 0$.

Proposition. PCG extremely efficient if Hessian is large

$$\lim_{\substack{\nabla^2 f_i(x)\to+\infty \\ i=1,\dots,k}} \rho = 0$$

Example: solution of a large (10 million variables, 210000 constraints) with quadratic objective function $x^\top Qx$, for different $Q = \beta I$

		CPLEX-11		Specialized IPM			
Instance	β	it.	CPU	it.	PCG	CPU	f^*
CTA-100-100-1000	0.01	7	29939	10	36	66	-2.6715e+08
CTA-100-100-1000	0.1	7	31328	9	40	61	-2.6715e+09
CTA-100-100-1000	1	8	33367	8	38	56	-2.6715e+10
CTA-100-100-1000	10	9	35220	7	37	51	-2.6715e+11

Quadratic regularizations improve the preconditioner

Standard barrier, proximal-point and quadratic regularization

- $B(x,\mu) \triangleq f(x) + \mu\left(-\sum_{i=1}^n \ln x_i - \sum_{i=1}^n \ln(u_i - x_i)\right)$
- $B_P(x,\mu) \triangleq f(x) + \frac{1}{2}(x-\bar{x})^\top Q_P(x-\bar{x}) + \mu\left(-\sum_{i=1}^n \ln x_i - \sum_{i=1}^n \ln(u_i - x_i)\right)$
- $B_Q(x,\mu) \triangleq f(x) + \mu\left(\frac{1}{2}x^\top Q_R x - \sum_{i=1}^n \ln x_i - \sum_{i=1}^n \ln(u_i - x_i)\right)$

Regularization only affects to Θ matrices

$$\Theta = (ZX^{-1} + WS^{-1} + \nabla^2 f(x))^{-1} \quad \text{for } B$$
$$\Theta = (Q_P + ZX^{-1} + WS^{-1} + \nabla^2 f(x))^{-1} \quad \text{for } B_P$$
$$\Theta = (\mu Q_R + ZX^{-1} + WS^{-1} + \nabla^2 f(x))^{-1} \quad \text{for } B_Q$$

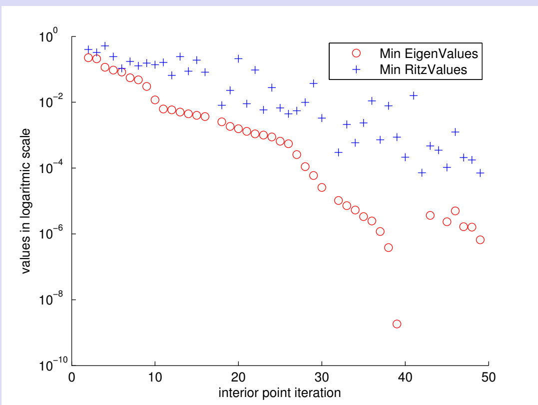
- μQ_R vanishes as we approach the solution, B_Q being equivalent to B.
- B_Q thus preferred to B_P.

Spectral radius ρ can be estimated from Ritz values

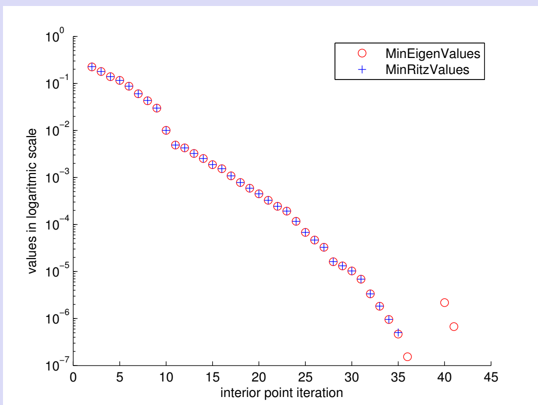
- Ritz values: eigenvalues of a certain tridiagonal matrix T_k associated to CG
- Proposition (EJOR 2013). For $h = 0$ (one term in preconditioner), if $r_{\min}$ is smallest Ritz value then ρ estimated as

$$1 - r_{\min}$$

$$T_k = \begin{bmatrix} \gamma_1 & \eta_2 & & & \\ \eta_2 & \gamma_2 & \eta_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \eta_{k-1} & \gamma_{k-1} & \eta_k \\ & & & \eta_k & \gamma_k \end{bmatrix}$$



loose PCG tolerance



tight PCG tolerance

Estimating spectral radius when $h > 0$

Proposition. Estimation of ρ (2014)

Let $M^{-1} = \left(\sum_{i=0}^{h} (D^{-1}(C^\top B^{-1} C))^i \right) D^{-1}$ be the preconditioner with h terms of the power series. And let $r_{\min}$ be the smallest Ritz value (easily computed).
Then the estimation of ρ is

$$\sqrt[h+1]{1 - r_{\min}}.$$

The BlockIP solver: some features

- Efficient implementation of the IPM for block-angular problems.
- For LO, QO, or CO problems.
- Problems in standard or general form.
- Uses Ng-Peyton Sparse Cholesky package (room for improvement).
- Fully written in C++, about 14000 lines of code.
- Many options: computation Ritz values, quadratic regularizations,...
- Comes with different types of matrices: General, oriented and non-oriented Network, Identity, Diagonal, $[I \quad I]$, $[D_1 \quad D_2]$.
  - Easy addition of other types of matrices.
  - Extension to Matrix-Free paradigm.

How to input a problem? 1. Callable library

The most efficient option

Example

```
...
// declare N (block constraints matrix) as a Matrix for BlockIP
MatrixBlockIP N;
// declare arc source and destination vectors
int *srcN, *dstN;
// N is created as network matrix
N.create_network_matrix(numArcs, numNodes, srcN, dstN);
// fill srcN and dstN; srcN and dstN allocated by create_network_matrix()
...
// declare L (linking constraints matrix) as a Matrix for BlockIP
MatrixBlockIP L;
// L is created as an identity matrix
L.create_identity_matrix(numArcs);

BlockIP bip; // declare BlockIP problem

double *cost, *qcost, *ub, *rhs;
// creation of BlockIP problem
bip.create_problem(BlockIP::QUADRATIC, cost, qcost, NULL, NULL, ub, rhs,
                   numBlocks, true, &N, true, &L);
// fill cost, qcost, ub, rhs ...
```


How to input a problem? 2. Input file in BlockIP format

Efficient format: vectors and sparse matrices

Example

```
#typeobj 0=linear 1=quadratic 2=nonlinear
1
#number of blocks
2
#sameN 1=yes 0=no
1
#Matrix:  first line m,n,nnz; next nnz lines i,j,a
3 5 7
1 1 1
1 2 1
1 3 1
2 1 -1
2 4 1
3 2 -1
3 5 1
...
```

How to input a problem? 3. Input file in Structured MPS

MPS extension for block-angular problems developed for BlockIP

Example

```
ROWS
E Block1:Cons1
...
E LinkCons1
...
COLUMNS
Block1:Var1 obj 1 Block1:Cons1 1
...
Slack1 LinkCons1 1
```


How to input a problem? 4. SML (Grothey et al. 2009)

- AMPL extension for structured problems.
- SML extended to separable nonlinear problems for BlockIP.

Example (multicommodity transportation problem)

```
block Prod{p in PROD}:
  var Trans {ORIG, DEST} >= 0; # units to be shipped
  minimize total_cost:
         sum {i in ORIG, j in DEST} cost[p,i,j] * Trans[i,j];
  subject to Supply {i in ORIG}:
         sum {j in DEST} Trans[i,j] = supply[p,i];
  subject to Demand {j in DEST}:
         sum {i in ORIG} Trans[i,j] = demand[p,j];
end block;
subject to Multi {i in ORIG, j in DEST}:
         sum {p in PROD} Prod[p].Trans[i,j] <= limit[i,j];
```

LP Multicommodity flow problems

Formulation

$$\min \quad \sum_{i=1}^{k} {c^i}^\top x^i$$

$$\text{s. to} \quad \begin{bmatrix} N & & & \\ & \ddots & & \\ & & N & \\ I & \dots & I & I \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^k \\ s \end{bmatrix} = \begin{bmatrix} b^1 \\ \vdots \\ b^k \\ u \end{bmatrix}$$

$$0 \le x^i \le u^i \quad i = 1, \dots, k, \qquad 0 \le s \le u.$$

N is network matrix, I is identity, u arc mutual capacities, x^i flows per commodity, s slacks of capacity constraints:

Results for some "small" difficult instances

Problem dimensions

Instance	k	constraints	variables
tripart1	16	3294	33774
tripart2	16	13301	135941
tripart3	20	25541	329161
tripart4	35	38004	869814
gridgen1	320	329831	985191

Computational results

	BlockIP			CPLEX 12.5	
Instance	Iter	CPU	PCG	Iter	CPU
tripart1	51	0.8	1260	19	0.3
tripart2	68	10	4034	17	4
tripart3	78	20	3363	19	13
tripart4	131	268	20791	24	34
gridgen1	199	253	4790	33	883

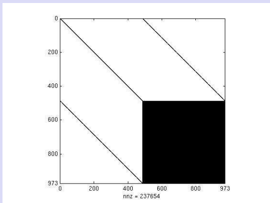
The minimum congestion problem

Goal: to make feasible a nonoriented multiflow problem

Minimize $\|y\|_\infty$, y is the vector of relative increments in arc capacities.

$$
\begin{array}{rlr}
\min & z & \\
\text{subject to} & Nx^{i^+} - Nx^{i^-} = b^i & i = 1,\ldots,k \\
& \sum_{i=1}^{k} (x_j^{i^+} + x_j^{i^-}) - y_j u_j \leq 0 & j = 1,\ldots,n \\
& y_j - z \leq 0 & j = 1,\ldots,n \\
& x^{i^+}, x^{i^-} \geq 0 & i = 1,\ldots,k \\
& y_j \geq 0 & j = 1,\ldots,n
\end{array}
$$

Dense column for variable z, matrix D of preconditioner is very dense

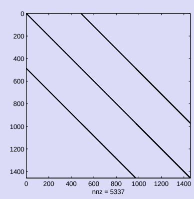


The minimum congestion problem: efficient formulation

Extra variables $z_i, i = 1,\ldots,n$, but no dense column

$$
\begin{array}{rlr}
\min & z_1 & \\
\text{subject to} & Nx^{i^+} - Nx^{i^-} = b^i & i = 1,\ldots,k \\
& \sum_{i=1}^{k} (x_j^{i^+} + x_j^{i^-}) - y_j u_j \leq 0 & j = 1,\ldots,n \\
& y_j - z_j \leq 0 & j = 1,\ldots,n \\
& z_j - z_{j+1} = 0 & j = 1,\ldots,n-1 \\
& x^{i^+}, x^{i^-} \geq 0 & i = 1,\ldots,k \\
& y_j \geq 0 & j = 1,\ldots,n
\end{array}
$$

Matrix D of the preconditioner of larger dimension but sparser



Results with efficient formulation

Problem dimensions

Instance	k	constraints	variables
M32-32	34	2449	33533
M64-64	66	5564	67962
M128-64	66	11640	155742
M128-128	130	19867	314243
M256-256	258	71891	1139467
M512-64	66	470075	634143
M512-128	130	79765	1249145

Computational results

	BlockIP			CPLEX 12.5	
Instance	Iter	CPU	PCG	Iter	CPU
M32-32	93	0.9	289	17	1.3
M64-64	94	2	183	17	4
M128-64	97	7	234	19	22
M128-128	97	15	213	20	52
M256-256	110	161	891	22	627
M512-64	131	95	1223	21	1071
M512-128	131	244	2090	25	2520

Minimum Distance Controlled Tabular Adjustment

Statistical table

- Vector $a \in \mathbb{R}^n$ of n cells.
- Satisfies constraints: $Aa = b, l_a \leq a \leq u_a$.

Goal: to find cell perturbations $x \in \mathbb{R}^n$ such that

- Minimizes $||x||_\ell$ for some distance ℓ
- Satisfies $A(x+a) = b,\ l_a \leq x+a \leq u_a \Longleftrightarrow Ax = 0,\ l \leq x \leq u$
- Satisfies protection requirements: $\alpha_i \leq x_i \leq \beta_i \quad i \in \mathcal{S} \subseteq \{1,\dots,n\}$, $0 \notin [\alpha_i, \beta_i]$.

Optimization problem

$$\begin{array}{ll} \min\limits_{x} & ||x||_\ell \\ \text{s. to} & Ax = 0 \\ & l \leq x \leq u \\ & \alpha_i \leq x_i \leq \beta_i \quad i \in \mathcal{S} \end{array}$$

Block-angular structure of 3D tables: cube/box of data

Example: Profession × County × Sex

A 2D table for each sex, plus a third table for totals

Males

	C1	C2	C3	Sums
P1	230	210	250	690
P2	180	190	210	580
P3	150	140	160	450
Sums	560	540	620	1720

Females

	C1	C2	C3	Sums
P1	184	168	200	552
P2	144	152	168	464
P3	120	112	128	360
Sums	448	432	496	1376

Sums

	C1	C2	C3	Sums
P1	414	378	450	1242
P2	324	342	378	1044
P3	270	252	288	810
Sums	1008	872	1116	3096

Different problems for three distances

Linear Problem: $\nabla^2 f(x) = 0$, twice the number of variables

$$\|x\|_{\ell_1} = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} (x_i^+ + x_i^-)$$

Quadratic Problem: $\nabla^2 f(x) = 2I$

$$\|x\|_{\ell_2}^2 = \sum_{i=1}^{n} x_i^2$$

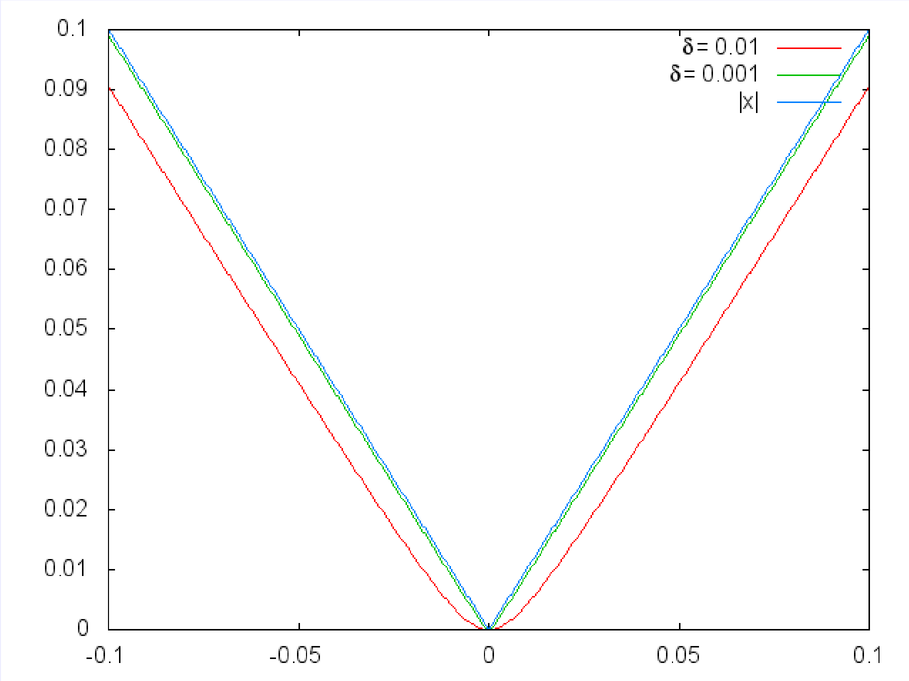
Nonlinear Problem: $\nabla^2 f(x) \succ 0$

$$\|x\|_{\ell_1} = \sum_{i=1}^{n} |x_i| \approx \sum_{i=1}^{n} \phi_\delta(x_i)$$

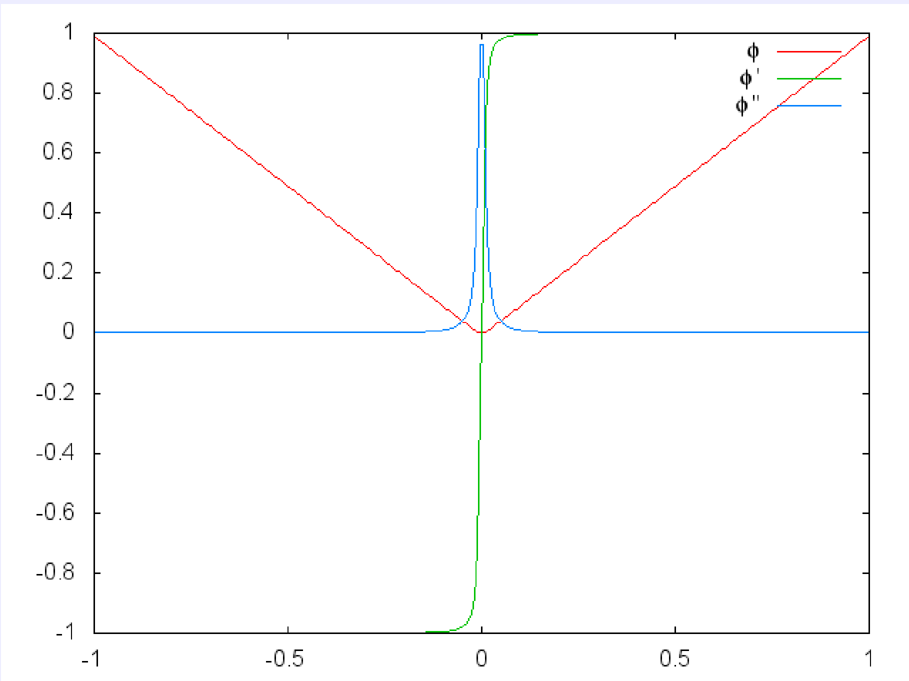
Pseudo-Huber function ϕ_δ approximates absolute value

$$\phi_\delta(x_i) = \sqrt{\delta^2 + x_i^2} - \delta \qquad \delta \approx 0$$

Plots of and $|x|$ and ϕ for some δ



Plots of ϕ, ϕ' and ϕ'' for $\delta = 0.01$



Results for ℓ_1

Instance	Dimensions constraints	Dimensions variables	BlockIP CPU	CPLEX 12.5 CPU
25-25-25	1850	31875	4	1
25-25-50	3075	63125	12	2
25-50-25	3100	63750	19	2
25-50-50	4950	126250	61	10
50-25-25	3100	63750	28	1
50-25-50	4950	126250	1	7
50-50-25	4975	127500	33	9
50-50-50	7450	252500	16	41
100-100-100	29900	2010000	8	986
100-100-200	49800	4010000	25	2262
200-100-200	79800	8020000	49	8789
200-200-200	119800	16040000	144	64521
500-500-50	299950	25250000	424	19595
500-50-500	299500	25025000	227	17415

Results for ℓ_2

Instance	Dimensions constraints	Dimensions variables	BlockIP CPU	CPLEX 12.5 CPU
25-25-25	1850	16250	0.0	0.8
25-25-50	3075	31875	0.1	1.4
25-50-25	3100	32500	0.1	1.2
25-50-50	4950	63750	0.1	5.8
50-25-25	3100	32500	0.1	1.2
50-25-50	4950	63750	0.1	4.2
50-50-25	4975	65000	0.1	5.1
50-50-50	7450	127500	0.2	19
100-100-100	29900	1010000	3	874
100-100-200	49800	2010000	6	1802
200-100-200	79800	4020000	11	7319
200-200-200	119800	8040000	29	65467
500-500-50	299950	12750000	91	15437
500-50-500	299500	12525000	28	14784

Results for pseudo-Huber in small instances

Pseudo-Huber more efficient since $\nabla^2 f \succ 0$

	Dimensions		BlockIP		BlockIP ℓ_1	
Instance	const.	variables	CPU	PCG	CPU	PCG
25-25-25	1850	16250	1	3285	4	16475
25-25-50	3075	31875	2	2940	12	22430
25-50-25	3100	32500	2	2525	19	34863
25-50-50	4950	63750	5	4658	61	57641
50-25-25	3100	32500	2	2404	28	53667
50-25-50	4950	63750	4	4392	1	526
50-50-25	4975	65000	4	3298	33	28669
50-50-50	7450	127500	6	1831	16	5523

- Other state-of-the-art convex solvers could not solve these instances.
- Larger instances neither could be solved with BlockIP.

Other applications under consideration

Routing in telecommunications networks

- Nonoriented multicommodity network.
- Many OD pairs
- Nonlinear Kleinrock delay function
- Already implemented: good results. Work in progress.

Transportation assignment problem in urban networks

- Similar to routing in telecommunications networks.
- Many OD pairs
- Nonlinear BPR (Bureau of Public Roads) function.
- To be tested soon.

Conclusions

- IP solver for block-angular problems.
- Shown to be very efficient for some applications.
- Many future applications to be tried.
- Soon available for research purposes from its web page.

Temporarily available from `www-eio.upc.edu/~jcastro/BlockIP.html`

Some references about the IPM and applications

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Thanks for your attention