On a Cohn's Embedding Theorem[1]

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Abstract

In 1956, P.M. Cohn gave necessary and sufficient conditions for a semigroup to be embeddable into a left simple semigroup. The conditions differ essentially according to whether or not the semigroup contains an idempotent element. The main purpose of our present paper is to show how to construct idempotent-free semigroups which can be embedded into left simple semigroups.

1 Introduction and motivation

In his paper [3], P.M. Cohn gave necessary and sufficient conditions for a semigroup to be embeddable into a left simple semigroup. He proved the following theorem.

Theorem 1.1 *([3, Theorem 1]) A semigroup S is embeddable into a left simple semigroup if and only if one of the two following sets of conditions holds:*

(i) *(a) for all $x, y, u, v \in S$, $ux = uy$ implies $vx = vy$,*

(b) S contains no idempotent.

(ii) *(a) the same as (i) (a),*

(b) S contains an idempotent, e say,

(c) the set eS is embeddable in a group,

(d) if in a chain $a_0, a_1, \ldots, a_n$ of elements of S, we have $ea_0 = ea_n$, and $a_{i-1}S \cap a_iS \neq \emptyset$ $(i = 1, \ldots, n)$, then $a_0 = a_n$.

In case (i) the embedding left simple semigroup can be chosen so as to contain no idempotent. □

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Using the terminology of [7], a semigroup S satisfying condition (i) (a) of the Cohn's embedding theorem is called a left equalizer simple semigroup. From the Cohn's embedding theorem it follows (see also the dual of Theorem 8.19 of [2]) that a semigroup S is embeddable into an idempotent-free left simple semigroup if and only if S is an idempotent-free left equalizer simple semigroup. In [7], a complete description of left equalizer simple semigroups are given. It is proved that a semigroup is left equalizer simple if and only if it is isomorphic to a semigroup defined in Construction 1 of [7]. This result together with part (i) of Cohn's embedding theorem motivate us to show how to construct all idempotent-free left equalizer simple semigroups, giving all semigroups which can be embedded into idempotent-free left simple semigroups.

2 Preliminaries

By a semigroup we shall mean a multiplicative semigroup, that is, a non-empty set together with an associative multiplication.

A non-empty subset L of a semigroup S is called a *left ideal* of S if $sl \in L$ for every $s \in S$ and $l \in L$. A semigroup is called a *left simple semigroup* if it does not properly contain any left ideal. It is known that a semigroup S is left simple if and only if $Sa = S$ is satisfied for every $a \in S$.

By [7, Definition 2.1], a semigroup S is called a *left equalizer simple semigroup* if it satisfies the following condition: for every $a, b \in S$, the assumption that $ua = ub$ is satisfied for some $u \in S$ implies that $va = vb$ is satisfied for all $v \in S$.

A semigroup S is called a *left cancellative semigroup* if, for every $x, a, b \in S$, the equation $xa = xb$ implies $a = b$. It is clear that every left cancellative semigroup is left equalizer simple. Applying [6, Theorem 2] for a special case, it is shown in [7] that all left equalizer simple semigroups can be obtained by a construction based on left cancellative semigroups ([7, Construction 1]). This construction play an important role in our investigation. Thus we cite it here.

Construction 2.1 *([7, Construction 1]) Let T be a left cancellative semigroup. For each $t \in T$, associate a nonempty set S_t such that*

$$S_t \cap S_r = \emptyset$$

is satisfied for every $t, r \in T$ with $t \neq r$.

For arbitrary couple $(t,r) \in T \times T$ with $r \in tT$, let $(\cdot)\varphi_{t,r}$ be a mapping of S_t into S_r. (As T is left cancellative, $x \mapsto tx$ is an injective mapping of T onto tT.) For all $t \in T$, $r \in tT$, $q \in rT \subseteq tT$ and $a \in S_t$, assume

$$(a)\left(\varphi_{t,r} \circ \varphi_{r,q}\right) = (a)\varphi_{t,q}. \tag{1}$$

On the set

$$S = \cup_{t \in T} S_t,$$

define an operation $\star$ as follows: for arbitrary $a \in S_t$ and $b \in S_x$, let

$$a \star b = (a)\varphi_{t,tx}. \tag{2}$$

If $a \in S_t$, $b \in S_x$, $c \in S_y$ are arbitrary elements then

$$a \star (b \star c) = a \star (b)\varphi_{x,xy} = (a)\varphi_{t,t(xy)} =$$

$$= (a)\left(\varphi_{t,tx} \circ \varphi_{tx,t(xy)}\right) = (a)\varphi_{t,tx} \star c = (a \star b) \star c.$$

Thus the operation $\star$ is associative, and so $S = \cup_{t \in T} S_t$ is a semigroup under the operation $\star$. □

The semigroup S defined in the above construction will be denoted by $(S; T, S_t, \varphi_{t,r}, \star)$.

For notations and notions not defined in this paper, we refer to the books [1], [4] and [5].

3 Semigroups which can be embedded into idempotent-free left simple semigroups

In this section we show how to construct semigroups which are embeddable into idempotent-free left simple semigroups. First we cite a theorem on left equalizer simple semigroups proved in [7] which will be used in the proof of Theorem 3.3.

Theorem 3.1 *([7, Theorem 2.2]) A semigroup is left equalizer simple if and only if it is isomorphic to a semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ defined in Construction 2.1.* □

It is known that, for an arbitrary semigroup S, the relation θ on S defined by
$$\theta = \{(a,b) \in S \times S : \ (\forall s \in S) \ sa = sb\}$$
is a congruence on S. By [6, Theorem 2], the θ-classes of the semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ defined in Construction 2.1 are the sets S_t $(t \in T)$. As $S_t \star S_r \subseteq S_{tr}$ for every $t, r \in T$, the mapping $t \mapsto S_t$ $(t \in T)$ is an isomorphism of the semigroup T onto the factor semigroup $(S; T, S_t, \varphi_{t,r}, \star)/\theta$. This fact will be used in the paper several times.

Theorem 3.2 *A semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ contains an idempotent element if and only if the left cancellative semigroup T has an idempotent element.*

Proof. Assume that the semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ contains an idempotent element e. If $e \in S_x$ $(x \in T)$, then S_x is a subsemigroup of $(S; T, S_t, \varphi_{t,r}, \star)$, because S_x is a θ-class of $(S; T, S_t, \varphi_{t,r}, \star)$. As $t \mapsto S_t$ $(t \in T)$ is an isomorphism of the semigroup T onto the factor semigroup $(S; T, S_t, \varphi_{t,r}, \star)/\theta$, x is an idempotent element of T.

Conversely, assume that the semigroup T has an idempotent element x. Then $S_x \star S_x \subseteq S_{x^2} = S_x$ and so S_x is a subsemigroup of the semigroup $(S; T, S_t, \varphi_{t,r}, \star)$. Thus, for every $a, b \in S_x$, we have
$$a \star b = (a)\varphi_{x,x}.$$

Let $a \in S_x$ be an arbitrary element. As S_x is a subsemigroup of the semigroup $(S; T, S_t, \varphi_{t,r}, \star)$, we have
$$a^2 \star a^2 = a \star a^3 = (a)\varphi_{x,x} = a \star a = a^2,$$
that is, a^2 is an idempotent element of $(S; T, S_t, \varphi_{t,r}, \star)$. □

In the next we prove our main theorem which show how to construct semigroups which can be embedded into idempotent-free left simple semigroups.

Theorem 3.3 *A semigroup is embeddable into an idempotent-free left simple semigroup if and only if it is isomorphic to a semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ defined in Construction 2.1, where the left cancellative semigroup T is idempotent-free.*

Proof. By Cohn's embedding theorem (Theorem 1.1), a semigroup S is embaddable into an idempotent-free left simple semigroup if and only if it is idempotent-free and left equalizer simple. By Theorem 3.1, a semigroup is left equalizer simple if and only if it is isomorphic to a semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ defined in Construction 2.1. Thus a semigroup is embeddable into an idempotent-free left simple semigroup if and only if it is isomorphic to an idempotent-free semigroup $(S; T, S_t, \varphi_{t,r}, \star)$. By Theorem 3.2, a semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ is idempotent-free if and only if the left cancellative semigroup T is idempotent-free. Thus the theorem is proved. $\square$

In the next we show how to construct semigroups satisfying special permutation identity which are embeddable into idempotent-free left simple semigroups.

Let σ be a non-identity permutation of degree n. We say that a semigroup S is *σ-permutable* if, for every $s_1, s_2 \dots, s_n \in S$, we have

$$s_1 s_2 \cdots s_n = s_{\sigma(1)} s_{\sigma(2)} \cdots s_{\sigma(n)}.$$

For a non-identity permutation σ of degree n, we shall say that a semigroup S is *σ'-permutable* if, for every $s_0, s_1, s_2, \dots, s_n \in S$, the equation

$$s_0 s_1 s_2 \cdots s_n = s_0 s_{\sigma(1)} s_{\sigma(2)} \cdots s_{\sigma(n)}.$$

is satisfied.

Lemma 3.4 *Let n be an integer and σ a non-identity permutation of degree n. A semigroup S is σ'-permutable if and only if the factor semigroup S/θ is σ-permutable.*

Proof. Let $s_1, s_2 \dots s_n \in S$ be arbitrary elements of S. The equation

$$s_0 s_1 s_2 \cdots s_n = s_0 s_{\sigma(1)} s_{\sigma(2)} \dots s_{\sigma(n)}$$

is satisfied for all $s_0 \in S$ if and only if

$$\left(s_1 s_2 \cdots s_n, s_{\sigma(1)} s_{\sigma(2)} \dots s_{\sigma(n)}\right) \in \theta$$

which proves the assertion of the lemma. $\square$

Theorem 3.5 *A semigroup is σ'-permutable and can be embedded into an idempotent-free left simple semigroup if and only if it is isomorphic to a semigroup $(S; T, S_t, \varphi_{t,r}, \star)$, where the left cancellative semigroup T is idempotent-free and σ-permutable.*

Proof. By Theorem 3.3, a semigroup is embeddable into an idempotent-free left simple semigroup if an only if it is isomorphic to a semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ in which the left cancellative semigroup is idempotent-free. By [6, Theorem 2], the semigroup T is isomorphic to the factor semigroup $(S; T, S_t, \varphi_{t,r}, \star)/\theta$. By Lemma 3.4, the semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ is σ'-permutable if and only if the semigroup T is σ-permutable. Thus the theorem is proved. □

A semigroup is called a *left [right] commutative semigroup* if it satisfies the identity $abc = bac$ [$abc = acb$]. A semigroup is called a *medial semigroup* if it satisfies the identity $abcd = acbd$.

Corollary 3.6 *A semigroup is right commutative [resp. medial] and can be embedded into an idempotent-free left simple semigroup if and only if it is isomorphic to a semigroup $(S; T, S_t, \varphi_{t,r}, \star)$, where the left cancellative semigroup T is idempotent-free and commutative [resp. left commutative].*

Proof. A semigroup S is right commutative [resp. medial] if and only if the factor semigroup S/θ is commutative [resp. left commutative] by Lemma 3.4. Thus the assertion of the corollary follows from Theorem 3.5. □

4 Examples

In this section we present two examples for left equalizer simple semigroups $(S; T, S_t, \varphi_{t,r}, \star)$ defined in Construction 2.1. Both of them are right commutative. In the first example, T is an idempotent-free commutative cancellative semigroup and so the semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ is embeddable into an idempotent-free left simple semigroup. The second example differs from the first only in that T has an identity element (that is, T is a commutative cancellative monoid). We show that, under this additional condition, the semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ satisfies all subcases of case (*ii*) of Theorem 1.1 and therefore it is embeddable into a left simple semigroup containing idempotent elements.

Example 4.1 We use the notations of Construction 2.1. Let T be an idempotent-free commutative cancellative semigroup. For every $t \in T$, let

$$S_t = \{(t, a) : \ a \in T\}.$$

For every $t \in T$ and $r = tm \in tT$, let $(\cdot)\varphi_{t,r}$ be the mapping of S_t into S_r defined by

$$((t, a))\varphi_{t,r} = (tm, am), \quad ((t, a) \in S_t). \tag{3}$$

Let $t \in T$, $r = tm \in tT$ and $q = rk \in rT$ be arbitrary elements. Then, for an arbitrary element $(t, a) \in S_t$,

$$((t, a))(\varphi_{t,r} \circ \varphi_{r,q}) = (((t, a))\varphi_{t,r})\varphi_{r,q} = ((tm, am))\varphi_{r,q} =$$

$$= (tmk, amk) = ((t, a))\varphi_{t,q}.$$

Thus condition (1) of Construction 2.1 is satisfied for the family of mappings $\varphi_{t,r}$ $(t \in T, r \in tT)$ defined by (3). Hence $S = \cup_{t\in T} S_t$ is a subsemigroup under the operation $\star$ defined by

$$(t, a) \star (m, b) = ((t, a))\varphi_{t,tm} = (tm, am)$$

(see (2) in Construction 2.1). Denoting this semigroup by $(S; T, S_t, \varphi_{t,r}, \star)$, Theorem 3.1 and Theorem 3.2 together imply that $(S; T, S_t, \varphi_{t,r}, \star)$ is an idempotent-free left equalizer simple semigroup. Thus $(S; T, S_t, \varphi_{t,r}, \star)$ can be embedded into an idempotent-free left simple semigroup by Theorem 1.1. As T is a commutative semigroup, $(S; T, S_t, \varphi_{t,r}, \star)$ is a right commutative semigroup by Theorem 3.5.

Example 4.2 Consider the semigroup $S = (S; T, S_t, \varphi_{t,r}, \star)$ which is the same as in Example 4.1, except that T has an identity element (that is, T is a commutative cancellative monoid). It is clear that S is also left equalizer simple, and so it satisfies condition (ii) (a) of Theorem 1.1.

S has idempotent elements; the idempotent elements of S are exactly the elements $(1, a)$, $a \in T$, where 1 denotes the identity element of T. Thus condition (ii) (b) of Theorem 1.1 is satisfied for S.

For every idempotent element $(1, a)$ of S,

$$(1, a) \star S = \{(m, am)| \ m \in T\}.$$

It is a matter of checking to see that $(1, a) \star S$ is a commutative cancellative subsemigroup of S. Thus $(1, a) \star S$ can be embedded into a group. Hence S satisfies condition (ii) (c) of Theorem 1.1.

Let

$$(t_0, a_0), (t_1, a_1), \ldots, (t_n, a_n)$$

be a sequence of elements of S such that

$$(1, a) \star (t_0, a_0) = (1, a) \star (t_n, a_n) \tag{4}$$

and

$$(t_{i-1}, a_{i-1}) \star S \cap (t_i, a_i) \star S \neq \emptyset \quad \text{for every} \quad i = 1, \ldots n. \tag{5}$$

From (4) we get

$$(t_0, at_0) = (t_n, at_n)$$

and so

$$t_0 = t_n.$$

From (5) we get that there are elements (α_i, β_i) and (γ_i, δ_i) of S $(i = 1, \ldots, n)$ such that

$$(t_{i-1}, a_{i-1}) \star (\alpha_i, \beta_i) = (t_i, a_i) \star (\gamma_i, \delta_i),$$

that is,

$$(t_{i-1}\alpha_i, a_{i-1}\alpha_i) = (t_i\gamma_i, a_i\gamma_i).$$

Thus

$$t_{i-1}\alpha_i = t_i\gamma_i$$

and

$$a_{i-1}\alpha_i = a_i\gamma_i.$$

From this it follows that

$$t_0\alpha_1\alpha_2 = t_1\gamma_1\alpha_2 = t_1\alpha_2\gamma_1 = t_2\gamma_2\gamma_1 = t_2\gamma_1\gamma_2.$$

We can prove

$$t_0\alpha_1 \cdots \alpha_k = t_k\gamma_1 \cdots \gamma_k$$

for every $k = 1, \ldots, n$. Thus

$$t_0\alpha_1\alpha_2 \cdots \alpha_n = t_n\gamma_1\gamma_2 \cdots \gamma_n. \tag{6}$$

We can prove in a similar way that

$$a_0\alpha_1\alpha_2 \cdots \alpha_n = a_n\gamma_1\gamma_2 \cdots \gamma_n. \tag{7}$$

Using the cancellativity of T, equations (6) and $t_0 = t_n$ together imply

$$\alpha_1\alpha_2\cdots\alpha_n = \gamma_1\gamma_2\cdots\gamma_n.$$

Using also the cancellativity of T, equation (7) implies $a_0 = a_n$. Consequently

$$(t_0, a_0) = (t_n, a_n).$$

Thus condition (ii) (d) of Theorem 1.1 is satisfied for S.

As all subcases of condition (ii) of Theorem 1.1 are satisfied for the semigroup $S = (S; T, S_t, \varphi_{t,r}, \star)$, it is embeddable into a left simple semigroup (containing idempotent elements).

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