On special Rees matrix rings related to the right annihilator of rings

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Abstract

In this paper we focuse on Rees matrix rings $\mathcal{M}(R; I, \Lambda; P)$ in which the set I has exactly one element. For a ring R, let $Ann_r(R)$ and $(Ann_r(R) :_r R)$ denote the right annihilator of R and the right colon ideal of $Ann_r(R)$, respectively. The main result of our paper is that, for every choice function P defined on the collection of all cosets of $Ann_r(R)$, the factor ring of the Rees matrix ring $\mathcal{M}(R; I, R/Ann_r(R); P)$ modulo its right annihilator is isomorphic to the Rees matrix ring $\mathcal{M}(R/(Ann_r(R) :_r R); I, R/Ann_r(R); P')$, in which P' is defined by $P' : a + Ann_r(R) \mapsto a + (Ann_r(R) :_r R);\ a \in R$.

Key words: Rings; Rees matrix rings; right annihilator of rings; right colon ideals of rings.

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1 Introduction

Let R be a ring, I and Λ be nonempty sets, P be a $\Lambda \times I$ matrix over R. Denote by $\mathcal{M}(R; I, \Lambda; P)$ the set of all $I \times \Lambda$-matrices over R having a finite number of nonzero entries endowed with the usual addition of matrices and with a multiplication $\circ$ defined by $A \circ B = APB$, where the multiplication on the right-hand side is the usual multiplication of matrices. $\mathcal{M}(R; I, \Lambda; P)$ is a ring, which is called the *Rees matrix ring* over R with sandwich matrix P (see [5] (for arbitrary R), or [1] (for R with unit element)). In this paper we only deal with Rees matrix rings $\mathcal{M}(R; I, \Lambda; P)$ in which I contains exactly

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one element. These Rees matrix rings will be denoted by $\mathcal{M}(R;\Lambda;P)$. In this case the elements of $\mathcal{M}(R;\Lambda;P)$ are $1\times\Lambda$-matrices, and the sandwich matrix P is a $\Lambda\times 1$-matrix. The sandwich matrix P and all of the elements A of $\mathcal{M}(R;\Lambda;P)$ can be considered as mappings of Λ into R. The entries of $A\in\mathcal{M}(R;\Lambda;P)$ and the entries of P will be denoted by $A(\lambda)$ and $P(\lambda)$, respectively ($\lambda\in\Lambda$). Thus, for arbitrary $A,B\in\mathcal{M}(R;\Lambda;P)$,

$$(A\circ B)(\lambda)=(\sum_{j\in\Lambda}A(j)P(j))B(\lambda).$$

Let J be an ideal of a ring R. By the *right colon ideal* of J we mean the ideal

$$(J:_r R)=\{r\in R|\ Rr\subseteq J\}$$

of R (see, for example, [4] or [6]). The ideal

$$Ann_r(R)=\{a\in R|\ Ra=\{0\}\}$$

of a ring R is called the *right annihilator* of R.

The cosets $a+Ann_r(R)$ and $a+(Ann_r(R):_r R)$ ($a\in R$) will be denoted by $[a]_{Ann_r(R)}$ and $[a]_{(Ann_r(R):_r R)}$, respectively. The factor rings of R by $Ann_r(R)$ will be denoted by R/Ann_r. It is easy to see that, for every ideal J of a ring R,

$$(R/J)/Ann_r\cong R/(J:_r R). \tag{1}$$

If J is an ideal of a ring R, then a mapping $P:R/J\to R$ will said to be a *choice function indicated by* J if $P(r+J)\in r+J$ for every coset $r+J$ of J.

Let R be a ring, $P:R/Ann_r\to R$ be an arbitrary choice function indicated by $Ann_r(R)$ and $P':R/Ann_r\to R/(Ann_r(R):_r R)$ be a mapping defined by

$$P':[a]_{Ann_r(R)}\mapsto[a]_{(Ann_r(R):_r R)};a\in R.$$

We can construct the following Rees matrix rings over the factor ring R/Ann_r:

$$\mathcal{M}=\mathcal{M}(R;R/Ann_r;P),\quad \mathcal{M}'=\mathcal{M}(R/(Ann_r(R):_r R);R/Ann_r;P').$$

Conider the following diagram:

$$\begin{array}{ccccc} R & \stackrel{P}{\longleftarrow} & R/Ann_r & \stackrel{P'}{\longrightarrow} & R/(Ann_r(R):_r R) \\ \Downarrow & & & & \Downarrow \\ \mathcal{M} & & & & \mathcal{M}' \end{array} \tag{2}$$

In Section 2, we show that diagram (2) can be supplemented by a surjective homomorphism Φ

$$\begin{array}{ccccc} R & \xleftarrow{P} & R/Ann_r & \xrightarrow{P'} & R/(Ann_r(R) :_r R) \\ \Downarrow & & & & \Downarrow \\ \mathcal{M} & & \xrightarrow[\Phi]{} & & \mathcal{M}' \end{array}$$

such that the kernel of Φ is the right annihilator of $\mathcal{M}$, and so

$$\mathcal{M}/Ann_r \cong \mathcal{M}'.$$

In Section 3, we define a property $(\star)$ of sequences of rings. Applying the result of Section 2, we show that, for an arbitrary ring R and an arbitrary ideal J of R, the sequence

$$R/(J :_r^{(0)} R)), R/(J :_r^{(1)} R), \ldots, R/(J :_r^{(n)} R), \ldots$$

of factor rings has the property $(\star)$, in which sequence $(J :_r^{(0)} R) = J$ and $(J :_r^{(n)} R) = ((J :_r^{(n-1)} R) :_r R)$ for every positive integer n.

For notations and notions not defined in this paper, we refer to [2] and [3].

2 On the Rees matrix rings $\mathcal{M}$ and $\mathcal{M}'$

Theorem 2.1 *Let R be a ring and P be an arbitrary choice function indicated by $Ann_r(R)$. Then*

$$\mathcal{M}(R; R/Ann_r; P)/Ann_r \cong \mathcal{M}(R/(Ann_r(R) :_r R); R/Ann_r; P'),$$

where P' is the mapping of R/Ann_r into $R/(Ann_r(R) :_r R)$ such that

$$P' : [a]_{Ann_r(R)} \mapsto [a]_{(Ann_r(R):_r R)}$$

for every $a \in R$.

Proof. Let R/Ann_r denoted by Λ. Let

$$\Phi : \mathcal{M}(R;\Lambda;P) \to \mathcal{M}(R/(Ann_r(R) :_r R);\Lambda;P')$$

be the following mapping. For arbitrary $A \in \mathcal{M}(R;\Lambda;P)$, let $\Phi(A)$ be the element of $\mathcal{M}(R/(Ann_r(R) :_r R);\Lambda;P')$ such that, for every $\lambda \in \Lambda$,

$$(\Phi(A))(\lambda) = [A(\lambda)]_{(Ann_r(R):_r R)}.$$

It is clear that Φ is surjective.

We show that Φ is a homomorphism. Let A and B be arbitrary elements of $\mathcal{M}(R;\Lambda;P)$. Then, for every $\lambda \in \Lambda$,

$$(A+B)(\lambda) = A(\lambda) + B(\lambda)$$

and

$$(\Phi(A) + \Phi(B))(\lambda) = (\Phi(A))(\lambda) + (\Phi(B))(\lambda).$$

Thus, for every $\lambda \in \Lambda$,

$$(\Phi(A+B))(\lambda) = [A(\lambda) + B(\lambda)]_{(Ann_r(R):_r R)} =$$

$$= [A(\lambda)]_{(Ann_r(R):_r R)} + [B(\lambda)]_{(Ann_r(R):_r R)} = (\Phi(A))(\lambda) + (\Phi(B))(\lambda) =$$
$$= (\Phi(A) + \Phi(B))(\lambda).$$

Thus

$$\Phi(A+B) = \Phi(A) + \Phi(B).$$

For every $\lambda \in \Lambda$,

$$(A \circ B)(\lambda) = \left(\sum_{j\in\Lambda} A(j)P(j)\right) B(\lambda)$$

and so

$$(\Phi(A \circ B))(\lambda) = \left[\left(\sum_{j\in\Lambda} A(j)P(j)\right) B(\lambda)\right]_{(Ann_r(R):_r R)} =$$

$$= \left(\sum_{j\in\Lambda}[A(j)]_{(Ann_r(R):_r R)}[P(j)]_{(Ann_r(R):_r R)}\right)[B(\lambda)]_{(Ann_r(R):_r R)}. \qquad (3)$$

If $j = [a]_{Ann_r(R)}$, then $[a]_{Ann_r(R)} = [P(j)]_{Ann_r(R)}$, because P is a choice function indicated by $Ann_r(R)$ and so $P(j) \in [a]_{Ann_r(R)}$. Thus

$$P'(j) = P'([a]_{Ann_r(R)}) = P'([P(j)]_{Ann_r(R)}) = [P(j)]_{(Ann_r(R):_r R)}$$

and so (3) equals

$$\left(\sum_{j\in\Lambda}[A(j)]_{(Ann_r(R):_r R)}P'(j)\right)[B(\lambda)]_{(Ann_r(R):_r R)} = (\Phi(A)\circ\Phi(B))(\lambda).$$

Hence

$$\Phi(A\circ B) = \Phi(A)\circ\Phi(B).$$

Consequently Φ is a homomorphism.

We show that the kernel of Φ is the right annihilator of $\mathcal{M}(R;\Lambda;P)$. Let $A \in \mathcal{M}(R;\Lambda;P)$ be an arbitrary element. $A \in ker_\Phi$ if and only if

$$\Phi(A) = 0 \quad \text{in} \quad \mathcal{M}(R/(Ann_r(R) :_r R);\Lambda;P')$$

if and only if

$$(\forall\lambda\in\Lambda)[A(\lambda)]_{(Ann_r(R):_r R)} = 0 \quad \text{in} \quad R,$$

that is,

$$(\forall x,y\in R,\lambda\in\Lambda) \quad xyA(\lambda) = 0. \tag{4}$$

We show that condition (4) is equivalent to the condition that A is in the right annihilator of $\mathcal{M}(R;\Lambda;P)$. Assume (4). Then, for every $C \in \mathcal{M}(R;\Lambda;P)$, we have

$$(C\circ A)(\lambda) = \left(\sum_{j\in\Lambda}C(j)P(j)\right)A(\lambda) = \sum_{j\in\Lambda}C(j)P(j)A(\lambda) = 0$$

and so A is in the right annihilator of $\mathcal{M}(R;\Lambda;P)$.

Conversely, assume that A is in the right annihilator of $\mathcal{M}(R;\Lambda;P)$. Let $r \in R$ and $j \in \Lambda$ be arbitrary elements. Let $C_{j,r}$ be the $1\times\Lambda$ matrix over R, in which $C_{j,r}(j) = r$ and the other entries are the zero of R. Then, for every $\lambda \in \Lambda$,

$$0 = (C_{j,r}\circ A)(\lambda) = rP(j)A(\lambda).$$

Thus

$$(\forall r\in R)(\forall j,\lambda\in\Lambda)\ rP(j)A(\lambda) = 0. \tag{5}$$

Let $a_1, a_2 \in R$ be arbitrary elements. Then there are $j_1, j_2 \in \Lambda$ such that

$$a_1 \equiv P(j_1) \bmod Ann_r(R), \quad a_2 \equiv P(j_2) \bmod Ann_r(R),$$

and so

$$a_1 = P(j_1) + \xi_1 \quad \text{and} \quad a_2 = P(j_2) + \xi_2$$

for some elements $\xi_1, \xi_2 \in Ann_r(R)$. Applying (5) and the fact that $\xi_1, \xi_2 \in Ann_r(R)$, we have

$$a_1 a_2 A(\lambda) = P(j_1)P(j_2)A(\lambda) + P(j_1)\xi_2 A(\lambda) + \xi_1 P(j_2)A(\lambda) + \xi_1\xi_2 A(\lambda) = 0$$

and so

$$A(\lambda) \in (Ann_r(R) :_r R).$$

Hence $\Phi(A) = 0$ in $\mathcal{M}(R/(Ann_r(R) :_r R); R/Ann_r; P')$ and so $A \in ker_\Phi$. Consequently ker_Φ is the right annihilator of $\mathcal{M}(R; R/Ann_r; P)$. By the homomorphism theorem,

$$\mathcal{M}(R; R/Ann_r; P)/Ann_r \cong \mathcal{M}(R/(Ann_r(R) :_r R); R/Ann_r; P')$$

$\square$

3 Sequences of rings with a special property

Definition 3.1 *We shall say that a sequence*

$$R_0, R_1, \ldots, R_n, \ldots$$

of rings R_i $(i = 0, 1, 2, \ldots)$ has the property $(\star)$ if, for every positive integer n, there are mappings

$$P_{n,n-1} : R_n \mapsto R_{n-1} \quad \textit{and} \quad R_{n,n+1} : R_n \mapsto R_{n+1}$$

such that

$$\mathcal{M}(R_{n-1}; R_n; P_{n,n-1})/Ann_r \cong \mathcal{M}(R_{n+1}; R_n; R_{n,n+1}).$$

In the next theorem, we shall use the following notations. For an ideal J of a ring R, let

$$(J :_r^{(0)} R) = J,$$

and let

$$(J :_r^{(n)} R) = ((J :_r^{(n-1)} R) :_r R)$$

for arbitrary positive integer n.

Theorem 3.1 *For an arbitrary ring R and an arbitrary ideal J of R, the sequence*

$$R/(J :_r^{(0)} R)), R/(J :_r^{(1)} R), \ldots, R/(J :_r^{(n)} R), \ldots$$

of factor rings has the property ($\star$).

Proof. Let n be a positive integer, and let

$$T_{n-1} = R/(J :_r^{(n-1)} R).$$

Using (1), we have

$$T_n = R/(J :_r^{(n)} R) \cong R/((J :_r^{(n-1)} R) :_r R) \cong$$

$$\cong (R/(J :_r^{(n-1)} R))/Ann_r \cong T_{n-1}/Ann_r$$

and

$$T_{n+1} = R/(J :_r^{(n+1)} R) \cong R/((J :_r^{(n)} R) :_r R) \cong (R/(J :_r^{(n)} R))/Ann_r \cong$$

$$\cong (T_{n-1}/Ann_r)/Ann_r \cong T_{n-1}/(Ann_r(T_{n-1}) :_r T_{n-1}).$$

Consider the diagram (2) in that case when $R = T_{n-1}$:

$$\begin{array}{ccccc} T_{n-1} & \xleftarrow{P} & T_{n-1}/Ann_r & \xrightarrow{P'} & T_{n-1}/(Ann_r(T_{n-1}) :_r T_{n-1}) \\ \Downarrow & & & & \Downarrow \\ \mathcal{M} = \mathcal{M}(T_{n-1}; T_n; P) & & & & \mathcal{M}' = \mathcal{M}(T_{n+1}; T_n; P') \end{array}$$

in which P is a choice function indicated by $Ann_r(T_{n-1})$ and P' is defined by

$$P' : [a]_{Ann_r(T_{n-1})} \mapsto [a]_{(Ann_r(T_{n-1}) :_r T_{n-1})}; a \in T_{n-1}.$$

By Theorem 2.1,

$$\mathcal{M}(T_{n-1}; T_n; P)/Ann_r \cong \mathcal{M}(T_{n+1}; T_n; P').$$

Let

$$P_{n,n-1} = P \quad \text{and} \quad P_{n,n+1} = P'.$$

Then

$$\mathcal{M}(T_{n-1}; T_n, P_{i,i-1})/Ann_r \cong \mathcal{M}(T_{n+1}; T_n; P_{i,i+1})$$

which proves that the sequence

$$T_0, T_1, \ldots, T_n, \ldots$$

has the property $(\star)$. Thus the sequence

$$R/(J :_r^{(0)} R), R/(J :_r^{(1)} R), \ldots, R/(J :_r^{(n)} R), \ldots$$

of factor rings has the property $(\star)$. $\square$

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