Seshadri constants via functions on Newton–Okounkov bodies

M. Dumnicki*, **A. Küronya****, **C. Maclean, T. Szemberg**

The aim of this note is to establish a somewhat surprising connection between functions on Newton–Okounkov bodies and Seshadri constants of line bundles on algebraic surfaces.

Seshadri constants, Newton–Okounkov bodies

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1 Introduction

The aim of the short note is to study the connection between functions on Newton–Okounkov bodies defined by filtrations imposed by orders of vanishing, and Seshadri-type invariants.

Seshadri constants first arose as a tool created by Demailly in order to understand Fujita's conjecture [5] and they soon became an area of independent research. Given a smooth projective variety X and a nef line bundle L on X, the Seshadri constant of L at a point $P \in X$ is defined to be the real number

$$\varepsilon(L;P) =_{\mathrm{def}} \inf_{C} \frac{L \cdot C}{\operatorname{mult}_P C}\ , \tag{1}$$

where the infimum is taken over all irreducible curves C containing the point P. In this note we are interested in the case of surfaces. Although it is not going to be explicit in what follows, our motivation for this note is to investigate arithmetic properties of Seshadri constants.

Rationality of Seshadri constants has been an intriguing question ever since they entered the scene about thirty years ago. Although the general expectation is that irrational Seshadri constants should abound, to this day there is no example of such behaviour. This is quite striking as in a closely related setting one can easily find irrational s-invariants as presented in [4, Example 1.7]. Nevertheless, the case of Seshadri constants, especially in dimension two, when many asymptotic invariants end up taking on only rational values, is wide open (see [9, Remark 5.1.13]).

The recent work [6] links the rationality of Seshadri constants on blow-ups of $\mathbb{P}^2$ to Nagata's conjecture (and the more general SHGH conjecture), which certainly adds some extra significance to the issue. In this paper we open a new line of attack in that we relate one-point Seshadri constants to invariants of functions on Newton–Okounkov bodies.

It is an immediate consequence of their definition that whenever a Seshadri constant is irrational then it must be $\varepsilon(L;P) = \sqrt{L^2}$, see e.g. [1, Theorem 2.1.5].

Our main result is the following.

Theorem. *Let X be a smooth projective surface, $Y_\bullet$ an admissible flag, L a nef and big line bundle on X, and let $P \in X$ be an arbitrary point. Furthermore let ord_P be the geometric valuation defined by the order of vanishing at P and let $\varphi_{\operatorname{ord}_P}(x)$ be the associated Okounkov function.*

$$\textit{If} \max_{x \in \Delta_{Y_\bullet}(L)} \varphi_{\operatorname{ord}_P}(x) \in \mathbb{Q}, \ \textit{then}\ \varepsilon(L;P) \in \mathbb{Q}\ .$$

2 Rationality of Seshadri constants and functions on Okounkov bodies

The theory of Newton–Okounkov bodies has emerged recently with work by Okounkov [12], Kaveh–Khovanskii [7], and Lazarsfeld–Mustaţă [10]. Shortly thereafter, Boucksom–Chen [2] and Witt-Nyström [11] have shown ways of constructing geometrically significant functions on Okounkov

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bodies, that were further studied in [3]. In the context of this note the study of Okounkov functions was pursued by Boucksom and the last three authors in [3]. We refer to [3] for the construction and properties of Okounkov functions.

In this section we consider an arbitrary smooth projective surface X and an ample line bundle L on X. Let $P \in X$ be an arbitrary point and let $\pi : Y \to X$ be the blow up of P with the exceptional divisor E. Recall that the Seshadri constant of L at P can equivalently be defined as

$$\varepsilon(L;P) = \sup\{t > 0 \mid \pi^*L - tE \text{ is nef }\}\,.$$

There is a related invariant

$$\begin{aligned}\mu(L;P) &\stackrel{\text{def}}{=} \sup\{t > 0 \mid \pi^*L - tE \text{ is pseudo-effective}\}\\ &= \sup\{t > 0 \mid \pi^*L - tE \text{ is big}\}\ .\end{aligned}$$

The invariant $\varepsilon(L;P)$ is the value of the parameter λ where the ray $\pi^*L - \lambda E$ meets the boundary of the nef cone of Y, and $\mu(L;P)$ is the value of λ where that ray meets the boundary of the pseudo-effective cone. The following relation between the two invariants is important in our considerations.

Remark 2.1 It follows from [1, Theorem 2.1.5 and Theorem 6.1.1] that if $\varepsilon(L;P)$ is irrational, then

$$\varepsilon(L;P) \;=\; \mu(L;P)\,.$$

In particular, if $\mu(L;P)$ is rational, then so is $\varepsilon(L;P)$.

Thus rationality of $\mu(L;P)$ implies the rationality of the corresponding Seshadri constant. This invariant appears in the study of the concave function φ_{ord_P} associated to the geometric valuation on X defined by the order of vanishing ord_P at P. We fix some flag $Y_\bullet : X \supseteq C \supseteq \{x_0\}$ and consider the Okounkov body $\Delta_{Y_\bullet}(L)$ defined with respect to that flag. We define also a multiplicative filtration determined by the geometrical valuation ord_P on the graded algebra $V = \oplus_{k\geqslant 0} V_k$ with $V_k = H^0(X, kL)$ by

$$\mathcal{F}_t(V) = \{s \in V :\ \mathrm{ord}_P(s) \geqslant t\}\,.$$

(All the above remains valid in the more general context of graded linear series.) There is an induced filtration $\mathcal{F}_\bullet(V_k)$ on every summand of V and one defines the maximal jumping numbers of both filtrations as

$$e_{\max}(V,\mathcal{F}_\bullet) = \sup\{t \in \mathbb{R} :\ \exists\, k\ \mathcal{F}_{kt}V_k \neq 0\}$$

and

$$e_{\max}(V_k,\mathcal{F}_\bullet) = \sup\{t \in \mathbb{R} :\ \mathcal{F}_t V_k \neq 0\}$$

respectively. Let $\varphi_{\mathrm{ord}_P}(x) = \varphi_{\mathcal{F}_\bullet}(x)$ be the Okounkov function on $\Delta_{Y_\bullet}(L)$ determined by the filtration $\mathcal{F}_\bullet$, see [2, Definition 1.8]. It turns out that $\mu(L;P)$ is the maximum of the Okounkov function φ_{ord_P}.

Proposition 2.2 *With notation as above we have that*

$$\begin{aligned}\mu(L;P) &= \limsup_{m\to\infty} \frac{\max\left\{\mathrm{ord}_P(s)\mid s\in H^0(X,\mathcal{O}_X(mL))\right\}}{m}\\ &= \max\nolimits_{x\in\Delta_{Y_\bullet}(L)} \varphi_{\mathrm{ord}_P}(x).\end{aligned}$$

Proof. Observe that

$$\mathrm{ord}_P(s) \;=\; \mathrm{ord}_E(\pi^*s) \;=\; \max\{m \in \mathbb{N} \mid \mathrm{div}(\pi^*s) - mE \text{ is effective}\}\ .$$

Consequently,

$$\begin{aligned}\mu(L;P) &= \sup\{t \in \mathbb{R}_{\geqslant 0} \mid \pi^*L - tE \text{ is pseudo-effective}\}\\ &= \limsup_{m\to\infty} \frac{\max\left\{\mathrm{ord}_P(s) \mid s \in H^0\left(X, \mathcal{O}_X(mL)\right\}\right.}{m}\ ,\end{aligned}$$

which gives the first equality.

For the second equality, we observe first that

$$\max\left\{\mathrm{ord}_P(s) \mid s \in H^0\left(X, \mathcal{O}_X(mL)\right)\right\} \;=\; e_{\max}(V_m, \mathcal{F}_\bullet)\ ,$$

and hence

$$\limsup_{m\to\infty} \frac{\max\left\{\mathrm{ord}_P(s)\,|\,s\in H^0\left(X,\mathcal{O}_X(mL)\right)\right\}}{m} \;=\; e_{\max}(V,\mathcal{F}_\bullet)\;.$$

Since

$$e_{\max}(V,\mathcal{F}_\bullet) \;=\; \max_{x\in\Delta_{Y_\bullet}(L)} \varphi_{\mathrm{ord}_P}(x)$$

by Theorem 2.4, we are done. □

2.1 Independence of the maximum of an Okounkov function on the flag

In the course of this section the projective variety X can have arbitrary dimension.

Boucksom and Chen proved that though $\varphi_{\mathcal{F}_\bullet}$ and $\Delta(V_\bullet)$ depend on the flag $Y_\bullet$, the integral of $\varphi_{\mathcal{F}_\bullet}$ over $\Delta(V_\bullet)$ is independent of $Y_\bullet$, [2, Remark 1.12 (ii)]. We prove now that the maximum of the Okounkov function does not depend on the flag. This fact is valid in the general setting of arbitrary multiplicative filtration $\mathcal{F}$ defined on a graded linear series $V_\bullet$.

Remark 2.3 Note that in general the functions $\varphi_{\mathcal{F}_\bullet}$ are only upper-semicontinuous and concave, but not necessarily continuous on the whole Newton–Okounkov body as explained in [3, Theorem 1.1]. They are however continuous provided the underlying body $\Delta(V_\bullet)$ is a polytope (see again [3, Theorem 1.1]), which is the case for complete linear series on surfaces [8].

Theorem 2.4 (Maximum of Okounkov functions) *With the above notation, we have that*

$$\max_{x\in\Delta_{Y_\bullet}(L)} \varphi_{\mathcal{F}_\bullet}(x) = e_{\max}(V,\mathcal{F}_\bullet).$$

In particular the left hand side does not depend on the flag $Y_\bullet$.

Proof. For any real $t\geqslant 0$, we consider the partial Okounkov body $\Delta_{t,Y_\bullet}(L)$ associated the graded linear series $V_{t,k}\subset H^0(X,kL)$ given by

$$V_{t,k} \stackrel{\text{def}}{=} \mathcal{F}_{kt}(H^0(X,kL))\;.$$

Note that by definition

$$e_{\max}(V,\mathcal{F}_\bullet) = \sup\left\{t\in\mathbb{R}|\bigcup_k V_{t,k}\neq 0\right\}.$$

In other words,

$$e_{\max}(V,\mathcal{F}_\bullet) = \sup\left\{t\in\mathbb{R}|\;\Delta_{t,Y_\bullet}(L)\neq\emptyset\right\}.$$

Recall that by definition

$$\varphi_{\mathcal{F}_\bullet}(x) = \sup\{t\in\mathbb{R}|\;x\in\Delta_{t,Y_\bullet}(L)\}.$$

and it is therefore immediate that for all x

$$\varphi_{\mathcal{F}_\bullet}(x)\leqslant e_{\max}(V,\mathcal{F}_\bullet)$$

from which it follows that

$$\max_{x\in\Delta_{Y_\bullet}(L)} \varphi_{\mathcal{F}_\bullet}(x)\leqslant e_{\max}(V,\mathcal{F}_\bullet).$$

Since the bodies $\Delta_{t,Y_\bullet}(L)$ form a decreasing family of closed subsets of $\mathbb{R}^d$, we have that

$$\bigcap_{t|\;\Delta_{t,Y_\bullet}(L)\neq\emptyset} \Delta_{t,Y_\bullet}(L)\neq\emptyset.$$

Consider a point $y\in\bigcap_{t|\;\Delta_{t,Y_\bullet}(L)\neq\emptyset}\Delta_{t,Y_\bullet}(L)$. We have then

$$y\in\Delta_{t,Y_\bullet}(L)\Leftrightarrow\Delta_{t,Y_\bullet}(L)\neq\emptyset$$

and hence

$$\sup\{t \in \mathbb{R} |\ y \in \Delta_{t,Y_\bullet}(L)\} = \sup\{t \in \mathbb{R} |\ \Delta_{t,Y_\bullet}(L) \neq \emptyset\}$$

or in other words

$$\varphi_{\mathcal{F}_\bullet}(y) = e_{\max}(V, \mathcal{F}_\bullet)$$

from which it follows that

$$\max_{x \in \Delta_{Y_\bullet}(L)} \varphi_{\mathcal{F}_\bullet}(x) \leqslant e_{\max}(V, \mathcal{F}_\bullet).$$

This completes the proof of the Theorem. □

We conclude this note with an example illustrating the application of the Theorem and with a challenge heading in the direction of potentially irrational Seshadri constants. The example is an easy modification of [3, Example 3.4], therefor we omit all arguments and merely state facts.

Example 2.5 Let $f : X = \mathrm{Bl}_Q\,\mathbb{P}^2 \to \mathbb{P}^2$ be the blow up of the projective plane in a point Q with the exceptional divisor E. Let $H = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and let the flag $Y_\bullet$ be given by $X \supset L \ni P_0$, where L is (proper transform of) a line not passing through the point Q and P_0 is an arbitrary point on L. For positive integers $a > b$ consider the ample line bundle $L_{a,b} = aH - bE$ on X. Let $P \in X$ be a point not on the total transform of the line passing through Q and P_0. Then the Okounkov body $\Delta_{Y_\bullet}(L_{a,b})$ is the trapezoid indicated in Figure 2.5. The function φ_{ord_P} is then given by

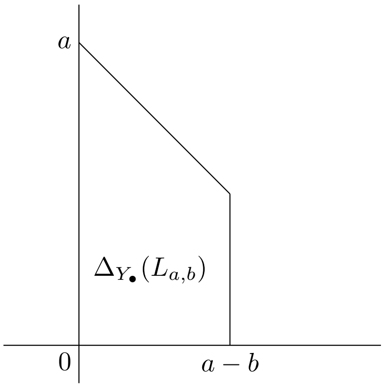

Fig. 1 The Newton-Okounkov body of $L_{a,b}$

$$\varphi_{\mathrm{ord}_P}(x_1,x_2) = \begin{cases} a - x_1 & \text{for} \quad x_1 + x_2 \leqslant a - b \\ 2a - 2x_1 - x_2 - b & \text{for} \quad a - b \leqslant x_1 + x_2 \leqslant a \end{cases}$$

Hence $\mu(L_{a,b}; P) = a$. On the other hand it is easy to see that

$$\varepsilon(L_{a,b}; P) = a - b,$$

the constant being computed by the proper transform of the line passing through P and Q.

Challenge. Similarly as in Example 2.5 we consider now $f : X = \mathrm{Bl}_{P_1,\ldots,P_9}\,\mathbb{P}^2 \to \mathbb{P}^2$ the blow up of $\mathbb{P}_2$ at 9 general points $P_1, \ldots, P_9$. Let $L = 22H - 7\mathbb{E}$ be an ample line bundle on X, where $\mathbb{E} = E_1 + \ldots + E_9$ is the union of the exceptional divisors E_i of f. Let P be a general point on X. Compute φ_{ord_P} with respect to a flag as in Example 2.5.

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Marcin Dumnicki, Jagiellonian University, Institute of Mathematics, Łojasiewicza 6, PL-30-348 Kraków, Poland.

E-mail address: `Marcin.Dumnicki@im.uj.edu.pl`

Alex Küronya, FB12 - Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 6-8, 60325 Frankfurt am Main, Germany.

E-mail address: `kuronya@math.uni-frankfurt.de`

Catriona Maclean Institut Fourier, CNRS UMR 5582, Université de Grenoble, 100 rue des Maths, BP 74, 38402 Saint-Martin d'Heres cedex, France.

E-mail address: `catriona.maclean@ujf-grenoble.fr`

Tomasz Szemberg, Instytut Matematyki UP, Podchorążych 2, PL-30-084 Kraków, Poland.

Current Address: Polish Academy of Sciences, Institute of Mathematics, Śniadeckich 8, PL-00-656 Warszawa, Poland

E-mail address: `tomasz.szemberg@gmail.com`