Rationality of Seshadri constants and the Segre-Harbourne-Gimigliano-Hirschowitz Conjecture

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Abstract

In this paper we relate the SHGH Conjecture to the rationality of one-point Seshadri constants on blow ups of the projective plane.

Keywords Nagata Conjecture, SHGH Conjecture, Seshadri constants

Mathematics Subject Classification (2000) MSC 14C20

1 Introduction

Nagata's conjecture and its generalizations have been a central problem in the theory of surfaces for many years, and much work has been done towards verifying them, e.g. [10], [3], [8], [4]. In this paper we open a new line of attack in which we relate Nagata-type statements to the rationality of one-point Seshadri constants. As a consequence, our approach provides some evidence that certain Nagata-type questions might be false.

Seshadri constants were first introduced by Demailly in the course of his work on Fujita's conjecture [6] in the late 80's and have been the object of considerable interest ever since. Recall that given a smooth projective variety X and a nef line bundle L on X, the Seshadri constant of L at a point $x \in X$ is the real number

$$\varepsilon(L;x) =_{\text{def}} \inf_{C} \frac{L \cdot C}{\operatorname{mult}_x C} \ , \tag{1}$$

where the infimum is taken over all irreducible curves passing through x. An intriguing and notoriously difficult problem about Seshadri constants on surfaces is the question whether these invariants are rational numbers, see [9, Remark 5.1.13] It follows quickly from their definition that if a Seshadri constant is irrational then it must be $\varepsilon(L;x) = \sqrt{L^2}$, see e.g. [1, Theorem 2.1.5]. It is also known that Seshadri constants of a fixed line bundle L, take their maximal value on a subset in X which is a complement of at most countably many Zariski closed proper subsets of X, i.e. in *very general* points.

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We denote this maximum by $\varepsilon(L;1)$. Similar notation $\varepsilon(L;s)$ is used for multi-point Seshadri constants, see [1, Definition 1.9]. In particular, if $\varepsilon(L;x)=\sqrt{L^2}$ at some point the same holds in a very general point on X and $\varepsilon(L;1)=\sqrt{L^2}$.

From a slightly different point of view, Seshadri constants reveal information on the structure of the nef cone on the blow-up of X at x, hence their study is closely related to our attempts to understand Mori cones of surfaces, see also [5] for a somewhat different approach.

An even older problem concerning linear series on algebraic surfaces is the conjecture formulated by Beniamino Segre in 1961 and rediscovered, made more precise and reformulated by Harbourne 1986, Gimigliano 1987 and Hirschowitz 1988. (See [3] for a very nice account on this development and related subjects.) In particular it is known, [3, Remark 5.12] that the SHGH Conjecture implies the Nagata Conjecture. We now recall these conjectures. There are several equivalent statements of the SHGH Conjecture. We choose here the one due to Gimigliano [7, Conjecture 3.3] as it is the most convenient formulation for our purposes.

SHGH Conjecture. *Let X be the blow up of the projective plane $\mathbb{P}^2$ in s very general points with exceptional divisors $E_1,\dots,E_s$. Let H denote the pullback to X of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2$. Let the integers $d, m_1\geqslant\ldots\geqslant m_s\geqslant -1$ with $d\geqslant m_1+m_2+m_3$ be given. Then the line bundle*

$$dH-\sum_{i=1}^{s} m_iE_i$$

is non-special.

Also for the Nagata Conjecture, we choose a statement which best suits our needs.

Nagata Conjecture. *Let s be an integer with $s\geqslant 9$. Then the multi-point Seshadri constant of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ on the projective plane satisfies*

$$\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1),s)=\frac{1}{\sqrt{s}}.$$

The main result of this note is the following somewhat unexpected relation between the SHGH Conjecture and the rationality problem for Seshadri constants.

Main Theorem. *Let $s\geqslant 9$ be an integer for which the SHGH Conjecture holds true. Then*

a) either there exist points $P_1,\dots,P_s\in\mathbb{P}^2$, a line bundle L on $\mathrm{Bl}_{P_1,\dots,P_s}\mathbb{P}^2$ and a points $P\in X$ such that

$$\varepsilon(L;P)\ \text{ is irrational}$$

b) or the SHGH Conjecture fails for $s+1$ points.

Note that it is known that the SHGH conjecture holds true for $s\leqslant 9$, [3, Theorem 5.1]. It is also known that Seshadri constants of ample line bundles on del Pezzo surfaces (i.e. for $s\leqslant 8$) are rational, see [11, Theorem 1.6]. In any case, the statement of the Theorem is interesting (and non-empty) for $s=9$.

Corollary 1.1. *If all one–point Seshadri constants on the blow-up of $\mathbb{P}^2$ in nine very general points are rational, then the SHGH conjecture fails for ten points.*

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2 Rationality of one point Seshadri constants and the SHGH Conjecture

In this section we prove the Main Theorem. We start with notation and preliminary lemmas.

Let $f : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $s \geqslant 9$ general points $P_1, \ldots, P_s$ with exceptional divisors $E_1, \ldots, E_s$. We denote as usual by $H = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$ the pull back of the hyperplane bundle. It is well-known that $H, E_1, \ldots, E_s$ form a basis of the Néron-Severi space $N^1(X)_{\mathbb{R}}$ of X. We denote by $\mathbb{E} = E_1 + \cdots + E_s$ the sum of exceptional divisors of f.

The following Lemma allows us to work with divisors with equal multiplicities.

Lemma 2.1. *Let $P_1, \ldots, P_s$ be a finite set of general points in $\mathbb{P}^2$. Let D be an effective divisor of degree d with $m_i = \operatorname{mult}_{P_i} D$. Let $\sigma \in \Sigma_s$ be a permutation of s elements. Then there exists an effective divisor D_σ of the same degree with $\operatorname{mult}_{P_i} D_\sigma = m_{\sigma(i)}$.*

Proof. Let $\mathcal{I}(P)$ denote the ideal sheaf of a point in $\mathbb{P}^2$. By the semi-continuity the function

$$(\mathbb{P}^2)^s \ni (P_1, \ldots, P_s) \to h^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{I}(P_1)^{m_1} \otimes \ldots \otimes \mathcal{I}(P_s)^{m_s}) \in \mathbb{Z}$$

is positive on a Zariski open set U. Making this set a little bit smaller if necessary, we can assume that with $(P_1, \ldots, P_s) \in U$ also $(P_{\sigma(1)}, \ldots, P_{\sigma(s)}) \in U$ for any permutation $\sigma \in \Sigma_s$. The claim follows. $\square$

We consider now the blow up $g : Y \to X$ of X at P with exceptional divisor F. The following result is a well-known consequence of Lemma 2.1. We were not able to find a reference and we include a proof for the benefit of a reader.

Lemma 2.2. *If there exists a curve $C \subset X$ in the linear system $dH - \sum_{i=1}^s m_i E_i$ computing the Seshadri constant of a $\mathbb{Q}$-line bundle $L = H - \alpha\mathbb{E}$, then there exists a divisor Γ with $\operatorname{mult}_{P_1} \Gamma = \ldots = \operatorname{mult}_{P_s} \Gamma = M$ computing the Seshadri constant of L at P, i.e.*

$$\frac{L \cdot \Gamma}{\operatorname{mult}_P \Gamma} = \frac{L \cdot C}{\operatorname{mult}_P C} = \varepsilon(L; P).$$

Proof. Since the points $P_1, \ldots, P_s$ are general, there exist divisors

$$C_\sigma = dH - \sum_{j=1}^{s} m_{\sigma(j)} E_j$$

for all permutations $\sigma \in \Sigma_s$. Since the point P is general, we may take all these divisors to have the same multiplicity m at P. Summing over a cycle σ of length s in the symmetric group Σ_s, we obtain a divisor

$$\Gamma = \sum_{i=1}^{s} C_{\sigma^i} = sdH - \sum_{i=1}^{s} \sum_{j=1}^{s} m_{\sigma^i(j)} E_j = sdH - M\mathbb{E},$$

with $M = m_1 + \ldots + m_s$. Note that the multiplicity of Γ at P equals sm. Taking the Seshadri quotient for Γ we have

$$\frac{L \cdot \Gamma}{sm} = \frac{sd - \alpha s M}{sm} = \frac{d - \alpha M}{m} = \varepsilon(L; P)$$

hence Γ satisfies the assertions of the Lemma. $\square$

The following auxiliary Lemma will be used in the proof of the Main Theorem. We postpone its proof to the end of this section.

Lemma 2.3. *Let $s \geqslant 9$ be an integer. The function*

$$f(\delta) = (2\sqrt{s+1} - s)\sqrt{1 - s\delta^2} + s(1 - \sqrt{s+1})\delta + s - 2 \tag{2}$$

is non-negative for δ satisfying

$$\frac{1}{\sqrt{s+1}} < \delta < \frac{1}{\sqrt{s}}. \tag{3}$$

Proof of the Main Theorem. If part a) in the Theorem holds, then we are done.

Otherwise we are in the situation that for all points $P_1, \ldots, P_s \in \mathbb{P}^2$, for all line bundles L on $X = \mathrm{Bl}_{P_1,\ldots,P_s} \mathbb{P}^2$ and for all points $P \in X$

$$\varepsilon(L; P) \text{ is a rational number.} \tag{4}$$

We assume also to the contrary that part b) is false, i.e. that the SHGH Conjecture holds for $s + 1$ points.

Let $U(j)$ denote an open set in $(\mathbb{P}^2)^j$ such that SHGH holds for all j-tuples from $U(j)$. Let

$$\pi_{j+1,j} : (\mathbb{P}^2)^{j+1} \to (\mathbb{P}^2)^j$$

denote the projection onto the first j factors. Let

$$W(s) := \pi_{s+1,s}(U(s+1)) \cap U(s).$$

Since the projection is an open mapping $W(s)$ is an open set. Let the points

$$P_1, \ldots, P_s \in W(s) \tag{5}$$

be fixed. Let δ be a rational number satisfying (3). Note that the SHGH Conjecture implies the Nagata Conjecture [3, Remark 5.12] so that

$$\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); s) = \frac{1}{\sqrt{s}}$$

and hence the $\mathbb{Q}$–divisor $L = H - \delta\mathbb{E}$ is ample. Changing δ a little bit if necessary, we can assume that $\sqrt{L^2}$ is an irrational number.

By (4) $\varepsilon(L; P)$ is a rational number for all points $P \in \mathbb{P}^2$. From now on, we fix a point P in such a way that $(P_1, \ldots, P_s, P) \in U(s+1)$ (this is possible by (5)).

By Lemma 2.2 there is a divisor $\Gamma \subset \mathbb{P}^2$ of degree γ with $M = \mathrm{mult}_{P_1} \Gamma = \ldots = \mathrm{mult}_{P_s} \Gamma$ and $m = \mathrm{mult}_P \Gamma$ whose proper transform $\widetilde{\Gamma}$ on X computes the Seshadri constant

$$\varepsilon(L; P) = \frac{L \cdot \widetilde{\Gamma}}{m} = \frac{\gamma - \delta s M}{m} < \sqrt{1 - s\delta^2}.$$

This gives an upper bound on γ

$$\gamma < m\sqrt{1-s\delta^2} + \delta s M. \tag{6}$$

Since we work under assumption that SHGH holds for $s+1$, the Nagata Conjecture also holds for $s+1$ points and this gives a lower bound for γ, since for Γ we must have

$$\frac{\gamma}{sM+m} \geqslant \frac{1}{\sqrt{s+1}}. \tag{7}$$

We now claim that

$$\gamma \geqslant 2M + m. \tag{8}$$

Suppose not. We then have that

$$\gamma < 2M + m. \tag{9}$$

The real numbers

$$a := \frac{2\sqrt{s+1}-s}{2-\delta s} \quad \text{and} \quad b := \frac{s-\delta s\sqrt{s+1}}{2-\delta s}$$

are positive. Multiplying (6) by a and (9) by b and adding these inequalities we obtain

$$sM + m \leqslant \gamma\sqrt{s+1} < sM + (b + a\sqrt{1-s\delta^2})m,$$

where the first inequality follows from (7). Rearranging the right and left hand sides of this inequality we obtain that

$$(2\sqrt{s+1}-s)\sqrt{1-s\delta^2} + s - \delta s\sqrt{s+1} < 2 - \delta s,$$

which contradicts Lemma 2.3. Hence (8) holds.

It follows now from the SHGH conjecture for $s+1$ points (in the form stated in the introduction) that the linear system

$$\gamma H - M\mathbb{E} - mF$$

on Y is non-special. Indeed the condition $\gamma \geqslant 2M + m$ is (8) and the condition $\gamma \geqslant 3M$ is satisfied since $\frac{\gamma}{sM} > \frac{1}{\sqrt{s}}$ (because the Nagata Conjecture holds for s points by hypothesis) and because we have assumed that $s \geqslant 9$. This system is also non-empty because the proper transform of Γ under g is a member. Thus by a standard dimension count

$$0 \leqslant \gamma(\gamma+3) - sM(M+1) - m(m+1).$$

The upper bound on γ (6) together with the above inequality yields

$$0 \leqslant (s\delta M + m\sqrt{1-s\delta^2})(s\delta M + m\sqrt{1-s\delta^2}+3) - m^2 - m - sM - sM^2. \tag{10}$$

Note that the quadratic term in (10) is a negative semi-definite form

$$(s^2\delta^2 - s)M^2 + 2s\delta\sqrt{1-s\delta^2}Mm - s\delta^2 m^2.$$

Indeed, the restrictions on δ made in (3) imply that the term at M^2 is negative. The determinant of the associated symmetric matrix vanishes. These two conditions imply together that the form is negative semi-definite. In particular this term of (10) is non-positive. The linear part in turn is

$$(3s\delta - s)M + (3\sqrt{1-s\delta^2} - 1)m,$$

which is easily seen to be negative. This provides the desired contradiction and finishes the proof of the Theorem. $\square$

Remark 2.4. As it is well known, Nagata's conjecture can be interpreted in terms of the nef and Mori cones of the blow-up X of $\mathbb{P}^2$ at s general points. More precisely, consider the following question: for what $t \geqslant 0$ does the ray $H - t\mathbb{E}$ meet the boundary of the nef cone? The conjecture predicts that this ray should intersect the boundaries of the nef cone and the effective cone at the same time. We denote the value of t at which this ray leaves the effective cone by $\mu(L;\mathbb{E})$.

Considering the Zariski chamber structure of X (see [2]), we see that this is equivalent to requiring that $H - t\mathbb{E}$ crosses exactly one Zariski chamber (the nef cone itself). Surprisingly, it is easy to prove that $H - t\mathbb{E}$ cannot cross more than two chambers.

Proposition 2.5. *Let $f : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ in s general points with exceptional divisors $E_1, \ldots, E_s$. Let H be the pull-back of the hyperplane bundle and $\mathbb{E} = E_1 + \ldots + E_s$. The ray $R = H - t\mathbb{E}$ meets at most two Zariski chambers on X.*

Proof. If $\varepsilon = \varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); s) = \frac{1}{\sqrt{s}}$ i.e. this multi-point Seshadri constant is maximal, then the ray crosses only the nef cone.

If ε is submaximal, then there is a curve $C = dH - \sum m_i E_i$ computing this Seshadri constant, i.e. $\varepsilon = \frac{d}{\sum m_i}$.

If this curve is homogeneous, i.e. $m = m_1 = \cdots = m_s$, then we put $\Gamma := C$. Otherwise we define $\Gamma = \gamma H - M \cdot \mathbb{E}$ as a symmetrization of C as in Lemma 2.2, i.e. we sum the curves $C_k = dH - \sum m_{\sigma^k(i)} E_i$ over a length s cycle $\sigma \in \Sigma_s$.

Let $\mu = \frac{M}{\gamma}$. Note that $\mu = \mu(L;\mathbb{E})$. Indeed, it is in any case $\mu \leqslant \mu(L;\mathbb{E})$ by the construction of Γ and a strict inequality would contradict the fact that Γ computes the Seshadri constant.

Now, we claim that

$$H - t\mathbb{E} = \frac{\mu - t}{\mu - \varepsilon}(H - \varepsilon\mathbb{E}) + \frac{t - \varepsilon}{\gamma(\mu - \varepsilon)}\Gamma$$

is the Zariski decomposition of $H - t\mathbb{E}$ for $\varepsilon \leqslant t \leqslant \mu$. Indeed, $H - \varepsilon\mathbb{E}$ is nef by definition and it is orthogonal to all components of Γ, which together with the Index Theorem implies that the intersection matrix of Γ is negative definite.

Thus the ray R after leaving the nef chamber remains in a single Zariski chamber. □

It is interesting to compare this result with the following easy example, which constructs rays meeting a maximal number of chambers.

Example 2.6. Keeping the notation from Proposition 2.5 let $L = (\frac{s(s+1)}{2} + 1)H - E_1 - 2E_2 - \ldots - sE_s$. Then L is an ample divisor on X and the ray $R = L + \lambda\mathbb{E}$ crosses $s + 1 = \rho(X)$ Zariski chambers. Indeed, with λ growing, exceptional divisors $E_1, E_2, \ldots, E_s$ join the support of the Zariski decomposition of $L - \lambda\mathbb{E}$ one by one. We leave the details to the reader.

We conclude this section with the proof of Lemma 2.3.

Proof of Lemma 2.3. Since $f(1/\sqrt{s+1}) = 0$ it is enough to show that $f(\delta)$ is increasing for $1/\sqrt{s+1} \leqslant \delta \leqslant 1/\sqrt{s}$. Consider the derivative

$$f'(\delta) = s\left(1 + \frac{\delta}{\sqrt{1 - s\delta^2}}(s - 2\sqrt{s+1}) - \sqrt{s+1}\right). \tag{11}$$

The function $h(\delta) = \frac{\delta}{\sqrt{1-s\delta^2}}$ is increasing for $1/\sqrt{s+1} \leqslant \delta < 1/\sqrt{s}$ since the numerator is an increasing function of δ and the denominator is a decreasing function of δ. We have $h(\frac{1}{\sqrt{s+1}}) = 1$ so that $h(\delta) \geqslant 1$ holds for all δ. Since the coefficient of $h(\delta)$ in (11) is positive we have

$$f'(\delta) \geqslant s\left(1 + (s - 2\sqrt{s+1}) - \sqrt{s+1}\right) = s(1 + s - 3\sqrt{s+1}) > 0,$$

which completes the proof. $\square$

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