Supremum and infimum of positive operators

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Partial ordering of operators

Let $\mathcal{H}$ be a Hilbert space, and consider the $\mathcal{B}_+(\mathcal{H})$ cone of positive operators and $A, B \in \mathcal{B}_+(\mathcal{H})$ arbitrary operators.
We say $A \leq B$ if $B - A \in \mathcal{B}(\mathcal{H})$ is a positive operator.
Question: What is the necessary and sufficient condition for the existence of the $A \vee B$ supremum and the $A \wedge B$ infimum within $\mathcal{B}_+(\mathcal{H})$.

THEOREM

(Kadison; 1951) There exists an $A \vee B \in \mathcal{B}_+(\mathcal{H})$ supremum if and only if $A \sim B$ ($A \geq B$ of $A \leq B$).

Lebesgue type decomposition of operators

We say an operator B is absolutely continuous with respect to the operator A ($B \ll A$), if for any sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{H}$, $(Ax_n|x_n) \to 0$ and $(B(x_n - x_m)|x_n - x_m) \to 0$ imply $(Bx_n|x_n) \to 0$. We say A and B are mutually singular ($A \perp B$), if $C \leq A$ and $C \leq B$ imply $C = 0$ for any $C \in \mathcal{B}_+(\mathcal{H})$.
The Lebesgue type decomposition of operators to absolutely continuous and singular parts is not unique in general, but there is an extremal decomposition $B = [A]B + (B - [A]B)$, where $[A]B$ is the absolutely continuous and $B - [A]B$ is the singular part. This can be easily constructed using parallel addition.

THEOREM

(Ando; 1999) There exists an $A \wedge B \in \mathcal{B}_+(\mathcal{H})$ infimum if and only if $[B]A \sim [A]B$.

Anti-dual pairs

Let E and F be complex vector spaces and $\langle \cdot, \cdot \rangle : F \times E \to \mathbb{C}$ be a sesquilinear function which separates the points of E and F.
We call the triple $(E, F, \langle \cdot, \cdot \rangle)$ (shortly $\langle F, E \rangle$) and anti-dual pair, and denote by $\mathcal{L}(E, F)$ the set of continuous linear operators from E to F.

Anti-dual pairs, partial ordering

We say an operator $A \in \mathcal{L}(E, F)$ is positive if $\langle Ax, x \rangle \geq 0$ for all $x \in E$, and $A \leq B$ if $B - A$ is positive.
We denote by $\mathcal{L}_+(E, F)$ the cone of positive operators within $\mathcal{L}(E, F)$.

Anti-dual pairs, sequential completeness

We may endow E and F with the corresponding weak topologies $\sigma(E,F)$ resp. $\sigma(F,E)$ induced by families $\{\langle f,\cdot\rangle : f \in F\}$ resp. $\{\langle \cdot, e\rangle : e \in E\}$.

We call the sequence $(x_n)_{n\in\mathbb{N}} \subset F$ a Cauchy sequence if for any neighborhood U of 0 there exist such $N \in \mathbb{N}$ that for all $n, m > N$, $x_n - x_m \in U$.

We call the anti-dual pair $\langle F, E\rangle$ weak-* sequentially complete if $(F, \sigma(F,E))$ is sequentially complete, that means all Cauchy sequences are convergent.

Main results

Theorem

Let $\langle F, E\rangle$ be a weak- sequentially complete anti-dual pair and let $A, B \in \mathcal{L}_+(E, F)$ be positive operators. There exists an $A \vee B \in \mathcal{L}_+(E, F)$ supremum if and only if $A \sim B$.*

Theorem

Let $\langle F, E\rangle$ be a weak- sequentially complete anti-dual pair and let $A, B \in \mathcal{L}_+(E, F)$ be positive operators. There exists an $A \wedge B \in \mathcal{L}_+(E, F)$ infimum if and only if $[B]A \sim [A]B$.*

Remark

Lebesgue type decomposition of operators of weak-* sequentially complete anti-dual pairs can be defined and constructed similarly to the Hilbert space case.

Bibliography

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