Dynamics and Steering Control of an Autonomous Unicycle

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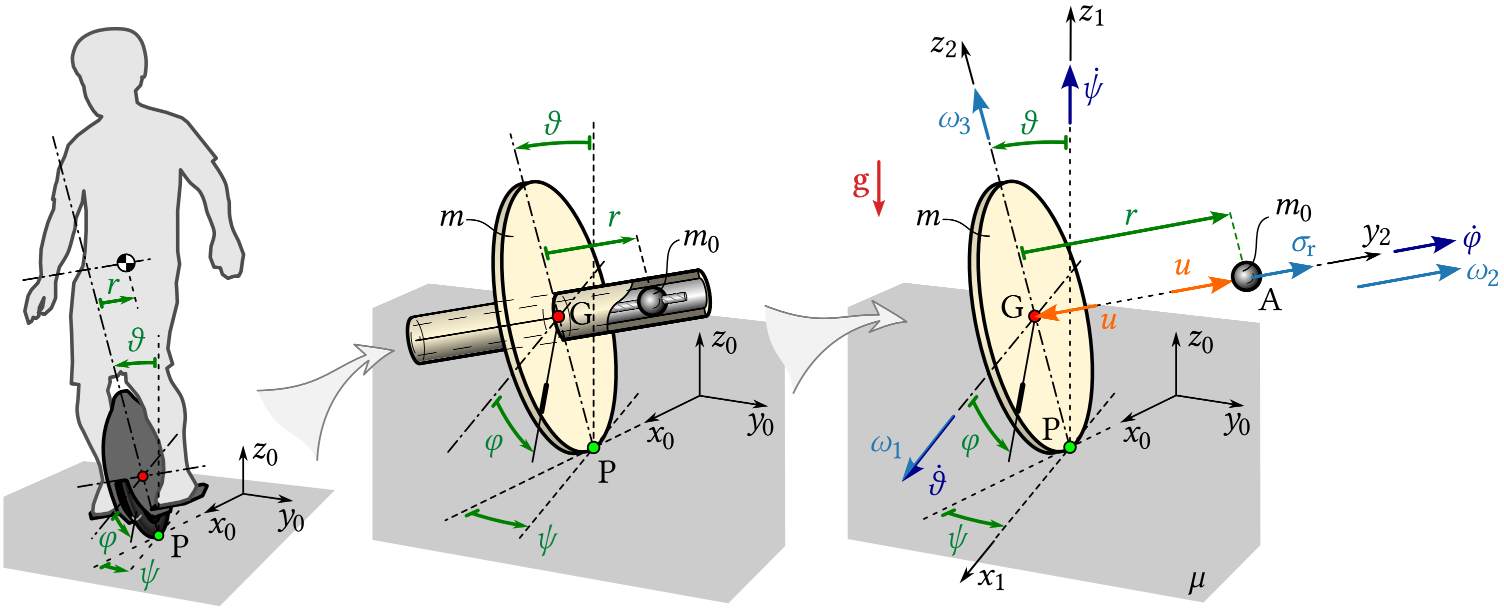
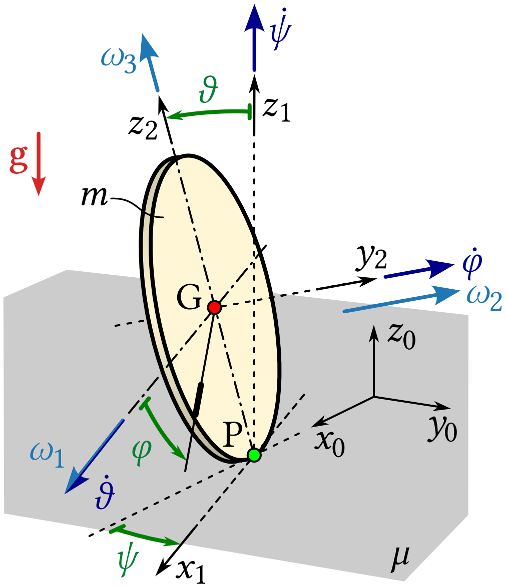
Motivation: unicycles (Onewheel car)



Mechanical model



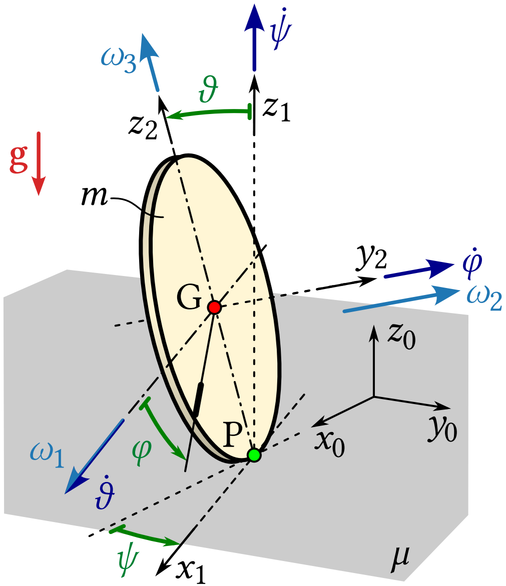
Overview

- Disc: m R $\mathbf{g}$
- Coordinate frames: $F_0 \xrightarrow[\psi]{} F_1 \xrightarrow[\vartheta]{} F_2 \xrightarrow[\varphi]{} F_3$
- Natural coordinates: *(6 db)*

  x_G y_G z_G ψ (yaw) ϑ (~~roll~~ tilt) φ (pitch)
- **Rolling**: *constraints*
  - P is the velocity pole: $\mathbf{v}_P = \mathbf{0}$ *kinematic*

$$\dot{x}_G = \dot{\psi} R \cos\psi \sin\vartheta + \dot{\vartheta} R \sin\psi \cos\vartheta + \dot{\varphi} R \cos\psi$$
$$\dot{y}_G = \dot{\psi} R \sin\psi \sin\vartheta - \dot{\vartheta} R \cos\psi \cos\vartheta + \dot{\varphi} R \sin\psi$$
$$\dot{z}_G = -R\dot{\vartheta} \sin\vartheta$$

Mechanical model



Constraints and coordinates

- Constraints: *2 kinematic* + *1 geometric*

$$\dot x_G = \dot\psi R\cos\psi\sin\vartheta + \dot\vartheta R\sin\psi\cos\vartheta + \dot\varphi R\cos\psi$$
$$\dot y_G = \dot\psi R\sin\psi\sin\vartheta - \dot\vartheta R\cos\psi\cos\vartheta + \dot\varphi R\sin\psi$$
$$\dot z_G = -R\dot\vartheta\sin\vartheta \quad \Rightarrow \quad z_G = R\cos\vartheta$$

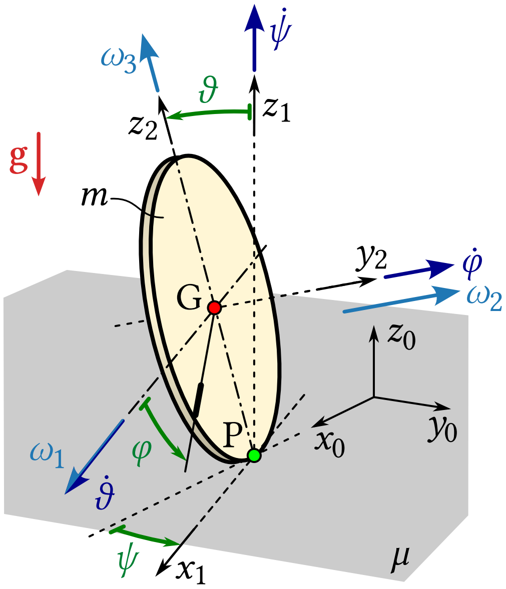
- Nonholonomic system → **Appellian approach!**
- Generalized coordinates: (*5 pcs*)

$$x_G \quad y_G \quad \psi\,(\text{yaw}) \quad \vartheta\,(\text{tilt}) \quad \varphi\,(\text{pitch})$$

- Pseudo velocities: (*3 pcs*)

$$\omega_1 := \dot\vartheta \qquad \omega_2 := \dot\psi\sin\vartheta + \dot\varphi \qquad \omega_3 := \dot\psi\cos\vartheta$$

Mechanical model



Equations of motion

Appell

$$\frac{\partial S}{\partial \dot\sigma_i} = \Pi_i$$

$$\begin{cases} \dot\omega_1 = \dfrac{6\omega_2\omega_3}{5} - \dfrac{\omega_3^2 \tan\vartheta}{5} + \dfrac{4g\sin\vartheta}{5R} \\ \dot\omega_2 = -\dfrac{2\omega_1\omega_3}{3} \\ \dot\omega_3 = -2\omega_1\omega_2 + \omega_1\omega_3\tan\vartheta \end{cases}$$

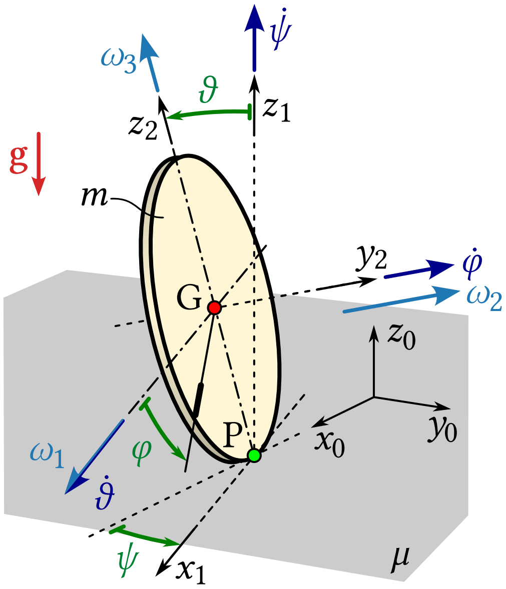
Pseudo velocities

$$\begin{cases} \dot\vartheta = \omega_1 \\ \dot\psi = \omega_3 \dfrac{1}{\cos\vartheta} \\ \dot\varphi = \omega_2 - \omega_3\tan\vartheta \end{cases}$$

Kinematic constraints

$$\begin{cases} \dot x_G = \omega_1 R\sin\psi\cos\vartheta + \omega_2 R\cos\psi \\ \dot y_G = -\omega_1 R\cos\psi\cos\vartheta + \omega_2 R\sin\psi \end{cases}$$

Mechanical model



Equations of motion

Essential motion

$$\begin{cases} \dot\omega_1 = \dfrac{6\omega_2\omega_3}{5} - \dfrac{\omega_3^2 \tan\vartheta}{5} + \dfrac{4g\sin\vartheta}{5R} \\ \dot\omega_2 = -\dfrac{2\omega_1\omega_3}{3} \\ \dot\omega_3 = -2\omega_1\omega_2 + \omega_1\omega_3\tan\vartheta \\ \dot\vartheta = \omega_1 \end{cases}$$

Hidden (cyclic) motion

$$\begin{cases} \dot\psi = \omega_3 \dfrac{1}{\cos\vartheta} \\ \dot\varphi = \omega_2 - \omega_3\tan\vartheta \\ \dot x_G = \omega_1 R\sin\psi\cos\vartheta + \omega_2 R\cos\psi \\ \dot y_G = -\omega_1 R\cos\psi\cos\vartheta + \omega_2 R\sin\psi \end{cases}$$

Steady state motion ↔ Fixed point of the essential motion

States:

$$\mathbf{x} = \begin{bmatrix}\omega_1 & \omega_2 & \omega_3 & \vartheta & \psi & \varphi & x_G & y_G\end{bmatrix}^T$$

Essential motion:

$$\begin{cases} \dot\omega_1 = \dfrac{6\omega_2\omega_3}{5} - \dfrac{\omega_3^2 \tan\vartheta}{5} + \dfrac{4g\sin\vartheta}{5R} \\ \dot\omega_2 = -\dfrac{2\omega_1\omega_3}{3} \\ \dot\omega_3 = -2\omega_1\omega_2 + \omega_1\omega_3\tan\vartheta \\ \dot\vartheta = \omega_1 \end{cases}$$

Equilibrium:

$$\omega_1(t) \equiv 0 \quad \omega_2(t) \equiv \omega_{2*} \quad \omega_3(t) \equiv \omega_{3*} \quad \vartheta(t) \equiv \vartheta_*$$

Relation of the states:

$$6\omega_{2*}\omega_{3*}R - \omega_{3*}^2 R\tan\vartheta_* + 4g\sin\vartheta_* = 0$$

$$\updownarrow$$

$$5\dot\psi_*^2 R\sin\vartheta_*\cos\vartheta_* + 6\dot\psi_*\dot\varphi_* R\cos\vartheta_* + 4g\sin\vartheta_* = 0$$

Solution:

- Straight rolling: $\dot\psi_* = 0 \quad \vartheta_* = 0 \quad \dot\varphi_* \in \mathbb{R}$
- Turning-rolling: $\dot\psi_* \neq 0 \quad \vartheta_* = -\pi/2 \ldots \pi/2$

$$\dot\varphi_* = -\frac{5}{6}\dot\psi_*\sin\vartheta_* - \frac{2g}{3R}\frac{\tan\vartheta_*}{\dot\psi_*}$$

  - Spinning: $\vartheta_* = 0 \quad \dot\psi_* \in \mathbb{R} \quad \dot\varphi_* = 0$
  - (Static : $\vartheta_* = 0 \quad \dot\psi_* = 0 \quad \dot\varphi_* = 0$)

Hidden motion

$$\psi_*(t) = \dot{\psi}_* t + \psi_0 \,,$$
$$\varphi_*(t) = \dot{\varphi}_* t + \varphi_0 \,,$$
$$x_{G*}(t) = \begin{cases} \left(\dfrac{\dot{\varphi}_*}{\dot{\psi}_*} + \sin\vartheta_*\right) R \sin(\psi_*(t)) + x_0 & \text{ha } \dot{\psi}_* \neq 0 \\ \dot{\varphi}_* t R \cos\psi_0 + x_0 & \text{ha } \dot{\psi}_* = 0 \end{cases}$$
$$y_{G*}(t) = \begin{cases} -\left(\dfrac{\dot{\varphi}_*}{\dot{\psi}_*} + \sin\vartheta_*\right) R \cos(\psi_*(t)) + y_0 & \text{ha } \dot{\psi}_* \neq 0 \\ \dot{\varphi}_* t R \sin\psi_0 + y_0 & \text{ha } \dot{\psi}_* = 0 \end{cases}$$

- G path: **straight**, **circle** or **not moving**

! In case of circular motion: $\dot{\varphi}_* = \dot{\varphi}_*(\vartheta_*, \dot{\psi}_*)$ **!** ρ_G, ρ_P

Relation of the states:

$$6\omega_{2*}\omega_{3*}R - \omega_{3*}^2 R \tan\vartheta_* + 4g\sin\vartheta_* = 0$$
$$\updownarrow$$
$$5\dot{\psi}_*^2 R \sin\vartheta_* \cos\vartheta_* + 6\dot{\psi}_*\dot{\varphi}_* R\cos\vartheta_* + 4g\sin\vartheta_* = 0$$

Solution:

- Straight rolling: $\dot{\psi}_* = 0 \quad \vartheta_* = 0 \quad \dot{\varphi}_* \in \mathbb{R}$
- Turning-rolling: $\dot{\psi}_* \neq 0 \quad \vartheta_* = -\pi/2 \ldots \pi/2$

$$\dot{\varphi}_* = -\frac{5}{6}\dot{\psi}_* \sin\vartheta_* - \frac{2g}{3R}\frac{\tan\vartheta_*}{\dot{\psi}_*}$$

  - Spinning: $\vartheta_* = 0 \quad \dot{\psi}_* \in \mathbb{R} \quad \dot{\varphi}_* = 0$
  - (Static : $\vartheta_* = 0 \quad \dot{\psi}_* = 0 \quad \dot{\varphi}_* = 0$)

Linear stability analysis

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 & \vartheta & \psi & \varphi & x_G & y_G \end{bmatrix}^\mathsf{T}$$

$$\mathbf{A} = \begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} & 0 & 0 & 0 & 0 \\ A_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{53} & A_{54} & 0 & 0 & 0 & 0 \\ 0 & 1 & A_{63} & A_{64} & 0 & 0 & 0 & 0 \\ A_{71} & A_{72} & 0 & 0 & A_{75} & 0 & 0 & 0 \\ A_{81} & A_{82} & 0 & 0 & A_{85} & 0 & 0 & 0 \end{bmatrix}$$

Characteristic polynomial:

$$\left(\lambda^2 - A_{12}A_{21} - A_{13}A_{31} - A_{14}\right)\lambda^6 = 0$$

Stability

- Straight rolling: $|\dot{\varphi}| > \dot{\varphi}_{\mathrm{crit}} = \sqrt{\frac{g}{3R}}$
- Spinning: $|\dot{\psi}| > \dot{\psi}_{\mathrm{crit}} = \sqrt{\frac{4g}{5R}}$
- Turning-rolling ($|\vartheta_*| < \mathscr{V}$):

$$|\dot{\psi}| < \dot{\psi}_{\mathrm{crit},1} \quad \text{or} \quad \dot{\psi}_{\mathrm{crit},2} < |\dot{\psi}|$$

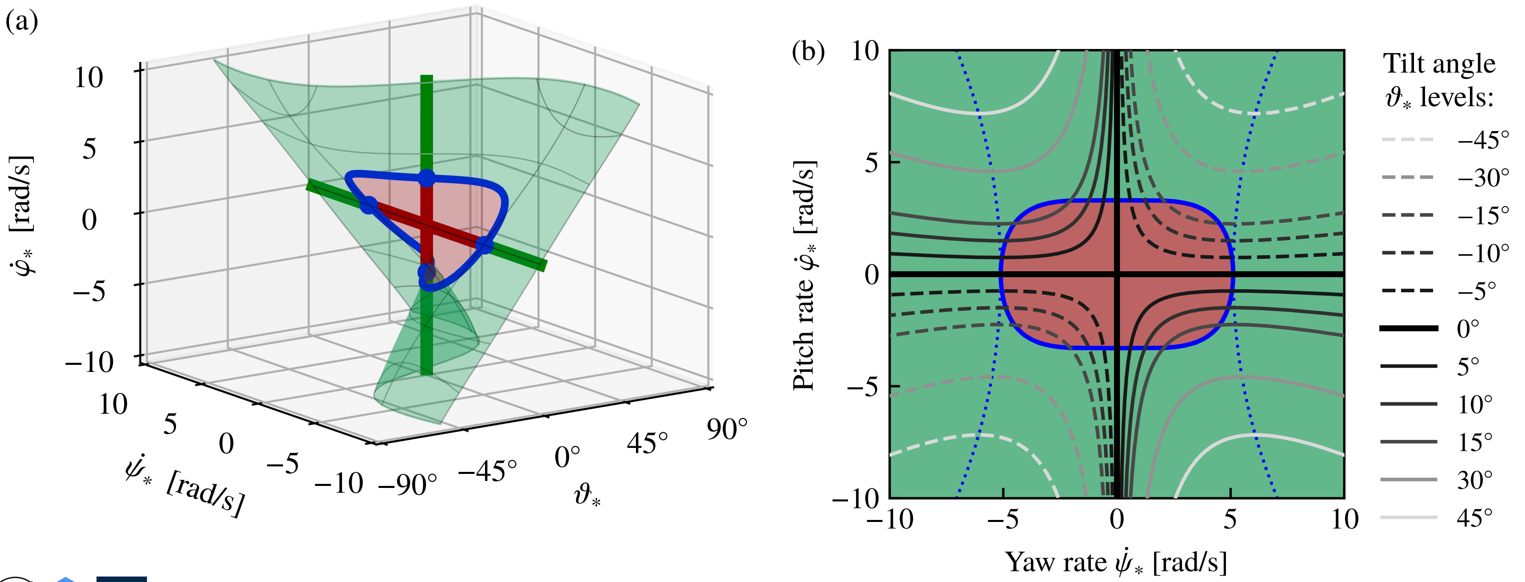
$$\dot{\psi}_{\mathrm{crit},1,2} = \sqrt{\frac{2g}{5R}}\sqrt{\frac{3-6\cos^2\vartheta_* \pm \sqrt{76\sin^4\vartheta_* - 96\sin^2\vartheta_* + 9}}{(2\sin^2\vartheta_* - 3)\cos\vartheta_*}}$$

Turning always stable if

$$|\vartheta_*| > \mathscr{V} = \arcsin\left(\sqrt{\frac{12}{19} - \frac{9\sqrt{5}}{38}}\right) \approx 18.62^\circ$$

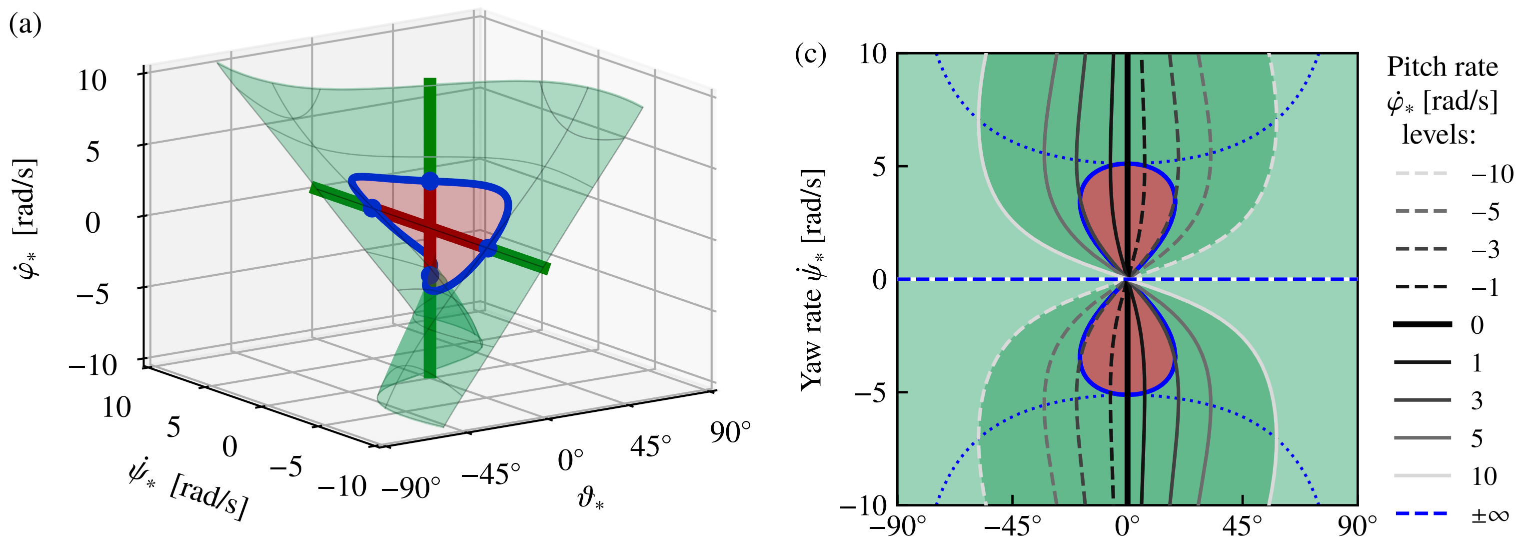
Steady state motion

$$6\omega_{2*}\omega_{3*}R - \omega_{3*}^2 R\tan\vartheta_* + 4g\sin\vartheta_* = 0 \quad\leftrightarrow\quad 5\dot\psi_*^2 R\sin\vartheta_*\cos\vartheta_* + 6\dot\psi_*\dot\varphi_* R\cos\vartheta_* + 4g\sin\vartheta_* = 0 \quad\to\quad \dot\varphi_*(\vartheta_*,\dot\psi_*) = \ldots$$



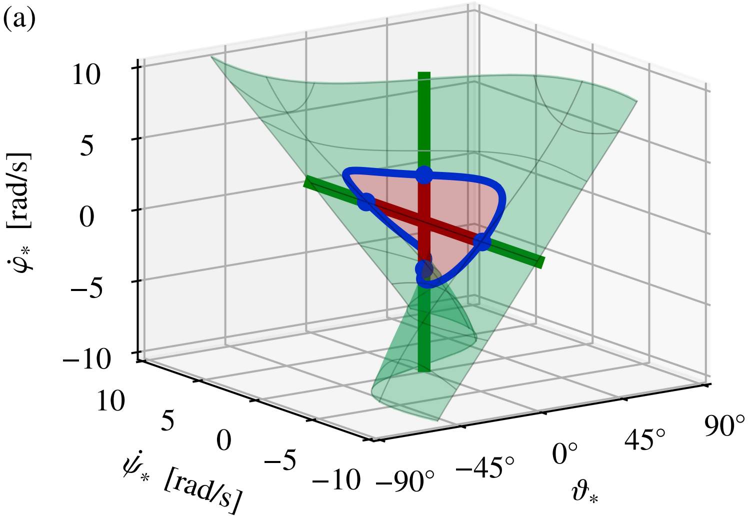
Steady state motion

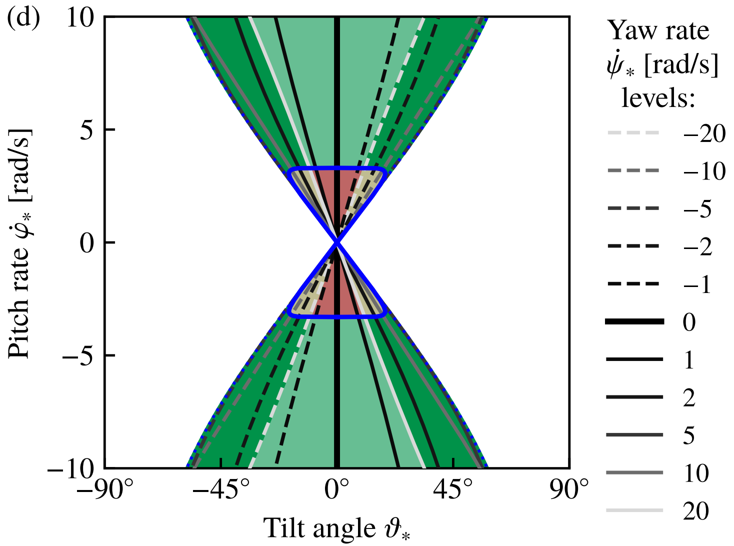
$$6\omega_{2*}\omega_{3*}R - \omega_{3*}^2 R\tan\vartheta_* + 4g\sin\vartheta_* = 0 \quad\leftrightarrow\quad 5\dot\psi_*^2 R\sin\vartheta_*\cos\vartheta_* + 6\dot\psi_*\dot\varphi_* R\cos\vartheta_* + 4g\sin\vartheta_* = 0 \quad\rightarrow\quad \dot\varphi_*(\vartheta_*,\dot\psi_*) = \dots$$



Steady state motion

$$6\omega_{2*}\omega_{3*}R - \omega_{3*}^2 R\tan\vartheta_* + 4g\sin\vartheta_* = 0 \quad\leftrightarrow\quad 5\dot\psi_*^2 R\sin\vartheta_*\cos\vartheta_* + 6\dot\psi_*\dot\varphi_* R\cos\vartheta_* + 4g\sin\vartheta_* = 0 \quad\rightarrow\quad \dot\varphi_*(\vartheta_*,\dot\psi_*) = \ldots$$





Gyroscopic effect in straight rolling

Using generalized velocities $\dot{q}$ and accelerations $\ddot{q}$:

$$\frac{5mR^2}{4}\ddot{\vartheta} - \frac{5mR^2}{4}\sin\vartheta\cos\vartheta\dot{\psi}^2 - \frac{3mR^2}{2}\cos\vartheta\dot{\psi}\dot{\varphi} - mgR\sin\vartheta = 0\,,$$

$$\frac{3mR^2}{2}\ddot{\varphi} + \frac{3mR^2}{2}\sin\vartheta\ddot{\psi} + \frac{5mR^2}{2}\cos\vartheta\dot{\psi}\dot{\vartheta} = 0\,,$$

$$\frac{mR^2\cos\vartheta}{4}\ddot{\psi} + \frac{mR^2}{2}\dot{\varphi}\dot{\vartheta} = 0\,.$$

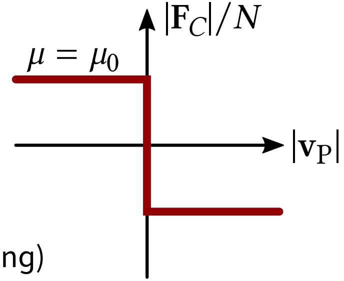
Linearized system: $\vartheta = 0\,,\quad \psi = 0\,,\quad \dot{\varphi} = \dot{\varphi}_*\,,\quad \varphi = \dot{\varphi}_* t\,,\quad \dot{\tilde{\varphi}} := \dot{\varphi} - \dot{\varphi}_*$

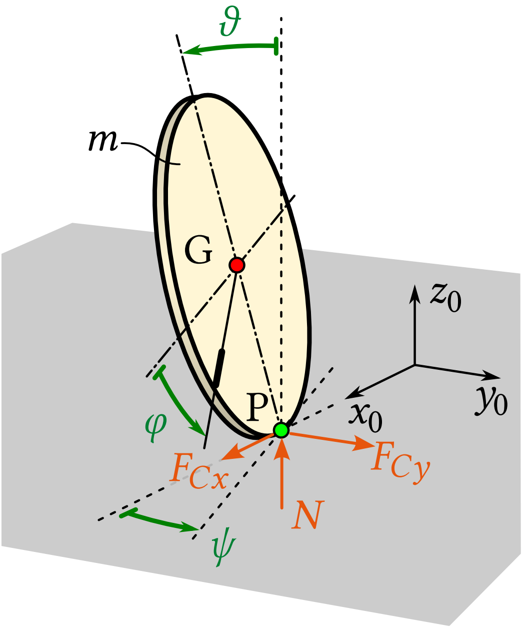
$$\underbrace{\frac{mR^2}{4}\begin{bmatrix}5 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 6\end{bmatrix}}_{\mathbf{M}}\begin{bmatrix}\ddot{\vartheta}\\ \ddot{\psi}\\ \ddot{\tilde{\varphi}}\end{bmatrix} + \underbrace{\frac{\dot{\varphi}_* mR^2}{2}\begin{bmatrix}0 & -3 & 0\\ 3 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{\mathbf{G}}\begin{bmatrix}\dot{\vartheta}\\ \dot{\psi}\\ \dot{\tilde{\varphi}}\end{bmatrix} + \underbrace{mgR\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{\mathbf{K}}\begin{bmatrix}\vartheta\\ \psi\\ \tilde{\varphi}\end{bmatrix} = \mathbf{0}$$

Non-smooth effects

- Lift-off ($N \leq 0$) and/or collision
- **Dry friction**
  - Rolling (8 first order ODE, Appell)
    $\omega_1,\ \omega_2,\ \omega_3,\ \vartheta,\ \psi,\ \varphi,\ x_G,\ y_G$
  - Sliding (10 first order ODE)
    $\omega_1,\ \omega_2,\ \omega_3,\ \vartheta,\ \psi,\ \varphi,\ x_G,\ y_G,\ \dot{x}_G,\ \dot{y}_G$

$$\mathbf{F}_C \begin{cases} \in [-\mu N,\ \mu N] & \text{if} \quad \mathbf{v}_P = \mathbf{0} \quad \text{(rolling)} \\ = -\mu N \frac{\mathbf{v}_P}{|\mathbf{v}_P|} & \text{if} \quad \mathbf{v}_P \neq \mathbf{0} \quad \text{(sliding)} \end{cases}$$

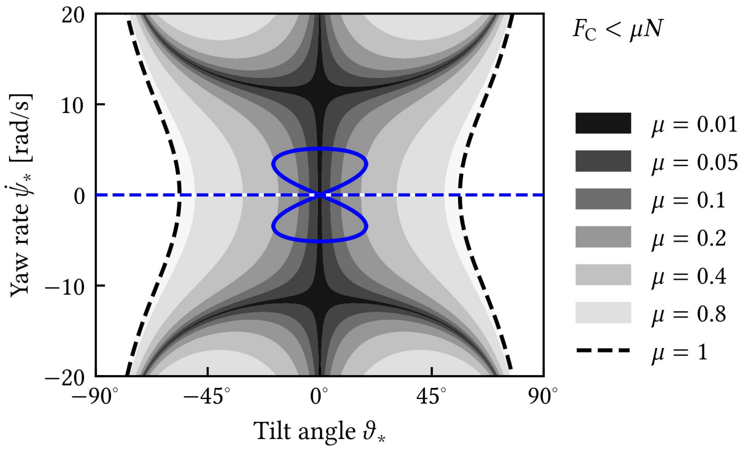


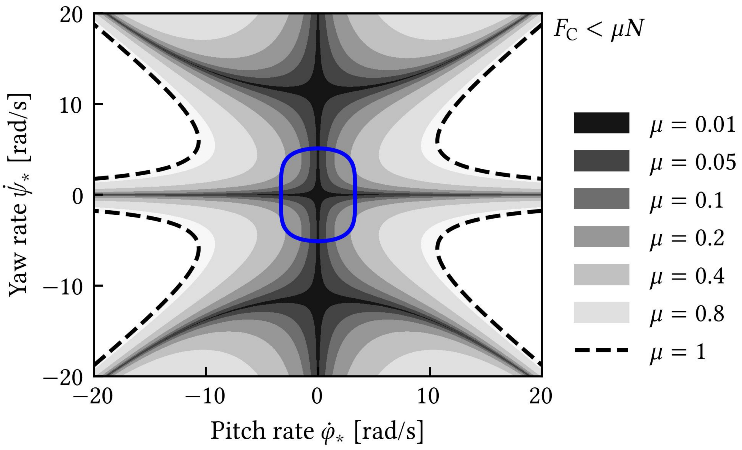


Literature recommendation about 3D rolling: ANTALI, M., STEPAN, G.: Nonsmooth analysis of three-dimensional slipping and rolling in the presence of dry friction. *Nonlinear Dyn* **97**, 1799–1817 (2019).

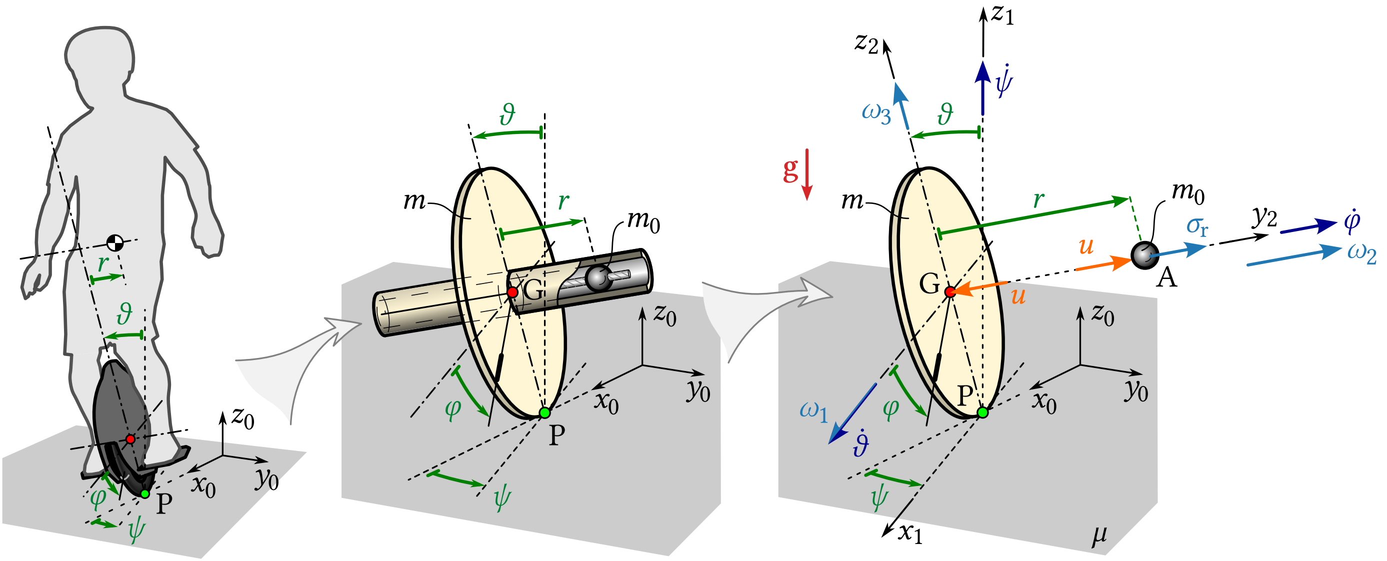
Dry friction and steady states

$$F_C = |\mathbf{F}_C| = \frac{m}{5}\left|5R\dot{\psi}^2\sin^3\vartheta - 4g\sin\vartheta\cos\vartheta + R\left(6\sin^2\vartheta - 1\right)\dot{\psi}\dot{\varphi}\right|$$
$$N = mg$$

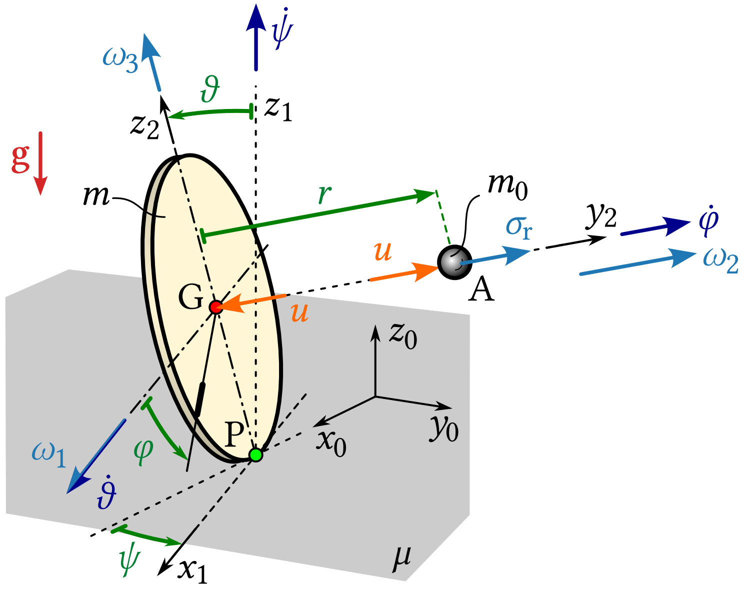




Unicycle model



Mechanical model



Overview

- Disc: $m \;\; R \;\; \mathbf{g}$ Point mass: m_0
- Coordinate frames: $\mathrm{F}_0 \underset{\psi}{\longrightarrow} \mathrm{F}_1 \underset{\vartheta}{\longrightarrow} \mathrm{F}_2 \underset{\varphi}{\longrightarrow} \mathrm{F}_3$
- Generalized coordinates: *(6 pcs)*

$$x_G \quad y_G \quad \psi \quad \vartheta \quad \varphi \quad r$$

- Pseudo velocities: *(4 pcs)*

$$\omega_1 := \dot\vartheta \qquad \omega_2 := \dot\psi \sin\vartheta + \dot\varphi \qquad \omega_3 := \dot\psi \cos\vartheta$$

$$\sigma := \dot r \quad \text{or} \quad \sigma := \dot r - \dot\vartheta R$$

- States: *(10 pcs)*

$$\mathbf{x} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 & \sigma & \vartheta & r & \psi & \varphi & x_G & y_G \end{bmatrix}^{\mathrm{T}}.$$

Essential motion with $\sigma := \dot r$

$$\dot\omega_1 = \frac{1}{5mR^2+4m_0r^2}\Big(4\omega_1^2 m_0 Rr - \omega_3^2(mR^2+4m_0r^2)\tan\vartheta - 8\omega_1\sigma m_0 r + \omega_2\omega_3(6mR^2+4m_0Rr\tan\vartheta) - 4m_0 gr\cos\vartheta + 4mgR\sin\vartheta + 4Ru\Big)\,,$$

$$\dot\omega_2 = \frac{2}{3mR^2+2m_0R^2+12m_0r^2}\Big(-2\omega_1\omega_2 m_0 Rr - \omega_1\omega_3(mR^2+m_0R^2+4m_0r^2) + 2\omega_3\sigma m_0 R\Big)\,,$$

$$\dot\omega_3 = \frac{1}{3mR^2+2m_0R^2+12m_0r^2}\Big(-24\omega_3\sigma m_0 r - \omega_1\omega_2(6mR^2+4m_0R^2) + \omega_1\omega_3\big((3mR^2+2m_0R^2+12m_0r^2)\tan\vartheta + 4m_0Rr\big)\Big)\,,$$

$$\dot\sigma = \frac{m_0}{5mR^2+4m_0r^2}\Big(\omega_1^2(5mR^2+4m_0(R^2+r^2))r - 8\omega_1\sigma R^2 r + \omega_3^2(5mR^2r - 4m_0Rr^2\tan\vartheta + 4m_0r^3 - mR^3\tan\vartheta) + \omega_2\omega_3R(mR^2+4m_0(Rr\tan\vartheta - r^2)) - (mR^2+4m_0r^2)g\sin\vartheta - 4m_0gRr\cos\vartheta + (5\tfrac{m}{m_0}R^2+4R^2+4r^2)u\Big)\,,$$

$$\dot\vartheta = \omega_1\,,$$

$$\dot r = \sigma\,,$$

Essential motion with $\sigma := \dot r - \dot\vartheta R$

$$\dot\omega_1 = \frac{1}{5mR^2+4m_0r^2}\Big(-4\omega_1^2 m_0 Rr - \omega_3^2(mR^2+4m_0r^2)\tan\vartheta - 8\omega_1\sigma m_0 r + \omega_2\omega_3(6mR^2+4m_0Rr\tan\vartheta) - 4m_0 gr\cos\vartheta + 4mgR\sin\vartheta + 4Ru\Big),$$

$$\dot\omega_2 = \frac{2}{3mR^2+2m_0R^2+12m_0r^2}\Big(-2\omega_1\omega_2 m_0 Rr + \omega_1\omega_3(m_0R^2-mR^2-4m_0r^2) + 2\omega_3\sigma m_0 R\Big),$$

$$\dot\omega_3 = \frac{1}{3mR^2+2m_0R^2+12m_0r^2}\Big(-24\omega_3\sigma m_0 r - \omega_1\omega_2(6mR^2+4m_0R^2) + \omega_1\omega_3\big((3mR^2+2m_0R^2+12m_0r^2)\tan\vartheta - 20m_0Rr\big)\Big),$$

$$\dot\sigma = \omega_1^2 r + \omega_3^2 r - \omega_2\omega_3 R - g\sin\vartheta + \frac{1}{m_0}u,$$

$$\dot\vartheta = \omega_1,$$

$$\dot r = \sigma + \omega_1 R,$$

Hidden motion (same)

$$\dot\psi = \omega_3\frac{1}{\cos\vartheta},$$

$$\dot\varphi = \omega_2 - \omega_3\tan\vartheta,$$

$$\dot x_G = \omega_1 R\sin\psi\cos\vartheta + \omega_2 R\cos\psi, \quad \text{or} \quad \dot x_P = \omega_2 R\cos\psi - \omega_3 R\tan\vartheta\cos\psi\,,$$

$$\dot y_G = -\omega_1 R\cos\psi\cos\vartheta + \omega_2 R\sin\psi, \quad \text{or} \quad \dot y_P = \omega_2 R\sin\psi - \omega_3 R\tan\vartheta\sin\psi\,.$$

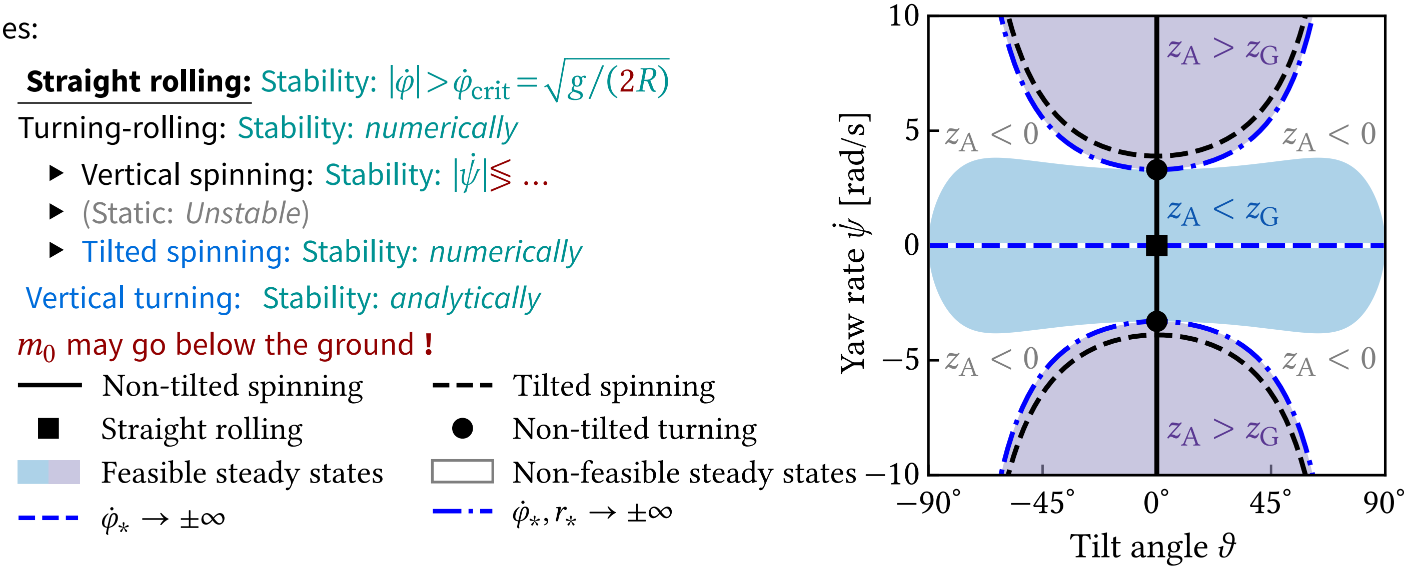
Steady state motion

$$\left.\begin{array}{l}\dot\psi_*^2\left(-4m_0Rr_*\sin^2\vartheta_* + \left(-5mR^2+4m_0r_*^2\right)\sin\vartheta_*\cos\vartheta_*\right)+\ldots \\ \dot\psi_*^2\left(4m_0R\sin\vartheta_*\cos\vartheta_* - 4m_0r_*\cos^2\vartheta_*\right)+4R\dot\psi_*\dot\varphi_*m_0\cos\vartheta_* + 4m_0g\sin\vartheta_*\end{array}\right\} \rightarrow \begin{array}{l}\dot\varphi_*(\vartheta_*,\dot\psi_*)=\ldots \\ r_*(\vartheta_*,\dot\psi_*)=\ldots\end{array}$$

Types:

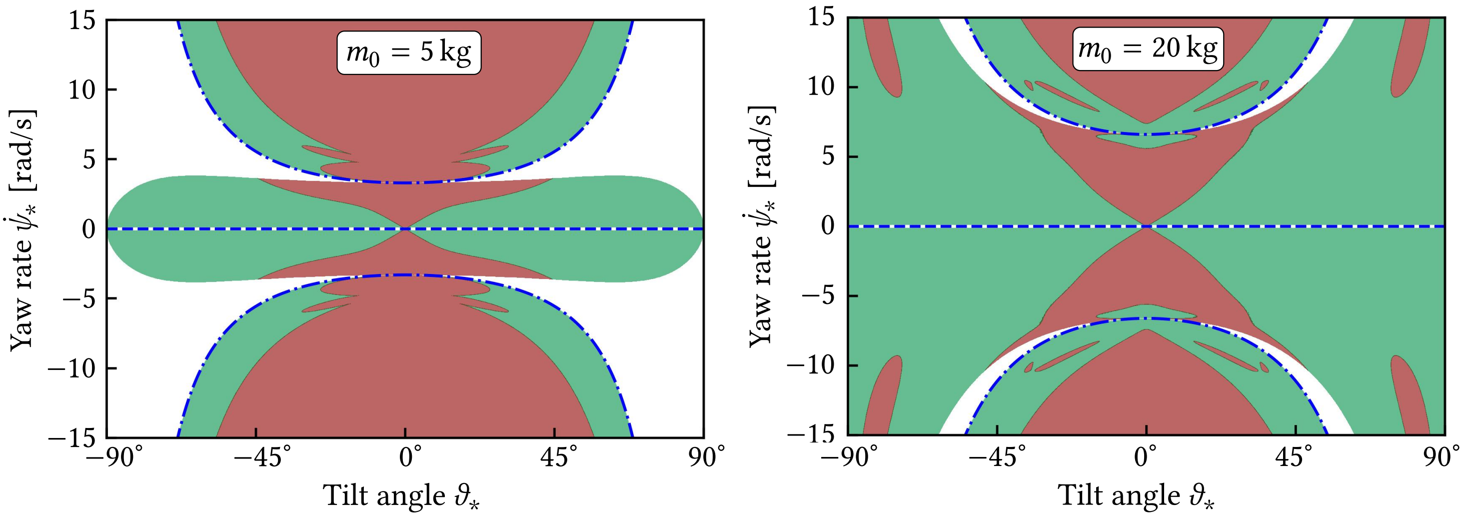
- **Straight rolling:** Stability: $|\dot\varphi| > \dot\varphi_{\text{crit}} = \sqrt{g/(2R)}$
- Turning-rolling: Stability: *numerically*
  - Vertical spinning: Stability: $|\dot\psi| \lessgtr \ldots$
  - (Static: *Unstable*)
  - Tilted spinning: Stability: *numerically*
- Vertical turning: Stability: *analytically*

! m_0 may go below the ground **!**

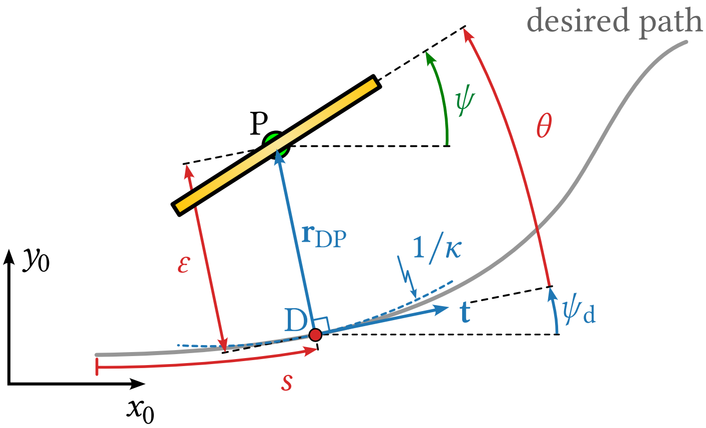


Stability of the steady state motion

$m = 10\,\mathrm{kg} \quad R = 0.3\,\mathrm{m}$



Path-following problem



Desired path:

- Curvature $\kappa(s)$

Coordinate transform: $x_\mathrm{P},\ y_\mathrm{P},\ \psi \quad \rightarrow \quad s,\ \varepsilon,\ \theta$

$$\dot{s} = \frac{1}{1-\kappa\varepsilon}\big(\omega_2 R\cos\theta - \omega_3 R\cos\theta\tan\vartheta\big)$$

$$\dot{\varepsilon} = \omega_2 R\sin\theta - \omega_3 R\sin\theta\tan\vartheta$$

$$\dot{\theta} = \frac{-1}{(1-\kappa\varepsilon)\cos\vartheta}\big(\omega_2 R\kappa\cos\theta\cos\vartheta - \omega_3\,(1-\kappa\varepsilon + R\kappa\sin\vartheta\,\cos\theta)\,\big)$$

Linear state space model

Based on straight rolling steady state: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$

$$
\begin{aligned}
\dot{\omega}_1 &= \frac{6}{5}\omega_3\dot{\varphi}_* + \frac{4g}{5R}\vartheta - \frac{4m_0 g}{5mR^2}r + \frac{4}{5mR}u\,,\\
\dot{\hat{\omega}}_2 &= 0\,,\\
\dot{\omega}_3 &= -2\omega_1\dot{\varphi}_*\,,\\
\dot{\sigma} &= -\omega_3\dot{\varphi}_* R - g\vartheta + \frac{1}{m_0}u\,,\\
\dot{\vartheta} &= \omega_1\,,\\
\dot{r} &= \omega_1 R + \sigma\,,\\
\dot{\hat{\psi}} &= \omega_3\,,\\
\dot{\varphi} &= \hat{\omega}_2 + \dot{\varphi}_*\,,\\
\dot{x}_P &= \hat{\omega}_2 R\cos\psi_* - \dot{\varphi}_* R\hat{\psi}\sin\psi_* + \dot{\varphi}_* R\cos\psi_*\,,\\
\dot{y}_P &= \hat{\omega}_2 R\sin\psi_* + \dot{\varphi}_* R\hat{\psi}\cos\psi_* + \dot{\varphi}_* R\sin\psi_*\,,\\
\dot{s} &= \hat{\omega}_2 R + \dot{\varphi}_* R\,,\\
\dot{\varepsilon} &= \dot{\varphi}_* R\theta\,,\\
\dot{\theta} &= \omega_3\,.
\end{aligned}
$$

$$
\mathbf{x} = \begin{bmatrix} \omega_1 & \hat{\omega}_2 & \omega_3 & \sigma & \vartheta & r & \hat{\varphi} & \hat{s} & \varepsilon & \theta \end{bmatrix}^{\mathsf{T}}
$$

$$
\mathbf{A} = \begin{bmatrix}
0 & 0 & \frac{6}{5}\dot{\varphi}_* & 0 & \frac{4g}{5R} & -\frac{4m_0 g}{5mR^2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2\dot{\varphi}_* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -R\dot{\varphi}_* & 0 & -g & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
R & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R\dot{\varphi}_*\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\mathbf{B} = \begin{bmatrix} \frac{4}{5mR} & 0 & 0 & \frac{1}{m_0} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}}
$$

Not controllable: $\mathrm{Rank}([\mathbf{B}\ \ \mathbf{AB}\ \ \dots\ \ \mathbf{A}^9\mathbf{B}]) = 6$

Reduced model for steering

- Reduced states: $\tilde{\mathbf{x}} = \begin{bmatrix}\omega_1 & \sigma & \vartheta & r & \varepsilon & \theta\end{bmatrix}^\mathsf{T}$

$$\dot{\omega}_1 = \left(-\frac{12\dot{\varphi}_*^2}{5} + \frac{4g}{5R}\right)\vartheta - \frac{4m_0 g}{5mR^2}r + \frac{4}{5mR}u$$

$$\dot{\sigma} = \left(2R\dot{\varphi}_*^2 - g\right)\vartheta + \frac{1}{m_0}u$$

$$\dot{\vartheta} = \omega_1$$

$$\dot{r} = \sigma + \omega_1 R$$

$$\dot{\varepsilon} = \dot{\varphi}_* R\theta$$

$$\dot{\theta} = -2\dot{\varphi}_*\vartheta$$

- Control input: $u := \mathbf{K}(\tilde{\mathbf{x}} - \tilde{\mathbf{x}}_{\text{des}})$

$$\mathbf{K} = [D_\vartheta \;\; D_\sigma \;\; P_\vartheta \;\; P_r \;\; P_\varepsilon \;\; P_\theta]$$

- Pole placement, Ackermann formula, $\lambda_i := -8$

$$\tilde{\mathbf{x}} = \begin{bmatrix}\omega_1 & \sigma & \vartheta & r & \varepsilon & \theta\end{bmatrix}^\mathsf{T}$$

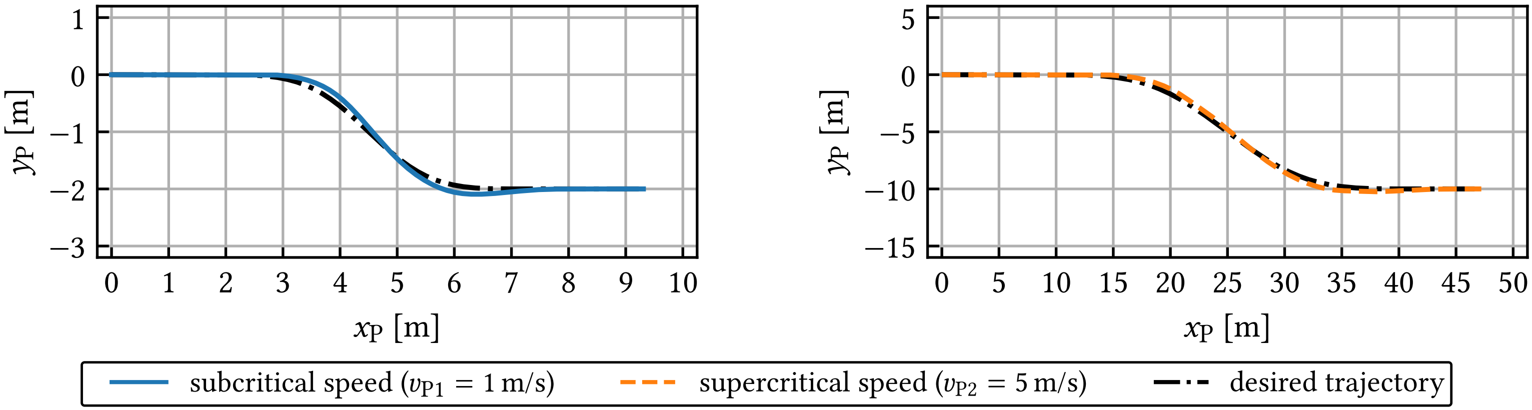
$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 0 & \frac{-12\dot{\varphi}_*^2 R + 4g}{5R} & -\frac{4m_0 g}{5mR^2} & 0 & 0 \\ 0 & 0 & 2\dot{\varphi}_*^2 R - g & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ R & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R\dot{\varphi}_* \\ 0 & 0 & -2\dot{\varphi}_* & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{B}} = \begin{bmatrix}\frac{4}{5mR} & \frac{1}{m_0} & 0 & 0 & 0 & 0\end{bmatrix}^\mathsf{T}$$

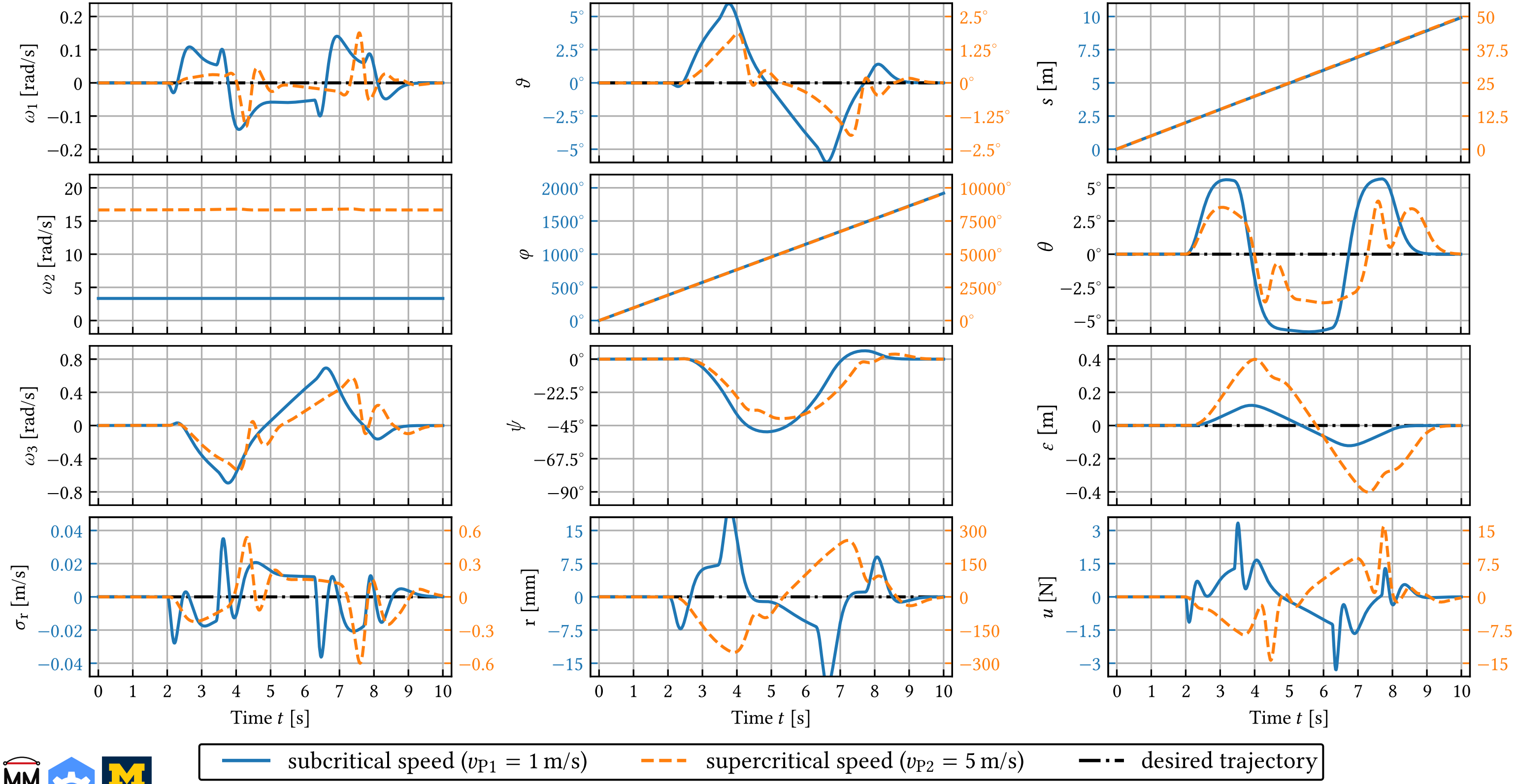
- **Controllable**: $\text{Rank}([\tilde{\mathbf{B}} \;\; \tilde{\mathbf{A}}\tilde{\mathbf{B}} \;\; \dots \;\; \tilde{\mathbf{A}}^5\tilde{\mathbf{B}}]) = 6$
- Everything depends on $\dot{\varphi}_*(\neq 0)$!
- Sub-or supercritical speed ?

$$|\dot{\varphi}_*| \lessgtr \dot{\varphi}_{\text{crit}} = \sqrt{g/(2R)}$$

Maneuver: Lane change



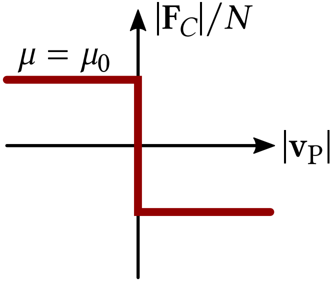
Maneuver: Lane change

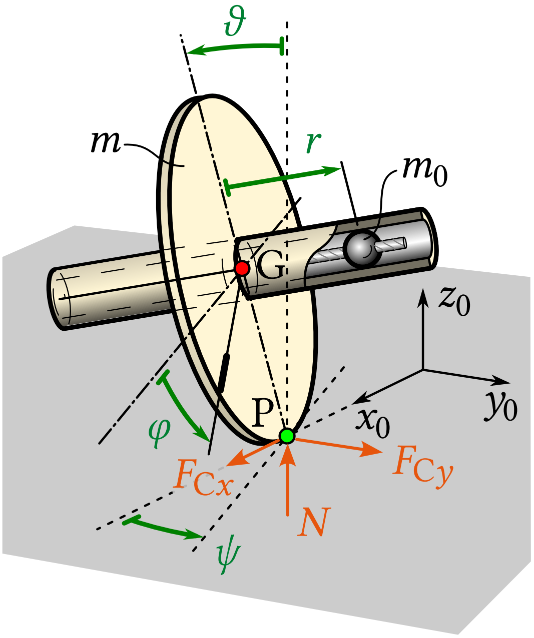


Non-smooth effects

- Lift-off ($N \leq 0$) of the wheel
- Collision of axle (point mass) and ground
- **Dry friction**
  - Rolling (10 first order ODE, Appell)
    $\omega_1, \omega_2, \omega_3, \vartheta, \sigma, r, \psi, \varphi, x_G, y_G$
  - Sliding (12 first order ODE)
    $\omega_1, \omega_2, \omega_3, \vartheta, \sigma, r, \psi, \varphi, x_G, y_G, \dot{x}_G, \dot{y}_G$

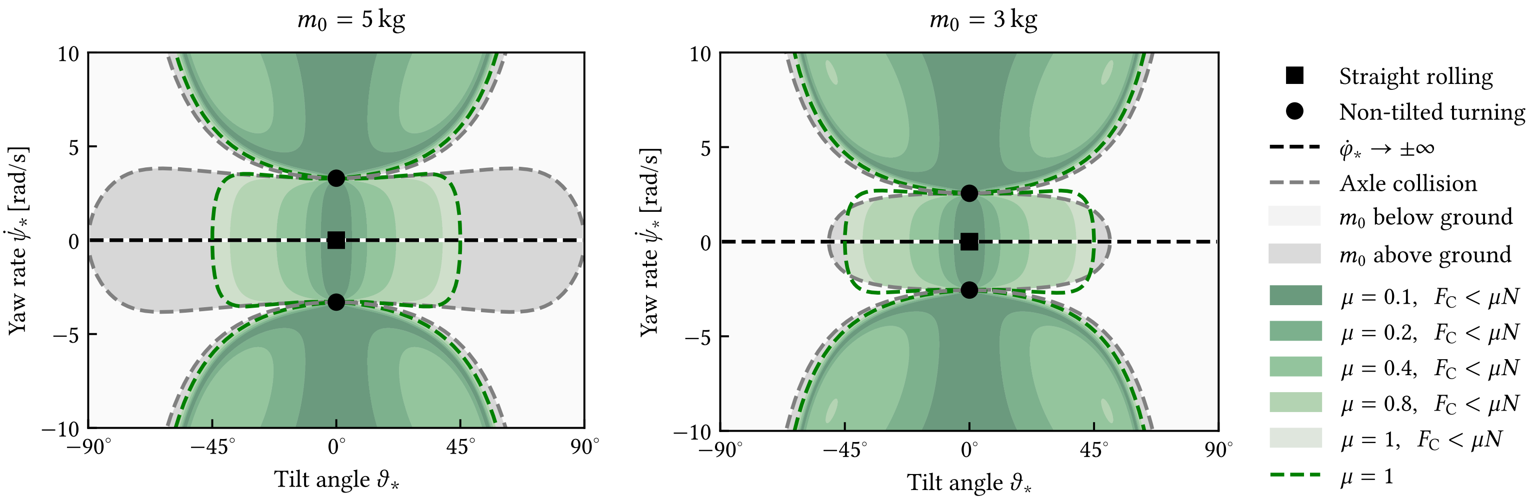
$$\mathbf{F}_C \begin{cases} \in [-\mu N,\ \mu N] & \text{if} \quad \mathbf{v}_P = \mathbf{0} \quad \text{(rolling)} \\ = -\mu N \frac{\mathbf{v}_P}{|\mathbf{v}_P|} & \text{if} \quad \mathbf{v}_P \neq \mathbf{0} \quad \text{(sliding)} \end{cases}$$



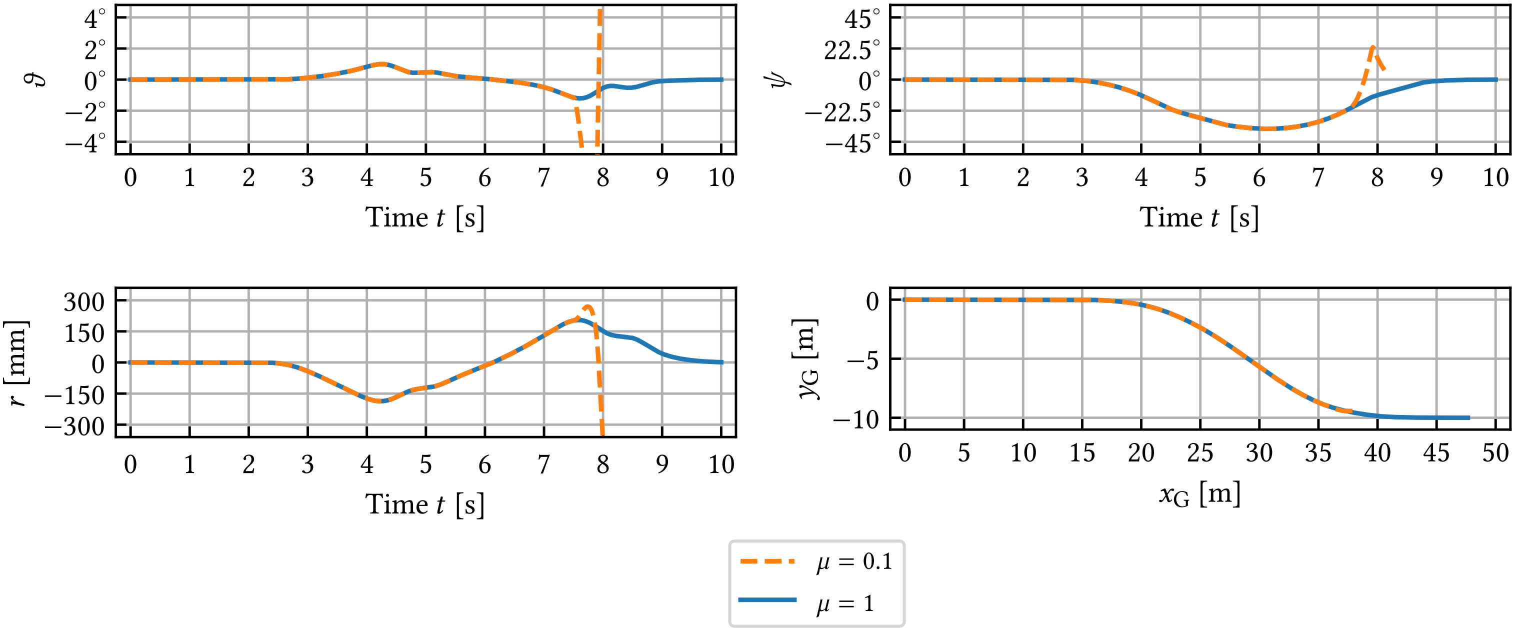


Dry friction and steady states

$$F_C = |\mathbf{F}_C| = \dots$$
$$N = \dots$$



Maneuver: Lane change



Thank you for your attention!