EXTENSIONS OF SYMPLECTIC NUMERICAL METHODS FOR SIMULATIONS OF IRREVERSIBLE SYSTEMS

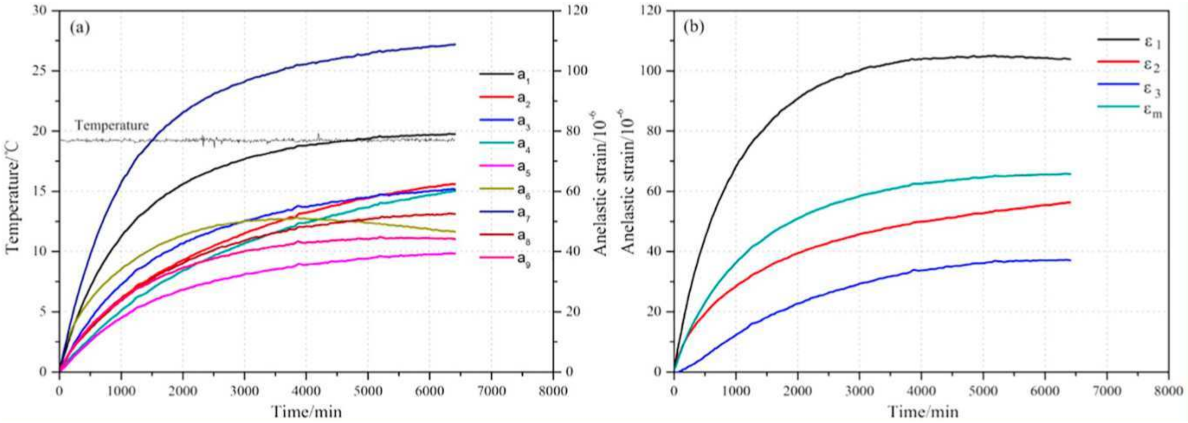
Tamás FÜLÖP

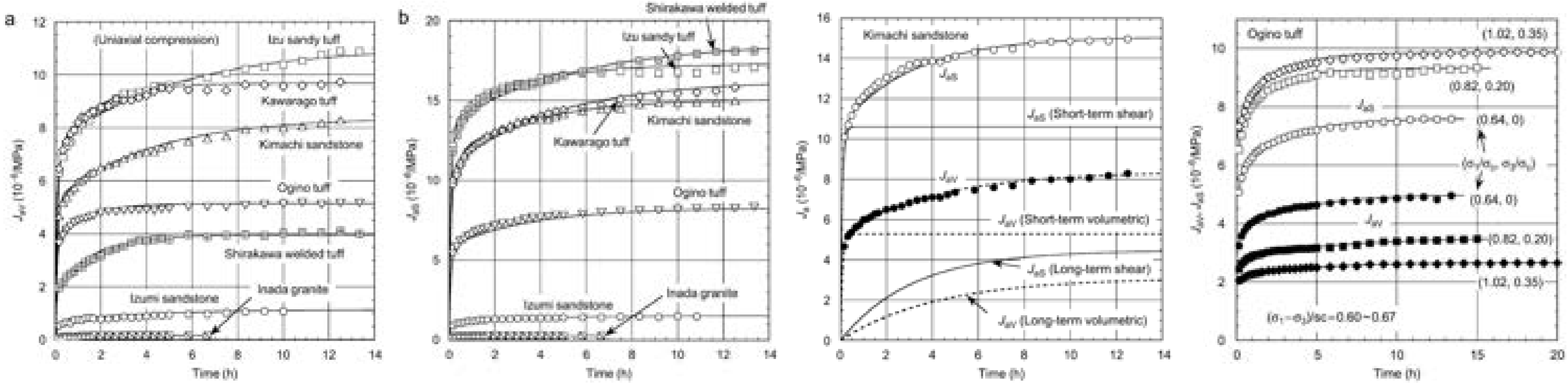
Department of Energy Engineering,
Budapest Univ. of Technology and Econ.

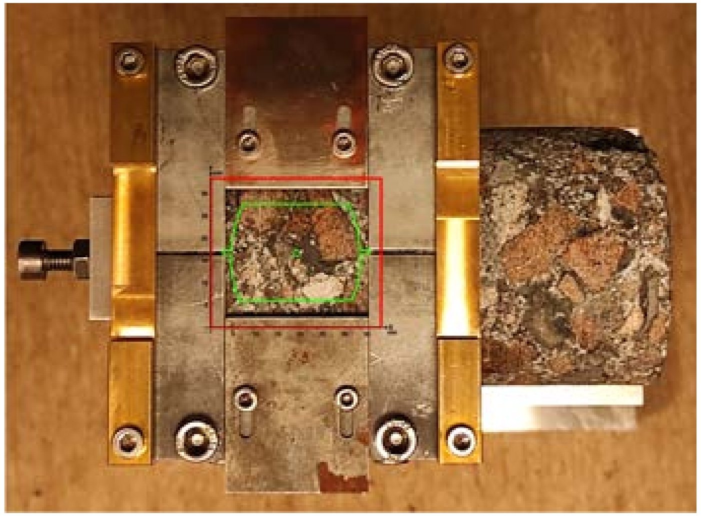
Montavid Thermodynamic Research Group,
Budapest, Hungary

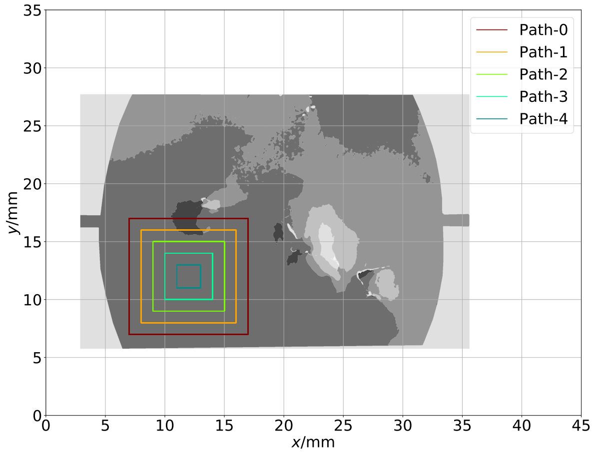
Miklós Farkas Seminar, BME, Budapest, 2024-02-22

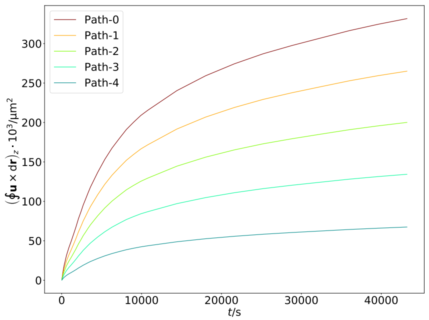
Motivations: Rocks

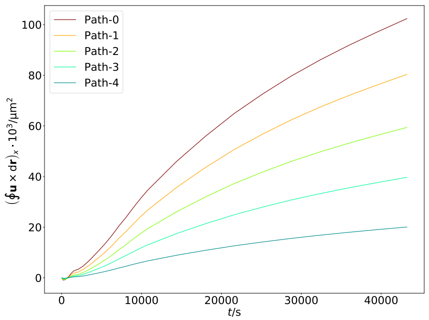


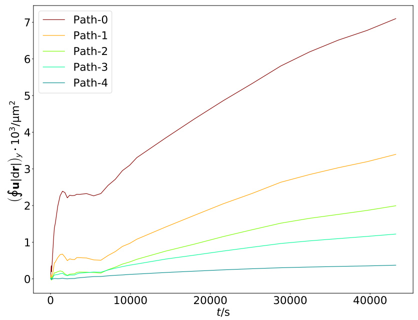


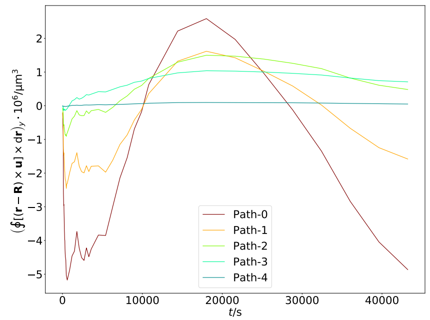



Path-0
Path-1
Path-2
Path-3
Path-4
y/mm
x/mm


Path-0
Path-1
Path-2
Path-3
Path-4
(∮u × dr)z · 10³/μm²
t/s


Path-0
Path-1
Path-2
Path-3
Path-4
(∮u × dr)x · 10³/μm²
t/s


Path-0
Path-1
Path-2
Path-3
Path-4
(∮u|dr|)y · 10³/μm²
t/s

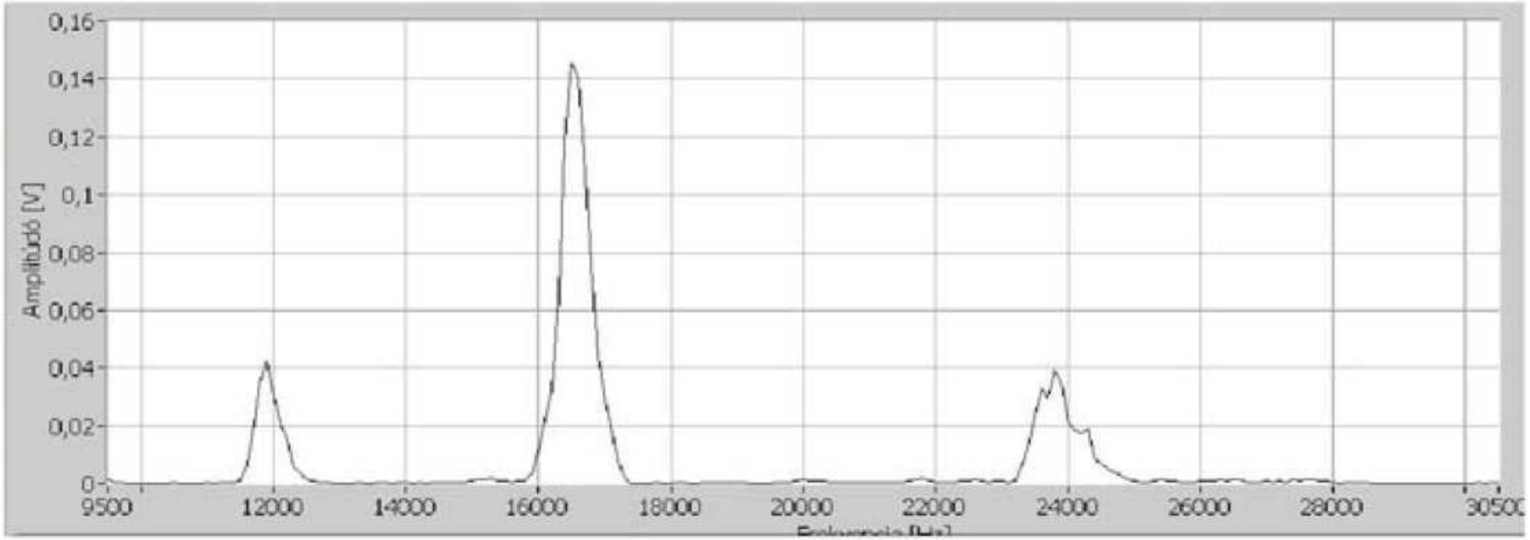

Path-0
Path-1
Path-2
Path-3
Path-4
(∮(r − R) × u) × dr)y · 10⁶/μm³
t/s

Wave propagation:

MEGJEGYZÉS: Felül/alul apróhomokkő/finomhomokkő; T: 24,6 C° Relatív páratartalom: 57,50 %

MÉRÉSI ALAPADATOK:

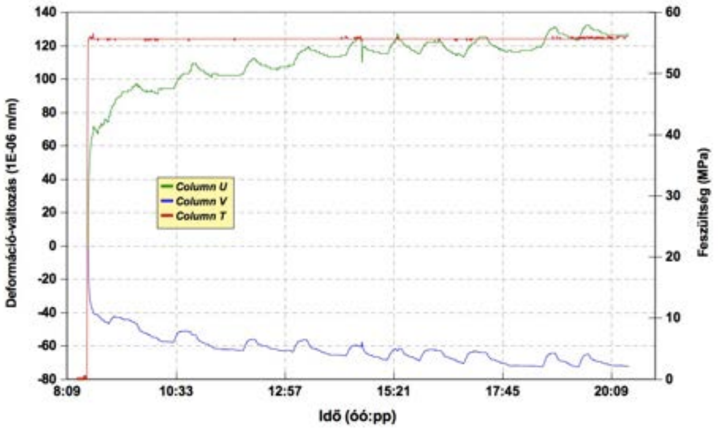
Minta jele:	Származási hely [fm]:	Hossz [mm]:	Átmérő [mm]:	Tömeg [kg]:	Poisson tényező [-]:
FP-40-4T	Szolnok_Magraktár	51.85	37.90	0.1451	0.175

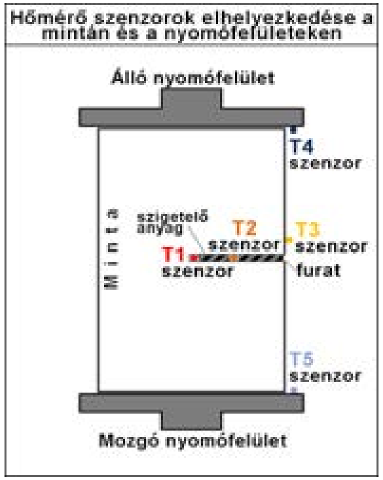


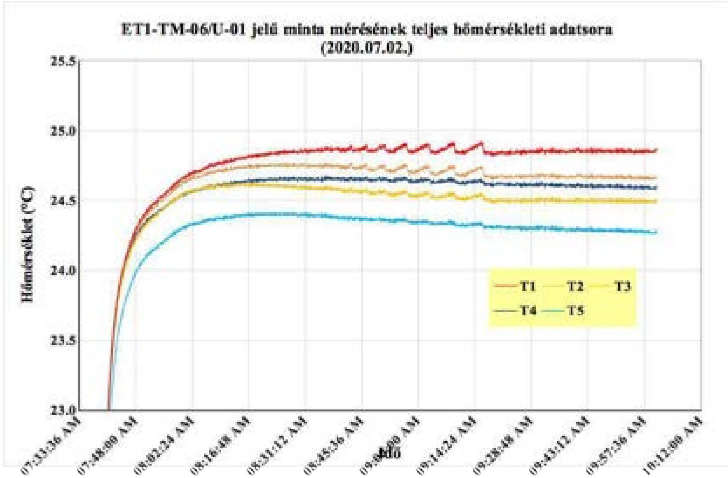
MÉRT ÉS SZÁMÍTOTT ÉRTÉKEK:

Térfogat [cm^3]:	Sűrűség [kg/m^3]:	Dilatációs rezonanciafrekvencia [Hz]:	Dilatációs rezgési sebesség [m/s]:	Dinamikus Young-modulus [GPa]:	Dinamikus nyírási modulus [GPa]:
58.49	2480.56	16500	1701.15	7.11	3.02

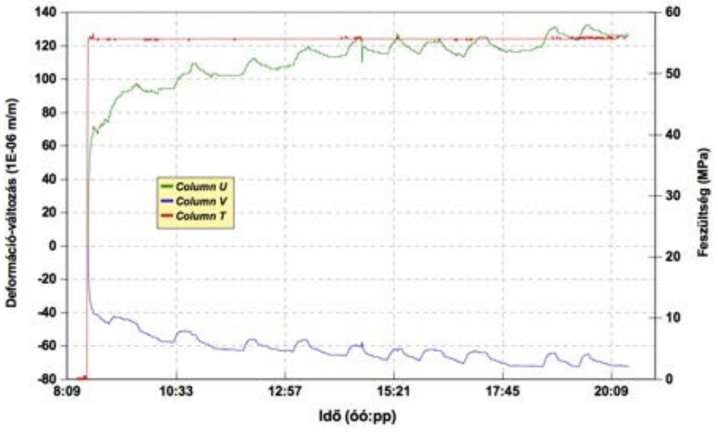
Temperature: thermal expansion, heat conduction:

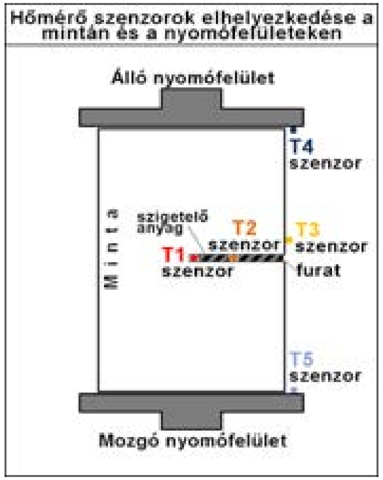


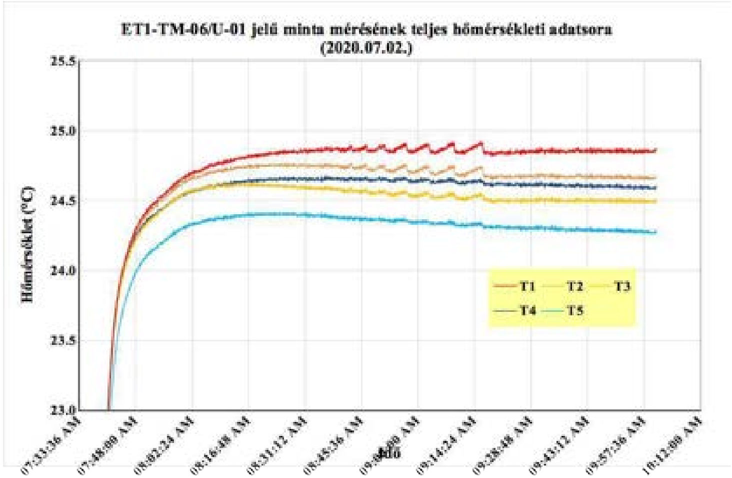


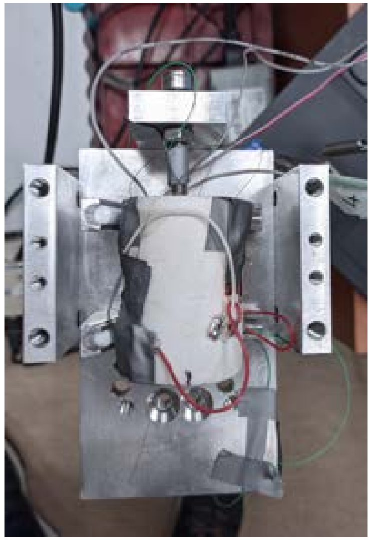


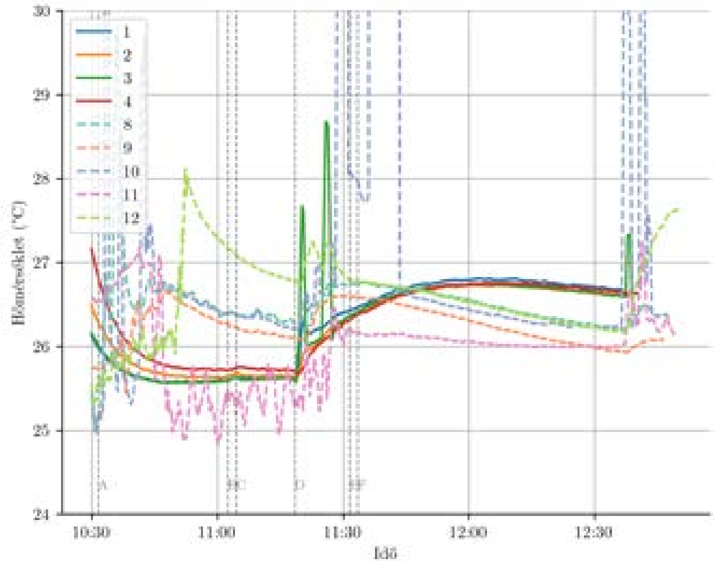
Temperature: thermal expansion, heat conduction:







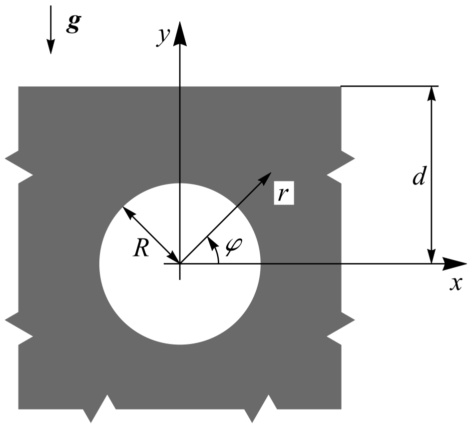


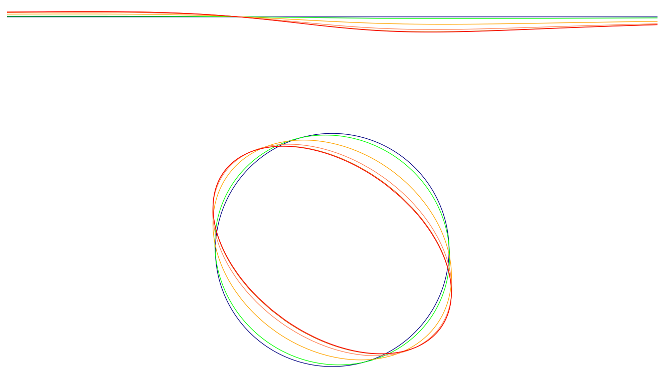


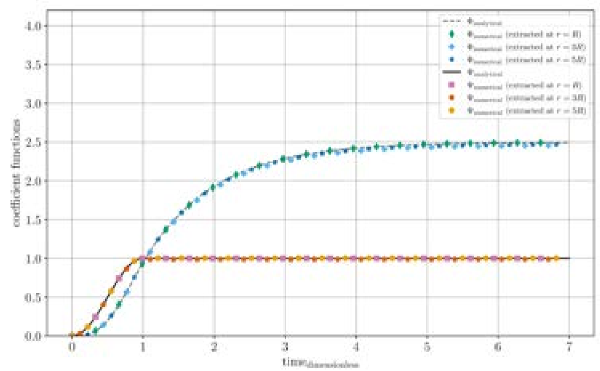
Motivations: Tunnels: an analytical solution method:

$$\mathscr{S}^{\text{dev}} \sum_{k=1}^{J} \left[\lambda_k(t) \sum_{j=1}^{J} c_j(\eta_k)\, \mathbf{s}_j^{\text{dev}}(\mathbf{r}) \right] = \mathscr{Z}^{\text{dev}} \sum_{k=1}^{J} \left[\kappa_k(t) \sum_{j=1}^{J} \frac{1}{\eta_k} c_j(\eta_k)\, \mathbf{s}_j^{\text{dev}}(\mathbf{r}) \right]$$

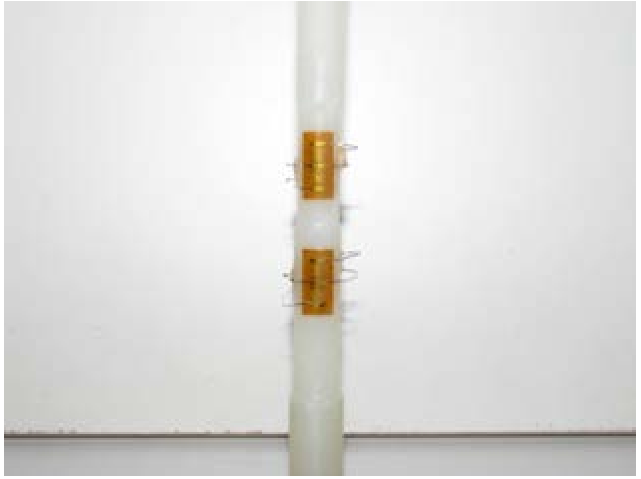
$$\mathscr{S}^{\text{sph}} \sum_{k=1}^{J} \left[\lambda_k(t) \sum_{j=1}^{J} c_j(\eta_k)\, \mathbf{s}_j^{\text{sph}}(\mathbf{r}) \right] = \mathscr{Z}^{\text{sph}} \sum_{k=1}^{J} \left[\kappa_k(t) \sum_{j=1}^{J} c_j(\eta_k)\, \mathbf{s}_j^{\text{sph}}(\mathbf{r}) \right]$$

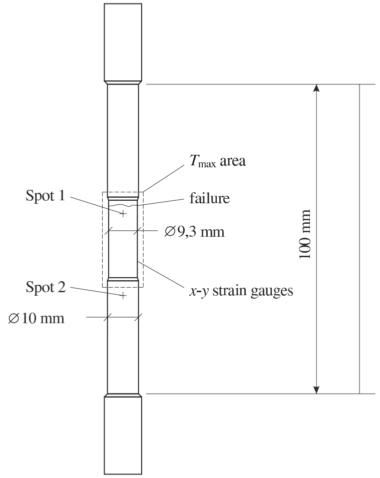


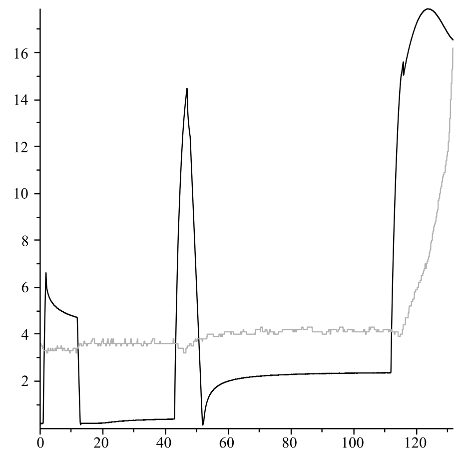


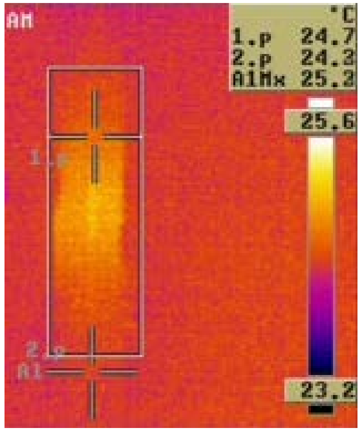


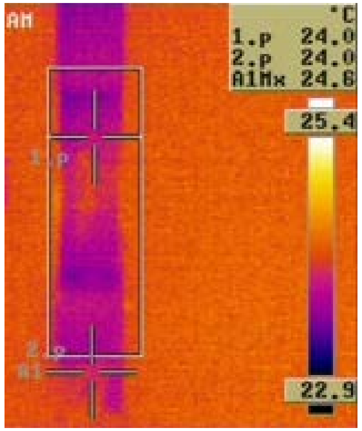
Motivations: Plastics:

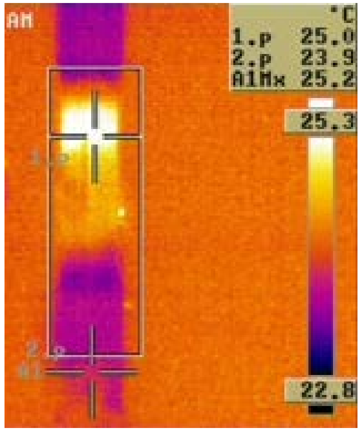


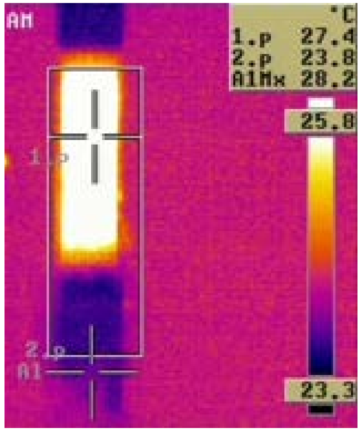


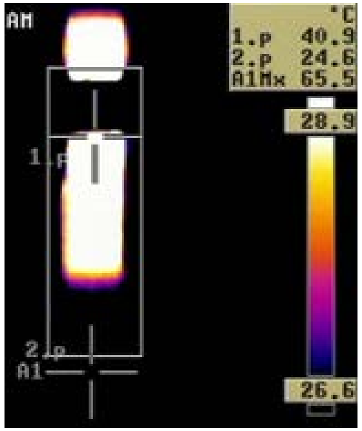








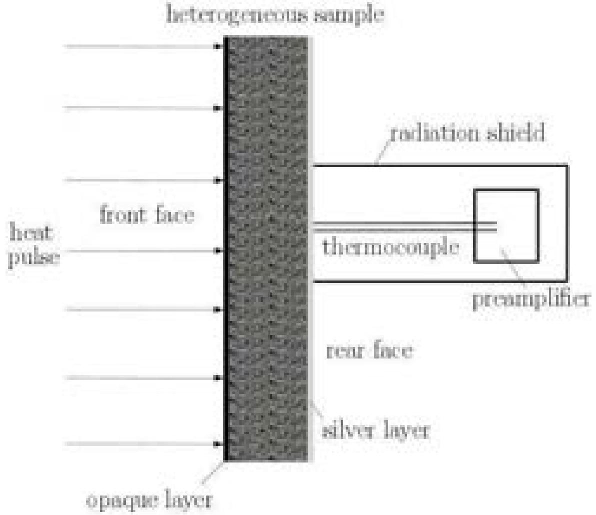


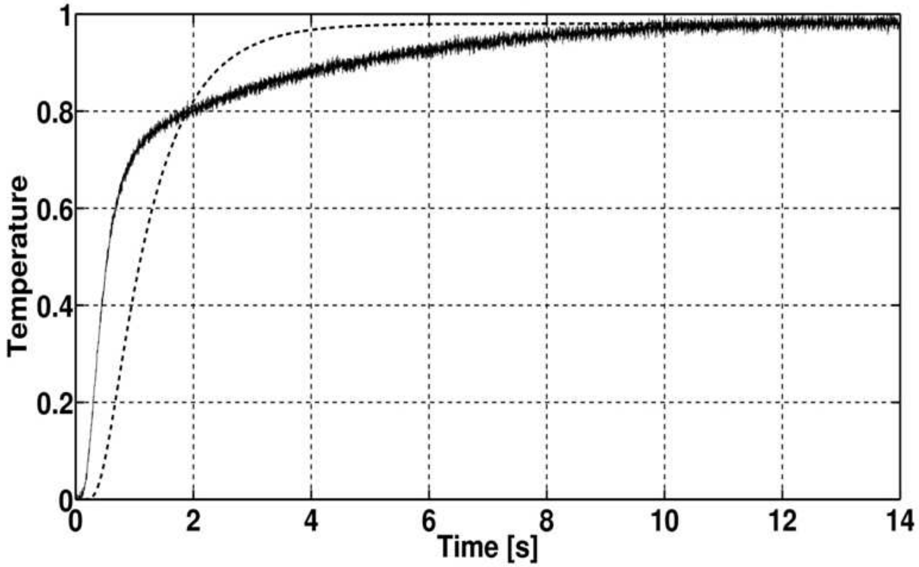


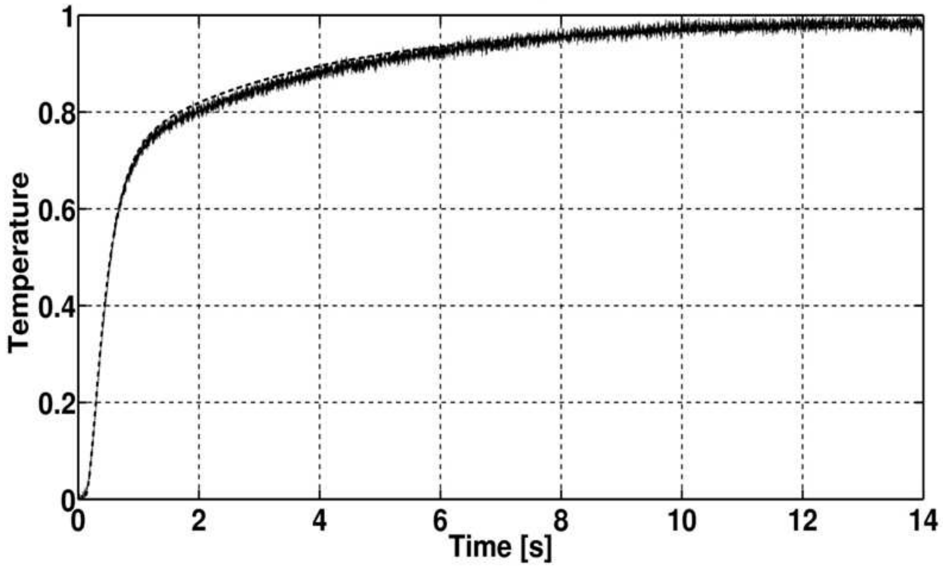
Motivations: Large-deformation elasticity + thermal expansion + plasticity: compatibility criterion:

$$
\begin{aligned}
&\mathrm{tr}_{1,5;3,4}\left[\mathbf{A}^{-1}\otimes\mathbf{A}^{-1}(\mathbf{A}\otimes\nabla)(\mathbf{A}\otimes\nabla)\right]+2\mathbf{A}^{-1}\,\mathrm{tr}_{1,2}\left[\mathbf{A}(\nabla\otimes\nabla\otimes\mathbf{A})\right]\mathbf{A}^{-1}\\
&+2\mathrm{tr}_{2,3}\left[\mathbf{A}^{-1}\otimes(\nabla\cdot\mathbf{A})(\nabla\otimes\mathbf{A})\right]\mathbf{A}^{-1}+2\mathrm{tr}_{1,2}\left[\mathbf{A}^{-1}(\mathbf{A}\otimes\nabla\otimes\nabla)\right]\\
&-2\mathbf{A}^{-1}\,\mathrm{tr}_{2,4}\left[(\mathbf{A}\otimes\nabla\otimes\nabla)\right]+2\mathrm{tr}_{1,4;3,5}\left[\mathbf{A}^{-1}\otimes\mathbf{A}^{-1}(\mathbf{A}\otimes\nabla)(\mathbf{A}\otimes\nabla)\right]\\
&+\mathrm{tr}_{1,2;3,5}\left[\mathbf{A}^{-1}(\mathbf{A}\otimes\nabla)(\mathbf{A}\otimes\nabla)\otimes\mathbf{A}^{-1}\right]\\
&-2\mathbf{A}^{-1}\,\mathrm{tr}_{2,5;3,6}\left[(\mathbf{A}\otimes\nabla)\mathbf{A}(\nabla\otimes\mathbf{A})\otimes\mathbf{A}^{-1}\right]\mathbf{A}^{-1}-2\left[\nabla\otimes(\nabla\cdot\mathbf{A})\right]\mathbf{A}^{-1}\\
&+2\mathrm{tr}_{2,4}\left[(\nabla\otimes\mathbf{A})\mathbf{A}^{-1}(\mathbf{A}\otimes\nabla)\right]\mathbf{A}^{-1}-3\mathrm{tr}_{2,6;3,5}\left[(\nabla\otimes\mathbf{A})\mathbf{A}^{-1}(\mathbf{A}\otimes\nabla)\otimes\mathbf{A}^{-1}\right]\\
&-\mathrm{tr}_{3,4;2,5}\left[\mathbf{A}^{-1}\otimes\mathbf{A}^{-1}(\mathbf{A}\otimes\nabla)\mathbf{A}(\nabla\otimes\mathbf{A})\right]\mathbf{A}^{-1}\\
&-2\mathbf{A}^{-1}\,\mathrm{tr}_{2,4;3,5}\left[(\mathbf{A}\otimes\nabla)\otimes(\nabla\otimes\mathbf{A})\right]\mathbf{A}^{-1}=\mathbf{0}
\end{aligned}
$$

Motivations: Beyond-Fourier heat conduction:



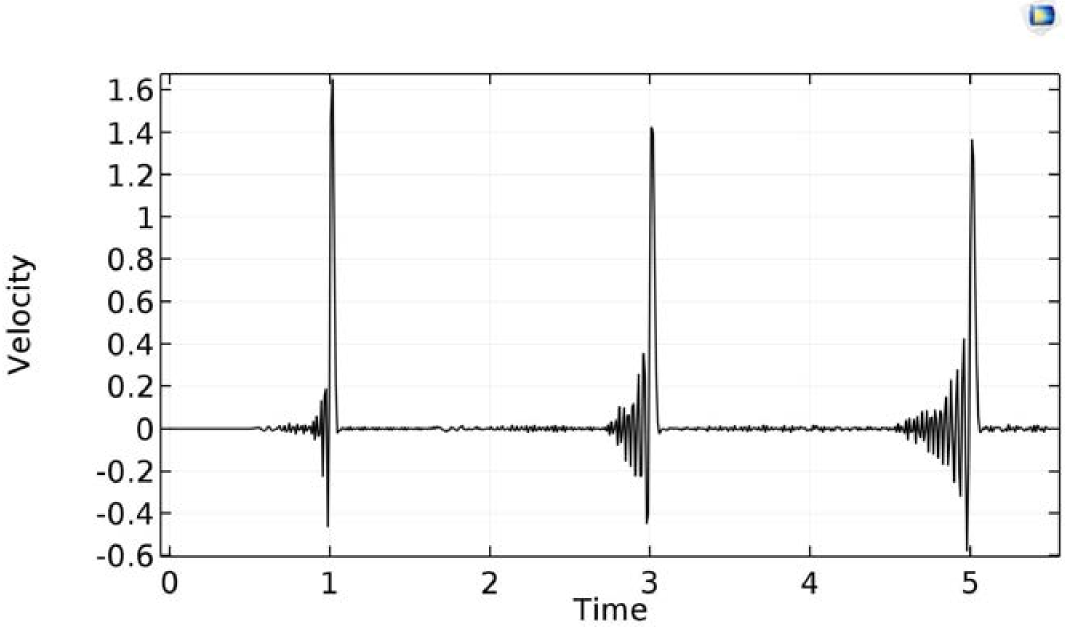


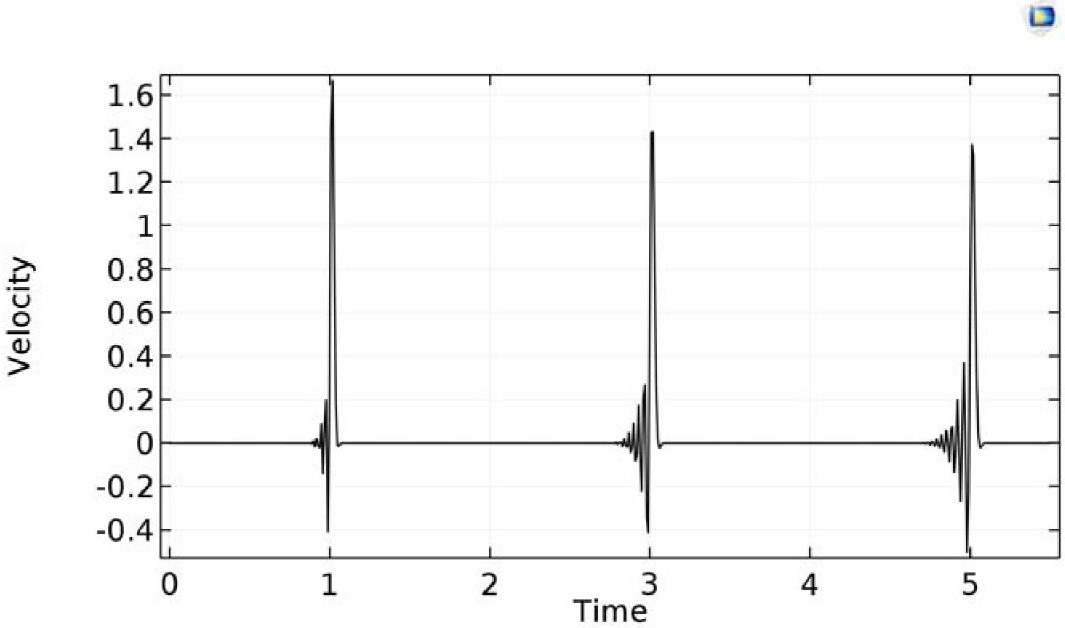


$$\frac{\partial T}{\partial t} = a\,\frac{\partial^2 T}{\partial x^2}$$

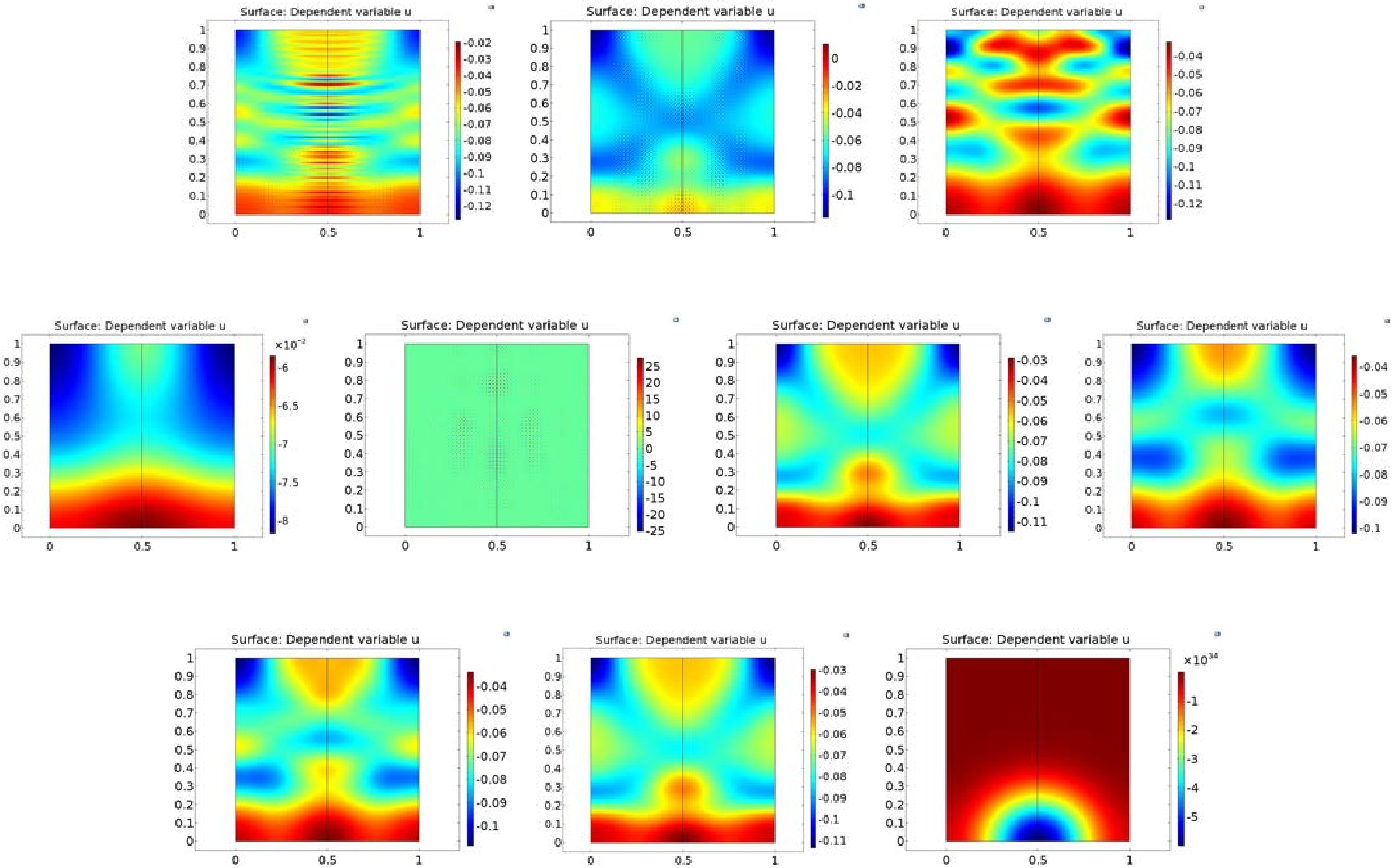
$$\frac{\partial T}{\partial t} + \tau\,\frac{\partial^2 T}{\partial t^2} = a\,\frac{\partial^2 T}{\partial x^2} + b\,\frac{\partial^3 T}{\partial t \partial x^2}$$

Commercial software: 1D wave propagation





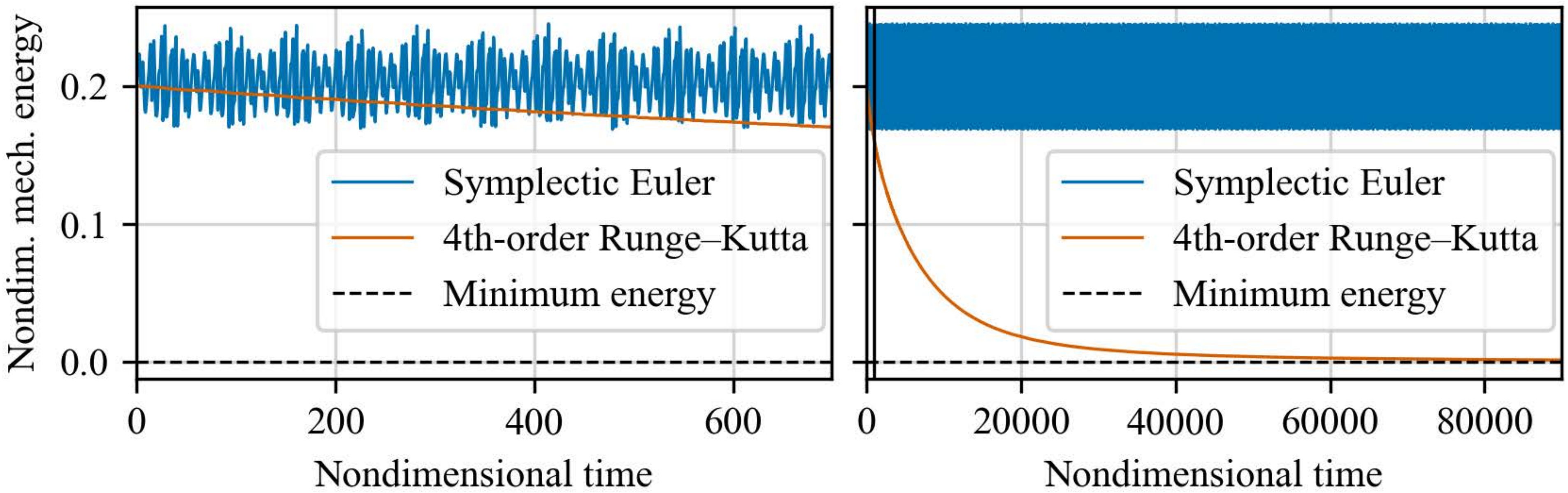
Commercial software: 2D wave propagation



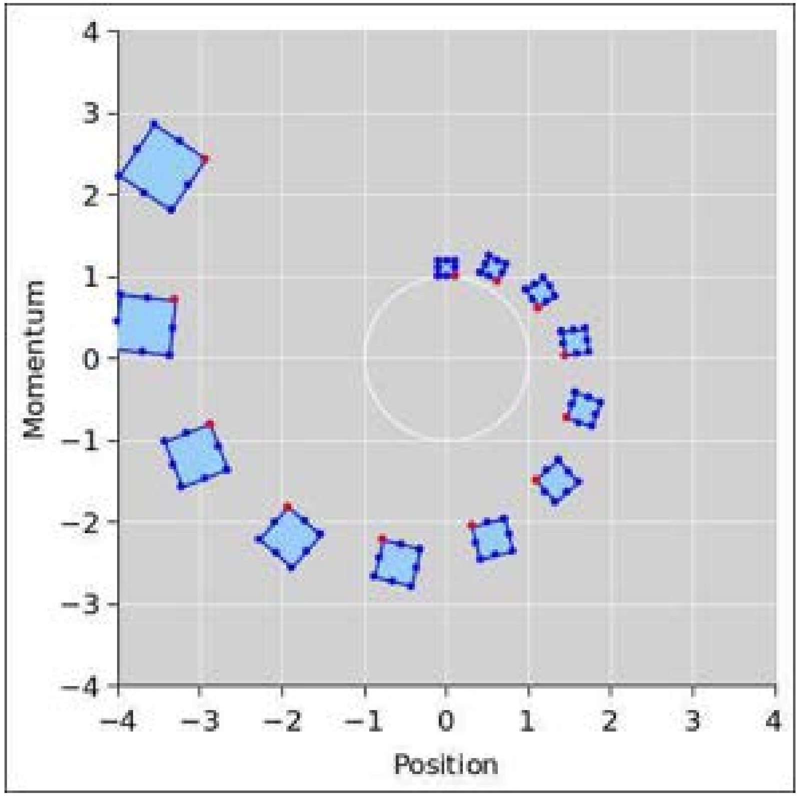
Self-developed numerical schemes: structure-preserving

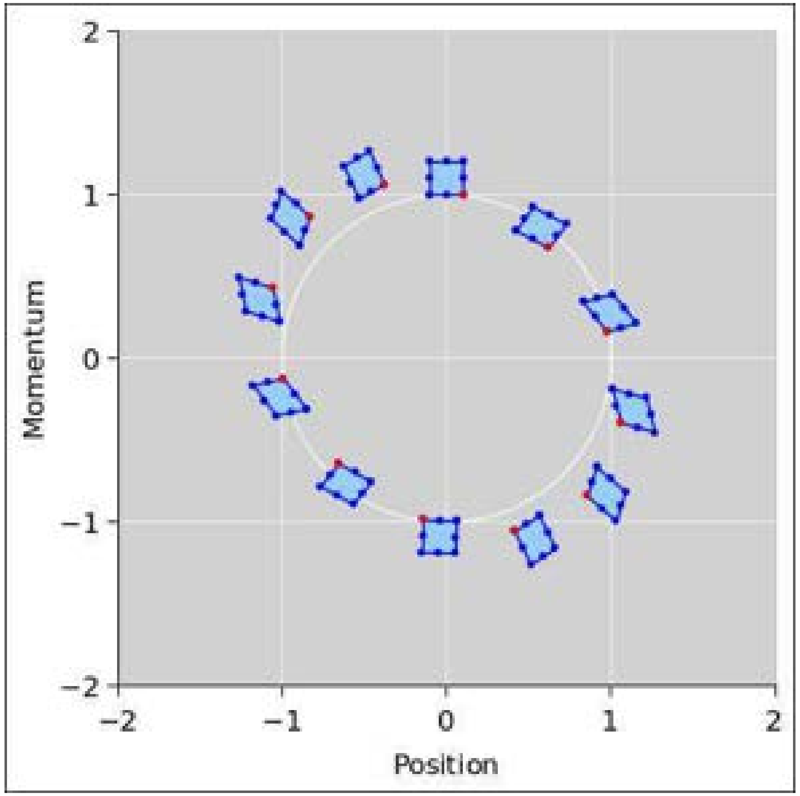
e.g., symplectic methods and irreversible extensions

On the example of the planar elastic pendulum:



Nonsymplectic vs. symplectic in phase space: [J. Denker]





Symplectic Euler: (1st-order accurate)

$$p_{n+1} = p_n - hH_q(p_{n+1}, q_n) \qquad\qquad p_{n+1} = p_n - hH_q(p_n, q_{n+1})$$
$$q_{n+1} = q_n + hH_p(p_{n+1}, q_n) \qquad\qquad q_{n+1} = q_n + hH_p(p_n, q_{n+1})$$

Störmer–Verlet: (2nd-order accurate)

$$\begin{aligned}
p_{n+1/2} &= p_n - \frac{h}{2} H_q(p_{n+1/2}, q_n) \\
q_{n+1} &= q_n + \frac{h}{2} \Big(H_p(p_{n+1/2}, q_n) + H_p(p_{n+1/2}, q_{n+1}) \Big) \\
p_{n+1} &= p_{n+1/2} - \frac{h}{2} H_q(p_{n+1/2}, q_{n+1})
\end{aligned}$$

$$\begin{aligned}
q_{n+1/2} &= q_n + \frac{h}{2} H_q(p_n, q_{n+1/2}) \\
p_{n+1} &= p_n - \frac{h}{2} \Big(H_p(p_n, q_{n+1/2}) + H_p(p_{n+1}, q_{n+1/2}) \Big) \\
q_{n+1} &= q_{n+1/2} + \frac{h}{2} H_q(p_{n+1}, q_{n+1/2})
\end{aligned}$$

Structure-preserving: *e.g.*,

symplectic (preserving a 2-form)

preserving energy, momentum, angular momentum etc.

variational

Tools: *e.g.*,

backward error analysis

modified vector field: $\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{f}\big(\mathbf{z}(t)\big) \;\longrightarrow\; \tilde{\mathbf{z}}(t+h) = \mathbf{\Phi}_h\big(\tilde{\mathbf{z}}(t)\big): \; \frac{\mathrm{d}\tilde{\mathbf{z}}}{\mathrm{d}t} = \tilde{\mathbf{f}}\big(\tilde{\mathbf{z}}(t)\big)$

shadowing: $\mathbf{z}(t_0) \longrightarrow \hat{\mathbf{z}}(t_0)$

Literature: *e.g.,*

G Benettin, A Giorgilli (1994) J. Stat. Phys. 74, 1117

C Kane, JE Marsden, M Ortiz, M West (2000) Int. J. Numer. Methods Eng. 49, 1295

WB Hayes, KR Jackson (2003) SIAM J. Numer. Anal. 41, 1948

PC Moan (2006) Journal of Physics A 39, 5545

E Hairer, C Lubich, G Wanner (2006) Geometric numerical integration, Springer, Berlin

X Shang, HC Öttinger (2020) Proc. R. Soc. A: Math. Phys. Eng. Sci. 476, 20190446

Own works:

T Fülöp, R Kovács, M Szücs, M Fawaier: Thermodynamical extension of a symplectic numerical scheme with half space and time shifts demonstrated on rheological waves in solids, Entropy 22 (2020) 155

Á Pozsár, M Szücs, R Kovács, T Fülöp: Four spacetime dimensional simulation of rheological waves in solids and the merits of thermodynamics, Entropy 22 (2020) 1376

T Fülöp: Wave Propagation in Rocks – Investigating the Effect of Rheology, Periodica Polytechnica Civil Engineering 65 (2021) 26

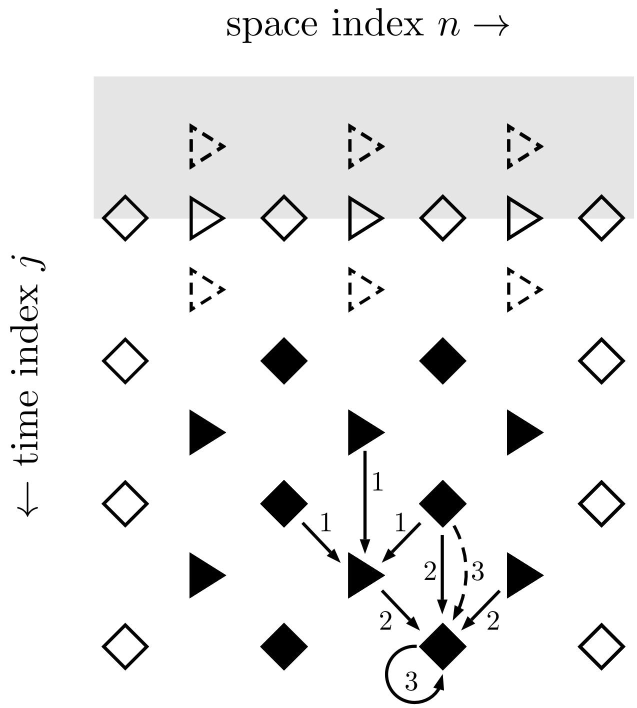
DM Takács, Á Pozsár, T Fülöp: Thermodynamically extended symplectic numerical simulation of viscoelastic, thermal expansion and heat conduction phenomena in solids, Continuum Mech. Thermodyn., Early access (2024) DOI:10.1007/s00161-024-01280-w

DM Takács, T Fülöp: Improving the accuracy of the Newmark method through backward error analysis, submitted; DOI:10.48550/arXiv.2403.02029

$$\varrho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial v}{\partial x}$$

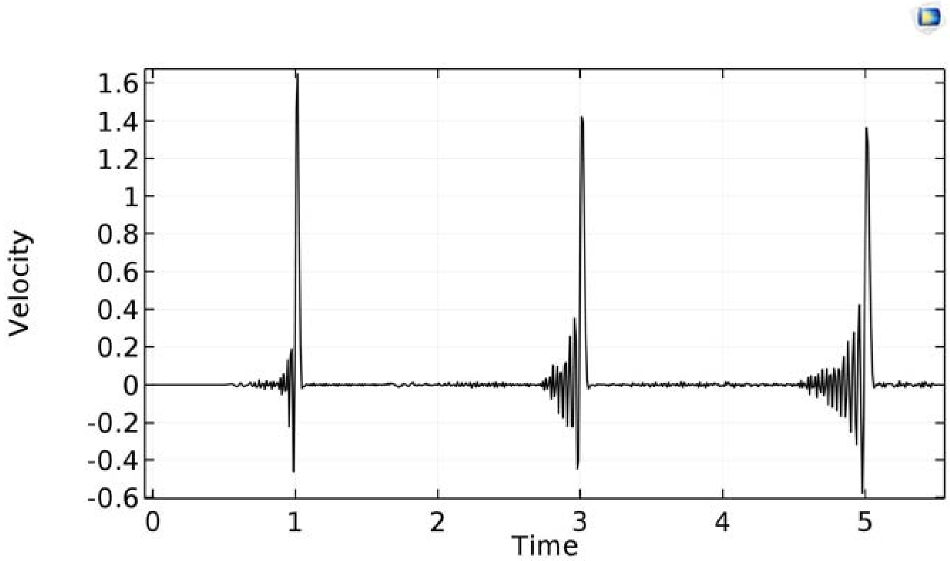
$$\sigma + \tau \frac{\partial \sigma}{\partial t} = E\varepsilon + \hat{E}\frac{\partial \varepsilon}{\partial t}$$

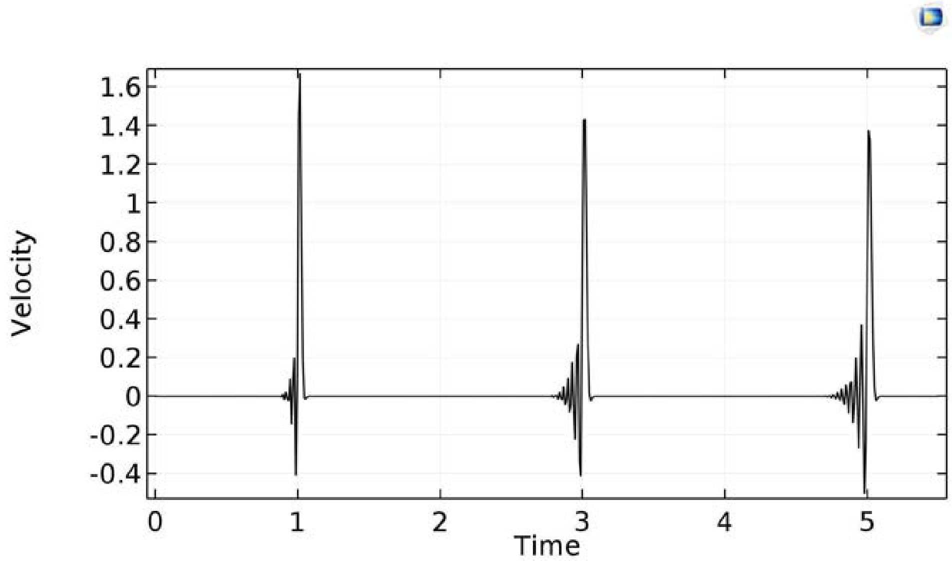


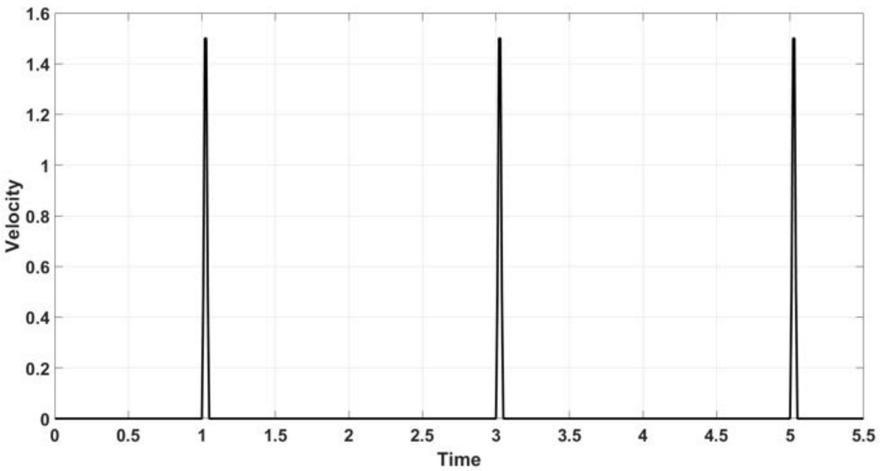
$$\varrho \frac{v_{n+1/2}^{j+1/2} - v_{n+1/2}^{j-1/2}}{\Delta t} = \frac{\sigma_{n+1}^{j} - \sigma_{n}^{j}}{\Delta x}$$

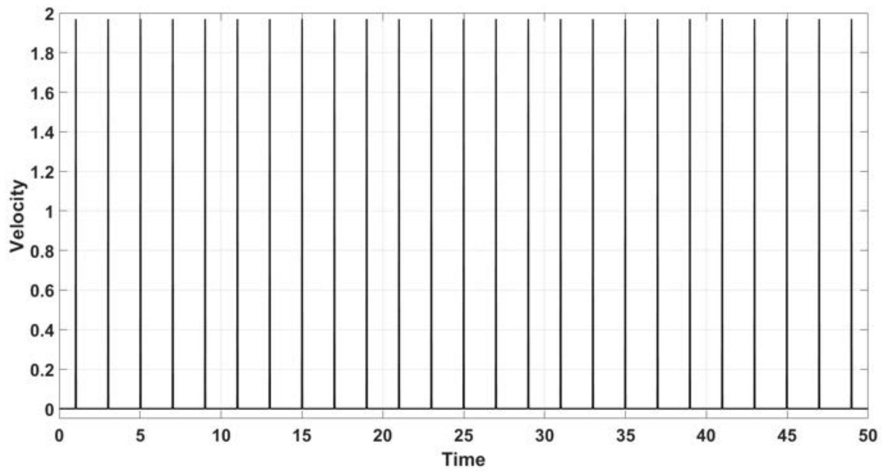
$$\frac{\varepsilon_{n}^{j+1} - \varepsilon_{n}^{j}}{\Delta t} = \frac{v_{n+1/2}^{j+1/2} - v_{n-1/2}^{j+1/2}}{\Delta x}$$

$$\sigma_{n}^{j+1} = \frac{1}{1 - \alpha + \frac{\tau}{\Delta t}} \left\{ \left(\frac{\tau}{\Delta t} - \alpha \right) \sigma_{n}^{j} + E \left[\alpha \varepsilon_{n}^{j} + (1-\alpha) \varepsilon_{n}^{j+1} \right] + \hat{E} \frac{\varepsilon_{n}^{j+1} - \varepsilon_{n}^{j}}{\Delta t} \right\}$$


Velocity
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
1
2
3
4
5
Time


Velocity
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
1
2
3
4
5
Time

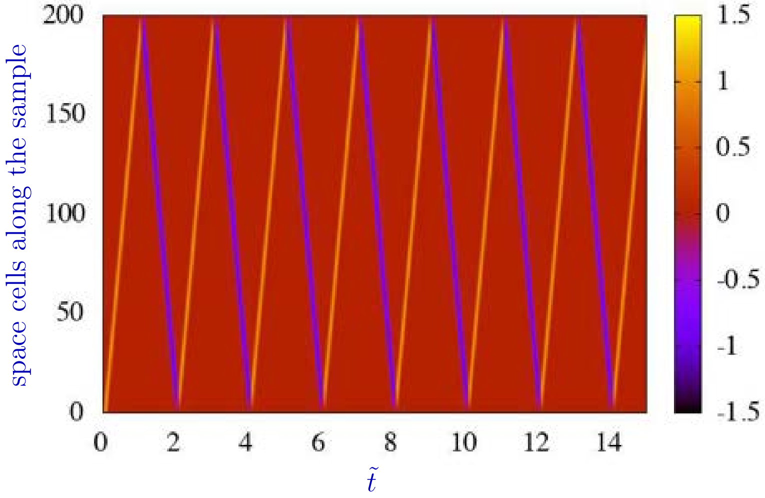

Velocity
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
Time

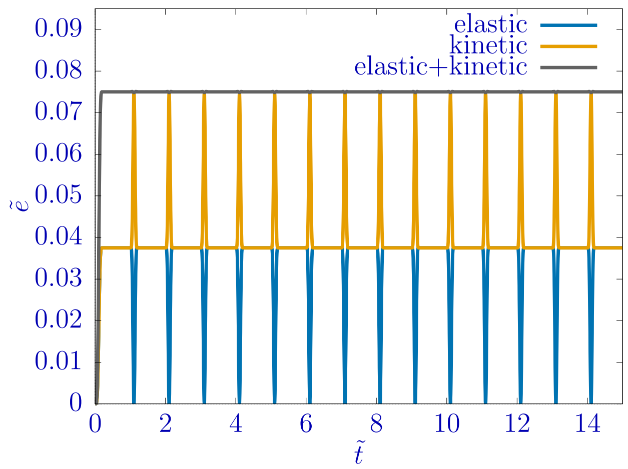

Velocity
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30
35
40
45
50
Time

spacetime picture:

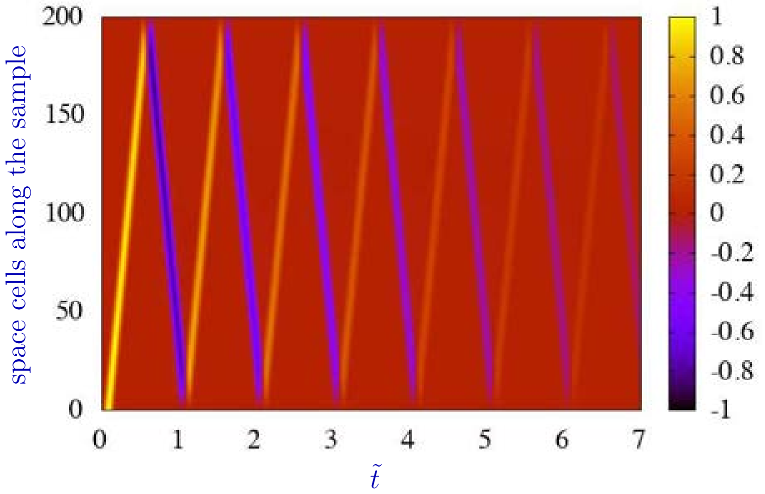
time dependence of energies:

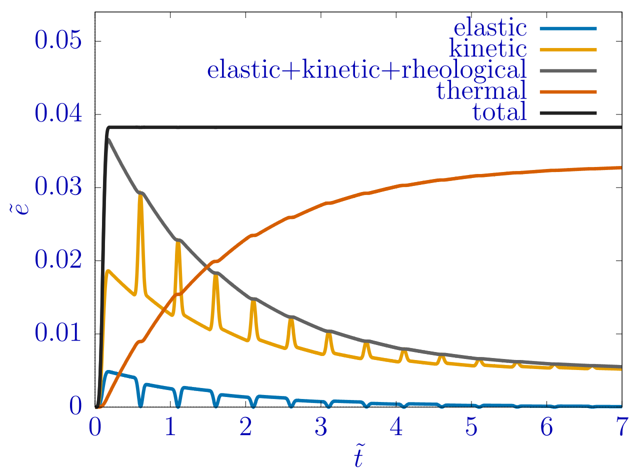
Hooke case:



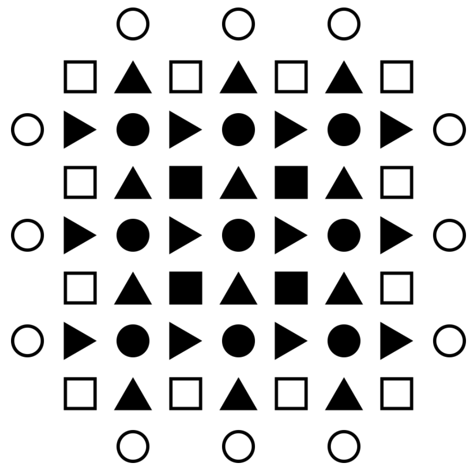


rheological case:

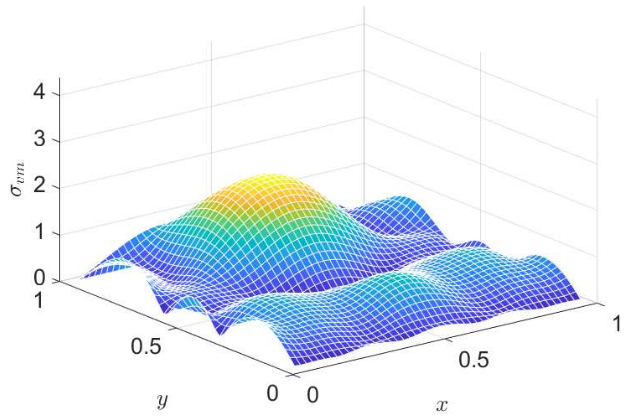


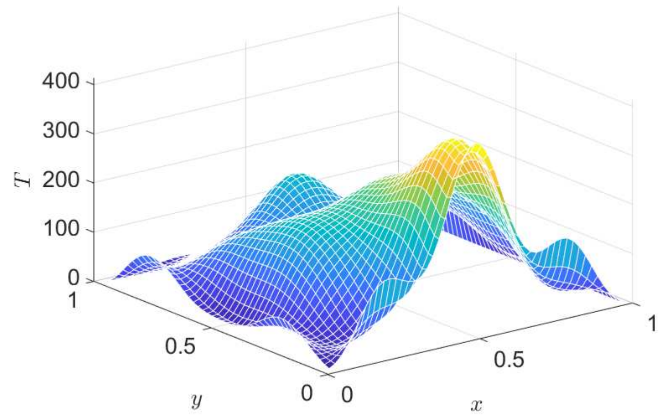


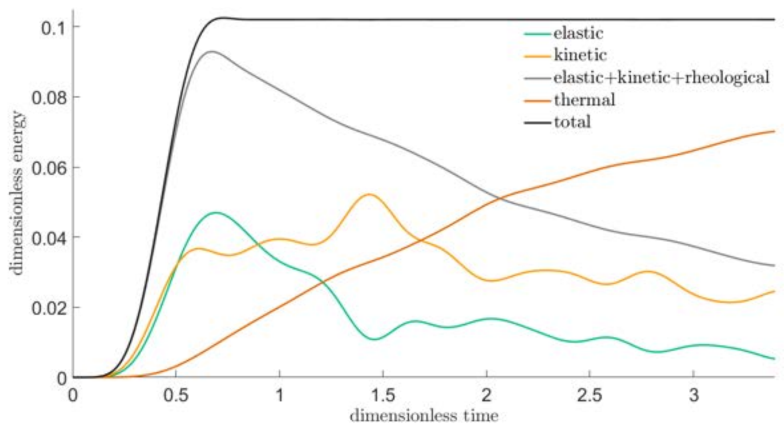
2 and 3 space dimensions:



different vector/tensor components:
centers, faces, edges, vertices

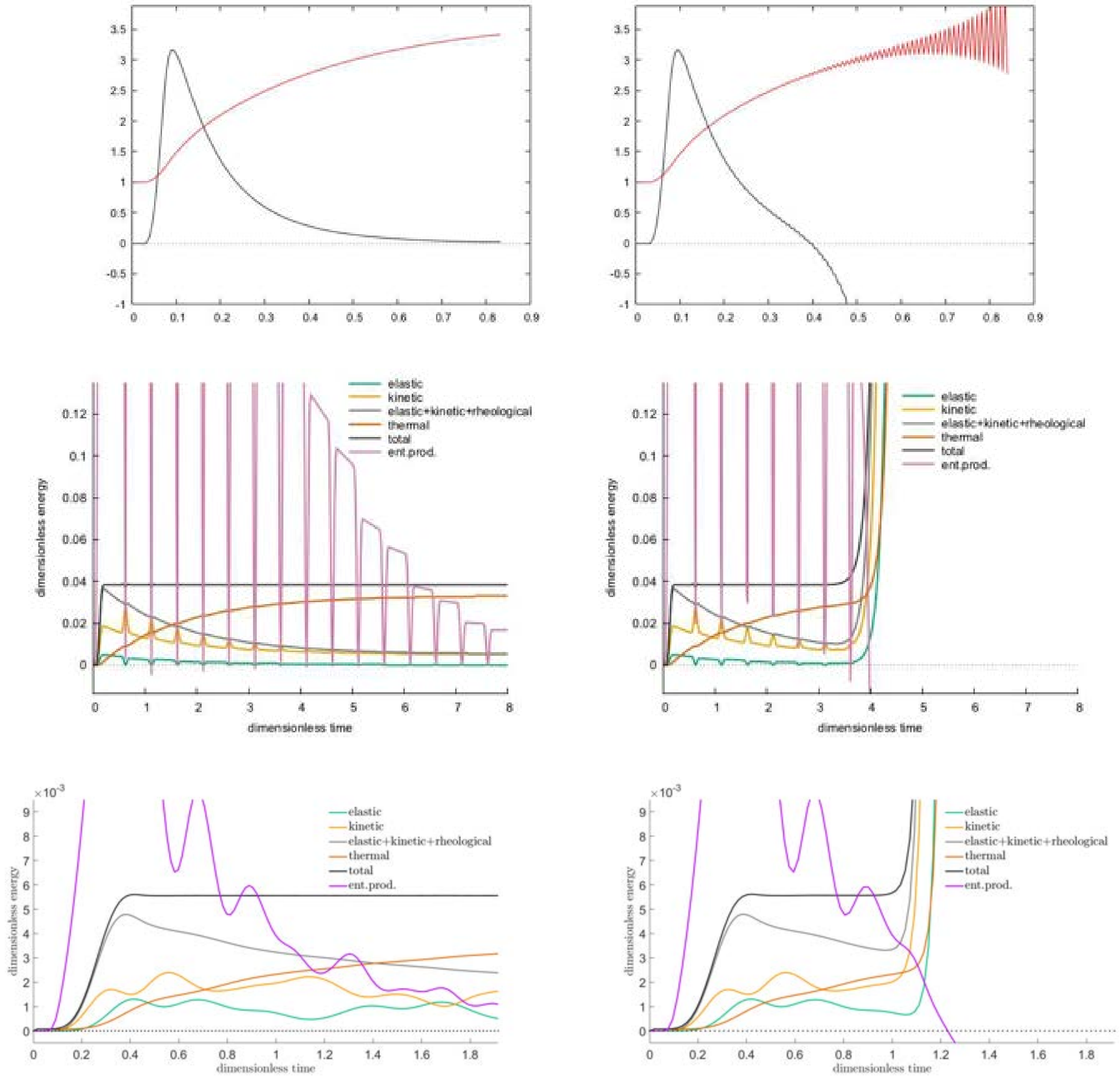




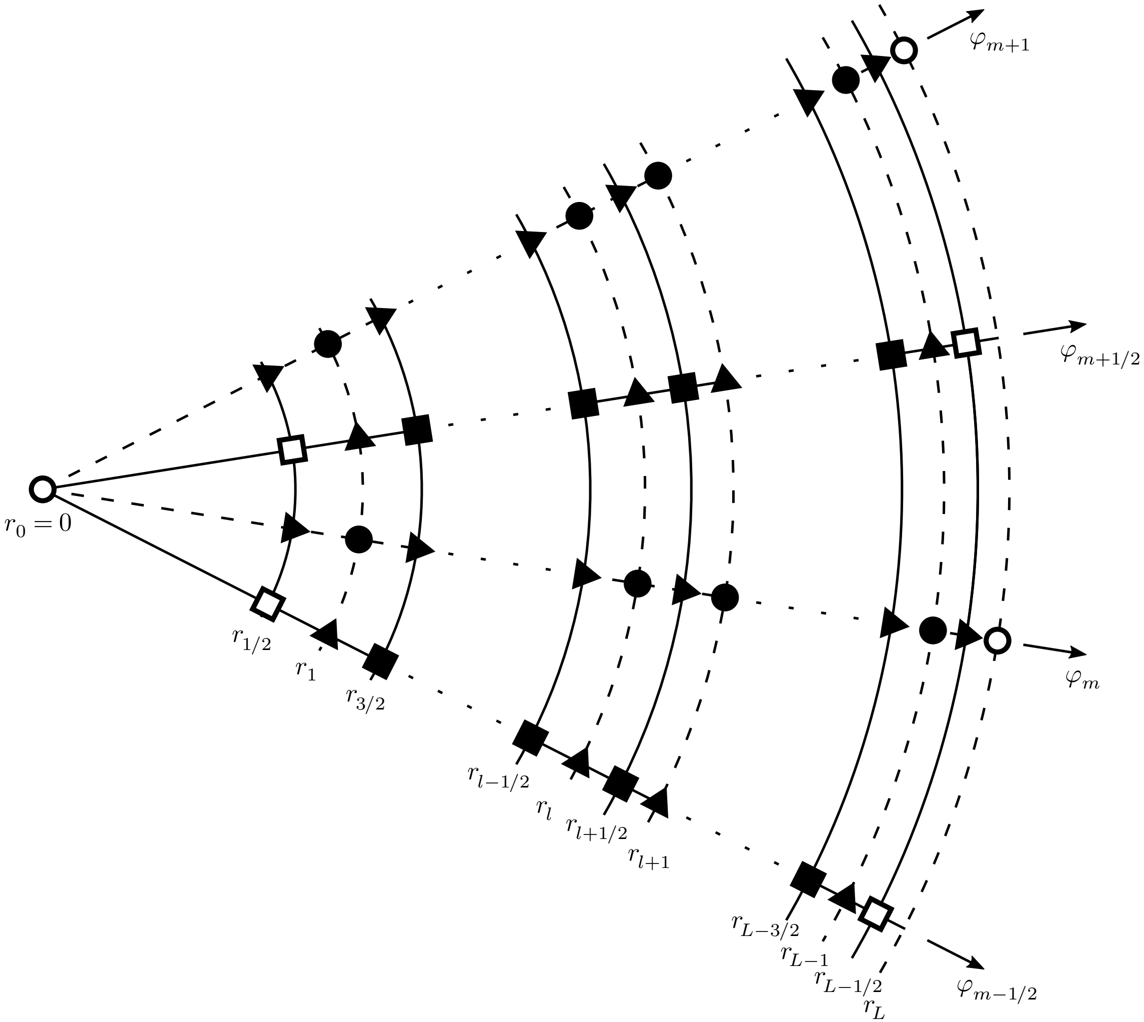


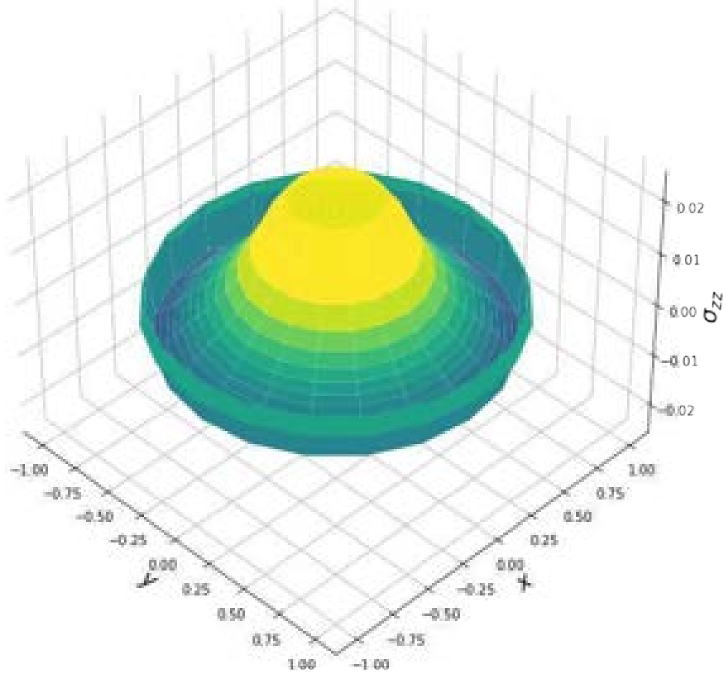
movie

Entropy production rate: discretizing a „raw”, not automatically non-negative, form of it:



Cylindrical geometry:





$$\rho \dot{v} = \frac{\partial \sigma}{\partial x} \quad (1)$$

$$\dot{\epsilon} = \frac{\partial v}{\partial x} \quad (2)$$

$$\sigma = \sigma_{\text{el}} + \hat{\sigma} \quad (3)$$

$$\hat{\sigma} + \tau \dot{\hat{\sigma}} = \hat{I} \dot{\epsilon} \quad (4)$$

$$\sigma_{\text{el}} = E D \quad (5)$$

$$e_{\text{int}} = e_{\text{thermal}} + e_{\text{elastic}} + e_{\text{rheol}} \quad (6)$$

$$\rho \dot{e}_{\text{int}} = -\frac{\partial q}{\partial x} + \sigma \dot{\epsilon} \quad (7)$$

$$\rho \dot{s} = -\frac{\partial \frac{q}{T}}{\partial x} + \underbrace{\frac{\partial \frac{1}{T}}{\partial x} q + \frac{\hat{\sigma}}{T} \left(\dot{\epsilon} - \frac{\tau}{\hat{I}} \dot{\hat{\sigma}} \right)}_{\pi_s} \quad (8)$$

$$q = -\lambda \frac{\partial T}{\partial x} \quad (9)$$

$$D = \epsilon - \alpha (T - T_{ex}) \quad (10)$$

$$e_{\text{total}} = \underbrace{\frac{v^2}{2}}_{e_{\text{kinetic}}} + \underbrace{c_{ex} (T - T_{ex})}_{e_{\text{thermal}}} + \underbrace{\frac{E}{2\rho} \epsilon (\epsilon + \alpha T_{ex})}_{e_{\text{elastic}}} + \underbrace{\frac{\tau}{2\rho \hat{I}} \hat{\sigma}^2}_{e_{\text{rheol}}} \quad (11)$$

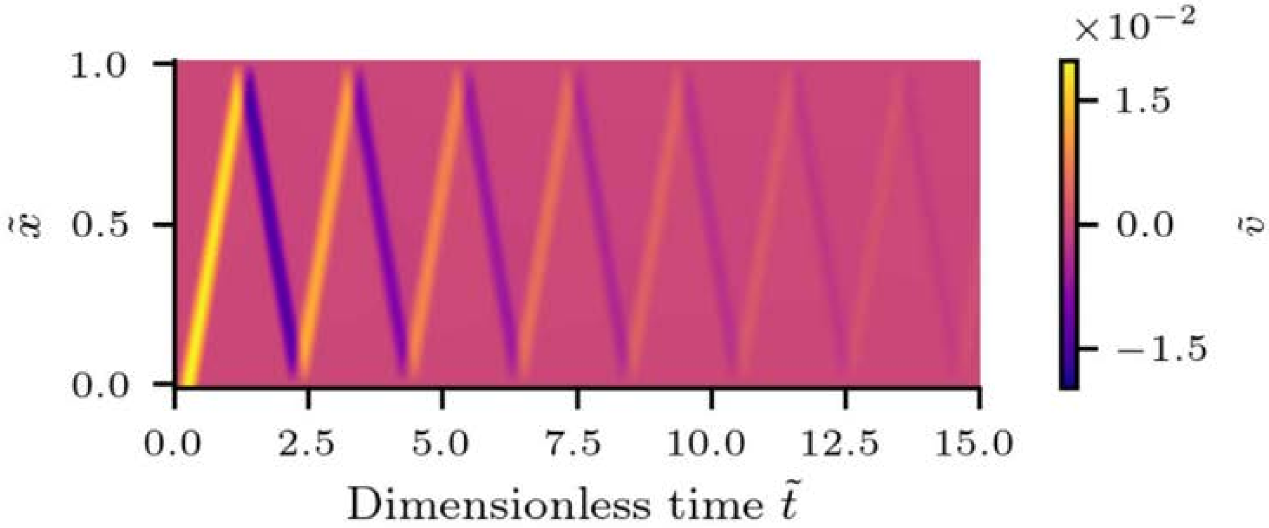
t \ x	$-\frac{1}{2}$	0	$+\frac{1}{2}$	$\cdots$	$n-\frac{1}{2}$	n	$n+\frac{1}{2}$	$\cdots$
$-\frac{1}{2}$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$
0	j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	j_e	$\cdots$	j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	j_e	$\cdots$
$+\frac{1}{2}$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$j-\frac{1}{2}$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$
j	j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	j_e	$\cdots$	j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	j_e	$\cdots$
$j+\frac{1}{2}$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$
$j+1$	j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	j_e	$\cdots$	j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	j_e	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

$$
\begin{aligned}
v_{n+1/2}^{j+1/2} &\overset{(1)}{=} v_{n+1/2}^{j-1/2} + \frac{1}{\varrho}\frac{\Delta t}{\Delta x}\left(\sigma_{n+1}^{j} - \sigma_{n}^{j}\right), \\
L_{n}^{j+1/2} &\overset{(3)}{=} \frac{1}{\Delta x}\left(v_{n+1/2}^{j+1/2} - v_{n-1/2}^{j+1/2}\right), \\
L_{n}^{j} &= \frac{1}{2}\left(L_{n}^{j+1/2} + L_{n}^{j-1/2}\right), \\
\epsilon_{n}^{j+1} &\overset{(13)}{=} \epsilon_{n}^{j} + \Delta t L_{n}^{j+1/2}, \\
\epsilon_{n}^{j+1/2} &= \frac{1}{2}\left(\epsilon_{n}^{j+1} + \epsilon_{n}^{j}\right), \\
\hat{\sigma}_{n}^{j+1} &\overset{(4)}{=} \frac{1}{\frac{\tau}{\Delta t} + \frac{1}{2}}\left[\hat{I}L_{n}^{j+1/2} + \left(\frac{\tau}{\Delta t} - \frac{1}{2}\right)\hat{\sigma}_{n}^{j}\right], \\
\hat{\sigma}_{n}^{j+1/2} &= \frac{1}{2}\left(\hat{\sigma}_{n}^{j+1} + \hat{\sigma}_{n}^{j}\right);
\end{aligned}
\tag{17}
$$

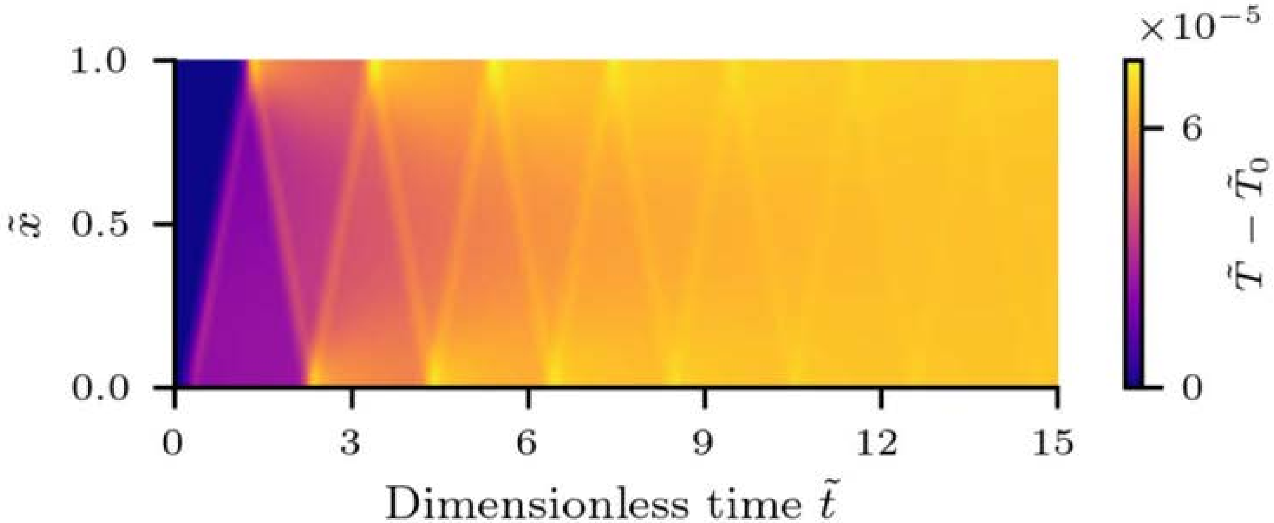
$$
\begin{aligned}
(e_{\mathrm{int}})_{n}^{j+1/2} &\overset{(11)}{=} (e_{\mathrm{int}})_{n}^{j-1/2} - \frac{1}{\varrho}\frac{\Delta t}{\Delta x}\left[(j_e)_{n+1/2}^{j} - (j_e)_{n-1/2}^{j}\right] + \frac{\Delta t}{\varrho}\sigma_{n}^{j}L_{n}^{j}, \\
T_{n}^{j+1/2} &\overset{(6)}{=} T_{\mathrm{ex}} + \frac{1}{c_{\mathrm{ex}}}\left[(e_{\mathrm{int}})_{n}^{j+1/2} - \frac{E}{2\varrho}\epsilon_{n}^{j+1/2}\left(\epsilon_{n}^{j+1/2} + 2\alpha T_{\mathrm{ex}}\right)\right. \\
&\qquad \left. - \frac{\tau}{2\varrho\hat{I}}\left({\hat{\sigma}_{n}^{j+1/2}}^{2}\right)\right], \\
(j_e)_{n+1/2}^{j+1/2} &\overset{(8)}{=} -\frac{\lambda}{\Delta x}\left(T_{n+1}^{j+1/2} - T_{n}^{j+1/2}\right), \\
D_{n}^{j+1/2} &\overset{(14)}{=} \epsilon_{n}^{j+1/2} - \alpha\left(T_{n}^{j+1/2} - T_{\mathrm{ex}}\right), \\
(\sigma_{\mathrm{el}})_{n}^{j+1/2} &\overset{(15)}{=} ED_{n}^{j+1/2}, \\
\sigma_{n}^{j+1/2} &\overset{(2)}{=} (\sigma_{\mathrm{el}})_{n}^{j+1/2} + \hat{\sigma}_{n}^{j+1/2};
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
(e_{\mathrm{int}})_{n}^{j+1} &\overset{(11)}{=} (e_{\mathrm{int}})_{n}^{j} - \frac{1}{\varrho}\frac{\Delta t}{\Delta x}\left\{\left[2\,(j_e)_{n+1/2}^{j} - (j_e)_{n+1/2}^{j-1/2}\right]\right. \\
&\qquad \left. - \left[2\,(j_e)_{n-1/2}^{j} - (j_e)_{n-1/2}^{j-1/2}\right]\right\} + \frac{\Delta t}{\varrho}\sigma_{n}^{j+1/2}L_{n}^{j+1/2}, \\
T_{n}^{j+1} &\overset{(6)}{=} T_{\mathrm{ex}} + \frac{1}{c_{\mathrm{ex}}}\left[(e_{\mathrm{int}})_{n}^{j+1} - \frac{E}{2\varrho}\epsilon_{n}^{j+1}\left(\epsilon_{n}^{j+1} + 2\alpha T_{\mathrm{ex}}\right) - \frac{\tau}{2\varrho\hat{I}}\left(\hat{\sigma}_{n}^{j+1}\right)^{2}\right], \\
(j_e)_{n+1/2}^{j+1} &\overset{(8)}{=} -\frac{\lambda}{\Delta x}\left(T_{n+1}^{j+1} - T_{n}^{j+1}\right), \\
D_{n}^{j+1} &\overset{(14)}{=} \epsilon_{n}^{j+1} - \alpha\left(T_{n}^{j+1} - T_{\mathrm{ex}}\right), \\
(\sigma_{\mathrm{el}})_{n}^{j+1} &\overset{(15)}{=} ED_{n}^{j+1}, \\
\sigma_{n}^{j+1} &\overset{(2)}{=} (\sigma_{\mathrm{el}})_{n}^{j+1} + \hat{\sigma}_{n}^{j+1}.
\end{aligned}
\tag{19}
$$

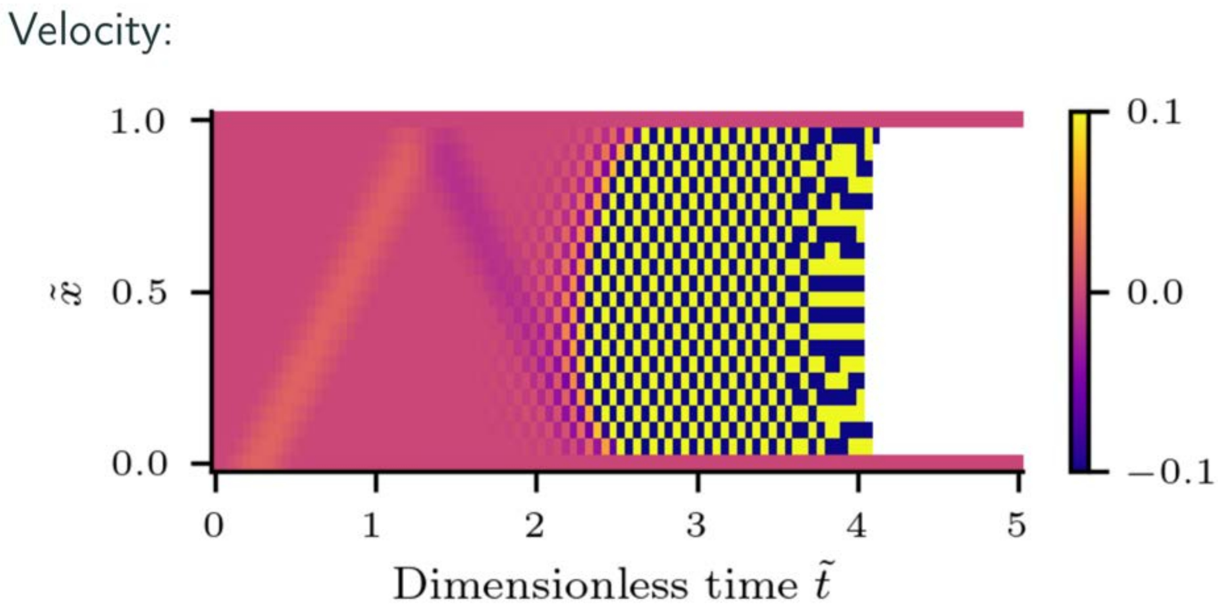
Velocity:



Temperature:



Loss of stability:



Conclusions:

symplectic and other structure-preseving schemes:

Conclusions:

symplectic and other structure-preseving schemes:

- direct realizations and extensions

Conclusions:

symplectic and other structure-preseving schemes:

- direct realizations and extensions
- heuristic extensions (inspiration)

Conclusions:

symplectic and other structure-preseving schemes:

- direct realizations and extensions
- heuristic extensions (inspiration)

benefits of not eliminating degrees of freedom

Conclusions:

symplectic and other structure-preseving schemes:

- direct realizations and extensions
- heuristic extensions (inspiration)

benefits of not eliminating degrees of freedom

benefits of quantities with a balance law

Conclusions:

symplectic and other structure-preseving schemes:

- direct realizations and extensions
- heuristic extensions (inspiration)

benefits of not eliminating degrees of freedom

benefits of quantities with a balance law

benefits of the spacetime view

Conclusions:

symplectic and other structure-preseving schemes:

- direct realizations and extensions
- heuristic extensions (inspiration)

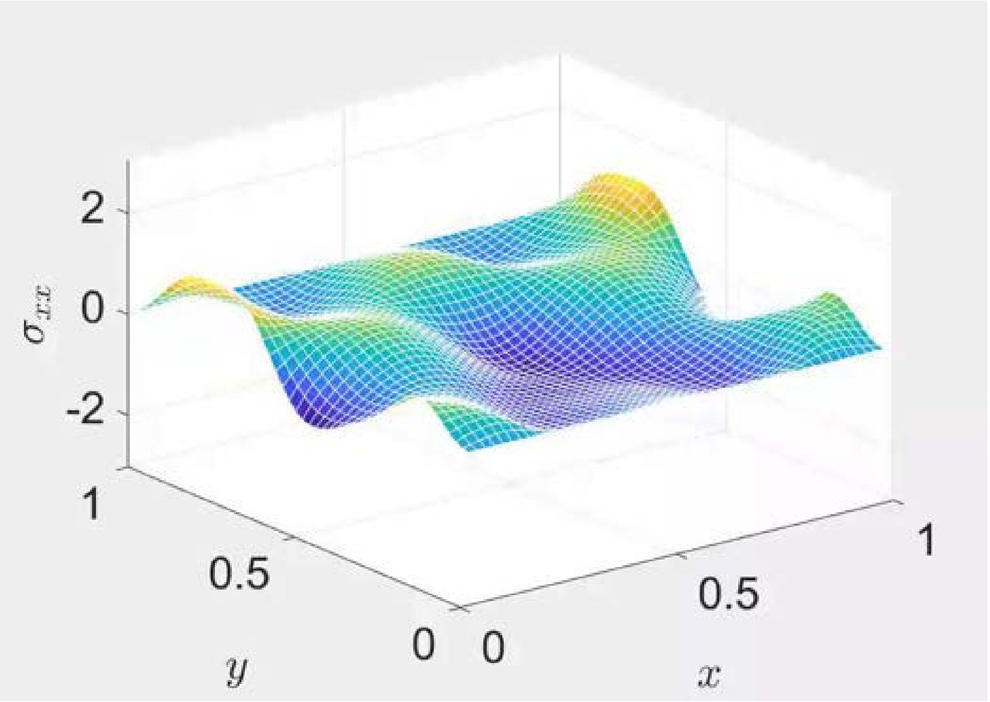
benefits of not eliminating degrees of freedom

benefits of quantities with a balance law

benefits of the spacetime view

benefits of strong mathematical results

THANK YOU FOR YOUR ATTENTION !



+ thanks to my co-authors

Róbert Kovács, Mátyás Szücs, Áron Pozsár, Donát M. Takács

(alphabetic order)