Classical and Quantum Systems and How to Find Them

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MOMENTUM OF INNOVATION

Operator algebraic models for single systems

Set of physical states:

$\Omega = \{\omega_1, \ldots, \omega_d\}$ finite set

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Common generalization

Observable algebra: $P_1, \ldots, P_r$ projections on $\mathcal{H}$, $\sum_i P_i = I$

$$\mathcal{A} := \{A \in \mathcal{B}(\mathcal{H}) : A = \textstyle\sum_{i=1}^{r} P_i A P_i\}$$

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$\mathcal{S}(\mathcal{A}) = \mathcal{S}(\mathcal{H}) \cap \mathcal{A}$, $M_x \in \mathcal{A}$, $x \in \mathcal{X}$

classical: $P_i = |e_i\rangle\langle e_i|$ quantum: $i = 1$, $P_1 = I$.

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$\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ norm-closed *-subalgebra: C^*-algebra

$\mathcal{S}(\mathcal{A}) \ni \varrho \equiv \varphi_\varrho : A \mapsto \operatorname{Tr} A\varrho$ positive linear functional, $\varphi_\varrho(I) = 1$

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$\mathcal{H}$ Hilbert space Observable algebra: $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ C^*-algebra

States: $\mathcal{S}(\mathcal{A}) = \{\varphi \in \mathcal{A}^* : \varphi(A^*A) \geq 0,\ A \in \mathcal{A},\ \ \varphi(I) = 1\}$

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Quantum bit

$\underline{v} = (\sin\theta\cos\gamma, \sin\theta\sin\gamma, \cos\theta)$ unit vector in $\mathbb{R}^3$

$\mapsto |\phi(\underline{v})\rangle := \cos\frac{\theta}{2}|0\rangle + e^{i\gamma}\sin\frac{\theta}{2}|1\rangle$ unit vector in $\mathbb{C}^2$

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perpendicular in $\mathbb{C}^2$ $\iff$ opposite in $\mathbb{R}^3$

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$\varrho(\underline{v}) = |\varphi(\underline{v})\rangle\langle\varphi(\underline{v})|, \; M(\underline{w})_+ = |\varphi(\underline{w})\rangle\langle\varphi(\underline{w})|, \; M(\underline{w})_- = |\varphi(-\underline{w})\rangle\langle\varphi(-\underline{w})|$

$P_{\varrho(\underline{v}),M(\underline{w})}(+) = \cos^2\frac{1}{2}\sphericalangle(\underline{v},\underline{w}), \quad P_{\varrho(\underline{v}),M(\underline{w})}(-) = \cos^2\frac{1}{2}\sphericalangle(\underline{v},-\underline{w})$

Composite systems

$$
\begin{aligned}
\Omega_A, \Omega_B \mapsto \Omega_A \times \Omega_B \mapsto \mathcal{H}_{\Omega_A \times \Omega_B} &= \ell^2(\Omega_A \times \Omega_B) \\
&= \operatorname{span}\{f \otimes g : \; f \in \mathcal{H}_{\Omega_A}, \, g \in \mathcal{H}_{\Omega_B}\}
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$\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle = \langle f_1, f_2 \rangle \cdot \langle g_1, g_2 \rangle$

Observable algebra:

$$\mathcal{A}_{AB} := \operatorname{span}\{X \otimes Y : \; X \in \mathcal{A}_A, \, Y \in \mathcal{A}_B\}$$

$(X \otimes Y) f \otimes g := Xf \otimes Yg$

Fully quantum: $\mathcal{A}_A = \mathcal{B}(\mathcal{H}_A)$, $\mathcal{A}_B = \mathcal{B}(\mathcal{H}_B) \Longrightarrow \mathcal{A}_{AB} = \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Entangled states

$$\mathcal{A}_{AB} = \mathcal{A}_A \otimes \mathcal{A}_B \subseteq \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

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Classical: All states are separable.

Quantum: $|\psi_{\max}\rangle\langle\psi_{\max}|$ not separable $\equiv$ entangled

$|\psi_{\max}\rangle := \frac{|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle}{\sqrt{2}}$, $|0\rangle, |1\rangle$ ONB in $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$.

maximally entangled state

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$= \frac{|\phi(\underline{a})\rangle\otimes|\phi(-\underline{a})\rangle - |\phi(-\underline{a})\rangle\otimes|\phi(\underline{a})\rangle}{\sqrt{2}}$ maximally entangled state

for all unit vector $\underline{a} \in \mathbb{R}^3$

Entangled states

Measurement $M(\underline{v})_+ := |\phi(\underline{v})\rangle\langle\phi(\underline{v})|$, $M(\underline{v})_- := |\phi(-\underline{v})\rangle\langle\phi(-\underline{v})|$ on system A

$$P_{|\psi_{\max}\rangle\langle\psi_{\max}|, M(\underline{v})\otimes I}(+) = P_{|\psi_{\max}\rangle\langle\psi_{\max}|, M(\underline{v})\otimes I}(-) = \frac{1}{2}$$

Post-measurement state:

$$\varrho_a = \frac{(M(\underline{v})_a \otimes I)|\psi_{\max}\rangle\langle\psi_{\max}|(M(\underline{v})_a \otimes I)}{\mathrm{Tr}(\dots)} = \begin{cases} |\phi(\underline{v})\rangle\langle\phi(\underline{v})| \otimes |\phi(-\underline{v})\rangle\langle\phi(-\underline{v})|, & a = +, \\ |\phi(-\underline{v})\rangle\langle\phi(-\underline{v})| \otimes |\phi(\underline{v})\rangle\langle\phi(\underline{v})|, & a = - \end{cases}$$

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Measurement $N(\underline{w})_+ := |\phi(\underline{w})\rangle\langle\phi(\underline{w})|$, $N(\underline{w})_- := |\phi(-\underline{w})\rangle\langle\phi(-\underline{w})|$ on system B

$$P_{|\psi_{\max}\rangle\langle\psi_{\max}|, M(\underline{v})\otimes N(\underline{w})}(a,b) = \frac{1}{2} \begin{cases} \cos^2 \frac{1}{2}\sphericalangle(-\underline{v}, \underline{w}), & a = +, b = +, \\ \cos^2 \frac{1}{2}\sphericalangle(\underline{v}, \underline{w}), & a = +, b = -, \\ \cos^2 \frac{1}{2}\sphericalangle(\underline{v}, \underline{w}), & a = -, b = +, \\ \cos^2 \frac{1}{2}\sphericalangle(\underline{v}, -\underline{w}), & a = -, b = -. \end{cases}$$

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Do they provide some benefit in some practical task?

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  with probability $\pi(x, y)$

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