Newton-Krylov methods for non-linear elliptic systems

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Outline

Background: linear problems (previous year's work)

- Let $d \geq 2$, $\Omega \subset \mathbb{R}^d$ be a bounded domain.
- We consider the elliptic problem

$$\begin{cases} -\mathrm{div}(G\nabla u) + \eta u = g, \\ u_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\eta = \eta(x)$ and G is a constant matrix.

Construction of the discretization

- Let $V_h \subset H_0^1(\Omega)$ be a FEM subspace. We look for u_h in V_h:

$$\int_\Omega (G \nabla u_h \cdot \nabla v + \eta u_h v) = \int_\Omega g v, \quad v \in V_h. \tag{1.2}$$

The corresponding linear algebraic system:

$$\underbrace{\mathbf{B}_h}_{(\mathbf{G}_h + \mathbf{D}_h)} \mathbf{c} = \mathbf{g}_h .$$

Preconditioned form:

$$\underbrace{\mathbf{G}_h^{-1} \mathbf{B}_h}_{(\mathbf{I}_h + \mathbf{G}_h^{-1} \mathbf{D}_h)} \mathbf{c} = \underbrace{\tilde{\mathbf{g}}_h}_{\mathbf{G}_h^{-1} \mathbf{g}_h} . \tag{1.3}$$

- Preconditioned conjugate gradient method (PCGM)= CGM for (1.3).

PCGM Algorithm

- Let $u_0 \in H$ arbitrary, $\rho_0 = \mathbf{B}_h u_0 - g$, $\mathbf{G}_h p_0 = \rho_0$, $r_0 = \rho_0$.
- For $k \in \mathbb{N}$:

$$\begin{cases} u_{k+1} = u_k + \alpha_k p_k, \\ r_{k+1} = r_k + \alpha_k \mathbf{G}_h^{-1} \mathbf{B}_h p_k, \\ p_{k+1} = r_{k+1} + \beta_k p_k. \end{cases}$$

- Auxiliary problems:

$$\mathbf{G}_h z_k = \mathbf{B}_{\boldsymbol{h}} p_k.$$

They can be solved easily with fast solvers due to the special structure of $\mathbf{G}_h$, [9], [5].

Results

- Derive mesh-independent superlinear convergence of the preconditioned CGM.

- Estimate the rate of superlinear convergence.

- Extend the results of [8] from $\eta \in \mathrm{C}(\overline{\Omega})$ to $\eta \in \mathrm{L}^q(\Omega)$.

Assumptions

(i) G is positive definite.

(ii) There exists $2 < p < \frac{2d}{d-2}$:

$$\eta \in \mathrm{L}^{p/(p-2)}(\Omega).$$

Main theorem

Theorem 1 (Superlinear convergence rate estimation)

Let $2 < p < \frac{2d}{d-2}$. *Then there exists* $\boldsymbol{C} > 0$ *such that*

$$\left(\frac{\|e_k\|_{\mathbf{A}_h}}{\|e_0\|_{\mathbf{A}_h}} \right)^{\frac{1}{k}} \leq \frac{\boldsymbol{C}}{k^{\alpha}} \to 0, \qquad \text{as } k \to \infty, \tag{2.4}$$

where $\alpha = \frac{1}{d} - \frac{1}{2} + \frac{1}{p}$.

Extension to elliptic systems

Systems of PDE's:

$$\begin{cases} -\Delta u_i + \eta_{i1} u_1 + \ldots \eta_{is} u_s = g_i, \\ u_i|_{\partial\Omega} = 0, \quad (i = 1, \ldots, s), \end{cases} \tag{2.5}$$

where $\boldsymbol{H} = \{\eta_{ij}\}_{i,j=1}^{s} : \Omega \to \mathbb{R}^{s\times s}_{symm}$ such that:

$$\forall i, j \in \{1, \ldots, s\} : \quad \eta_{ij} \in L^{p/(p-2)}(\Omega).$$

Main result

Theorem 1 holds for (2.5) as well. If **H** is not symmetric, we substitute CGM with GMRES, and Theorem 1 still holds.

Advantages of the preconditioner

The auxiliary problem $\mathbf{S}_h w_k = \mathbf{Q}_h p_k$ for the PCGM becomes

$$\begin{cases} -\Delta(w_k)_1 & = \sum_{j=1}^{s} \eta_{1j}(p_k)_j, \\ -\Delta(w_k)_2 & = \sum_{j=1}^{s} \eta_{2j}(p_k)_j, \\ & . \\ & . \\ & . \\ -\Delta(w_k)_s & = \sum_{j=1}^{s} \eta_{sj}(p_k)_j, \\ (w_i)|_{\partial\Omega} & = 0, \quad \forall i = 1, \ldots, s. \end{cases}$$

These equations are independent of one another, hence they can be solved in parallel.

Non-linear problems

Goal: Generalize previous results for nonlinear elliptic systems.

The nonlinear problem: elliptic transport system of the form

$$\begin{cases} -\mathrm{div}(K_i \nabla u_i) + \mathbf{b}_i \cdot \nabla u_i + f_i(x, u_1, \ldots, u_l) & = g_i; \\ u_i|_{\partial\Omega} & = 0, \end{cases} \tag{3.6}$$

where $i = 1, \ldots, l$.

Objective

Solve FEM discretization of (3.6) with a Newton iteration plus GMRES technique and prove superlinear convergence.

Operator equation for the weak form of the system

- Let $F : \mathrm{H}_0^1(\Omega)^l \to \mathrm{H}_0^1(\Omega)^l$ given by

$$\langle F(\mathbf{u}), \mathbf{v}\rangle_{\mathrm{H}_0^1(\Omega)} = \int_\Omega \sum_{i=1}^{l} K_i \nabla u_i \cdot \nabla v_i + (\mathbf{b}_i \cdot \nabla u_i) v_i + f_i(x, \mathbf{u}) v_i).$$

- There exists a unique $\overline{\mathbf{g}} \in \mathrm{H}_0^1(\Omega)^l$ such that

$$\int_\Omega \mathbf{g}\mathbf{v} = \langle \overline{\mathbf{g}}, \mathbf{v}\rangle_{\mathrm{H}_0^1(\Omega)} \quad (\mathbf{v} \in \mathrm{H}_0^1(\Omega)). \tag{3.7}$$

Altogether,

$$F(\mathbf{u}) = \overline{\mathbf{g}}, \qquad \text{in } \mathrm{H}_0^1(\Omega)^l. \tag{3.8}$$

FEM discretization

- Let $V_h \in H_0^1(\Omega)^l$ be a N-dimensional subspace. We look for $\mathbf{u}_h \in V_h$ such that

$$\langle F(\mathbf{u}_h), \mathbf{v}_h \rangle_{H_0^1(\Omega)} = \langle \mathbf{g}, \mathbf{v}_h \rangle_{H_0^1(\Omega)} \quad (\mathbf{v}_h \in V_h). \tag{3.9}$$

- Denote $F_h = P_h F$ and $\mathbf{g}_h = P_h \mathbf{g}$.

The projected problem:

$$F_h(\mathbf{u}_h) = \mathbf{g}_h \quad \text{in } V_h \tag{3.10}$$

is then solved using the *Damped Inexact Newton (DIN) method*

Construction of the DIN iteration

- Let $\mathbf{u}_0 \in V_h$ arbitrary. DIN iteration $(\mathbf{u}_n) \subset V_h$ constructed recursively as
$$\mathbf{u}_{n+1} = \mathbf{u}_n + \tau_n \mathbf{p}_n \quad (n \in \mathbb{N}),$$
where $0 < \tau_n \leq 1$.
- Here $\mathbf{p}_n \in V_h$ is the approximate solution of the linear auxiliary problem
$$\langle F_h'(\mathbf{u}_n)\mathbf{p}_n, \mathbf{v}_h \rangle_{H_0^1(\Omega)} = -\langle F_h(\mathbf{u}_n) - \mathbf{g}_h, \mathbf{v}_h \rangle_{H_0^1(\Omega)}. \tag{3.11}$$

DIN as an outer-inner iteration process

- Let $\mathbf{u}_n$ be constructed in the DIN iteration.
- Linearized problem (3.11) :

$$F'_h(\mathbf{u}_n)\mathbf{p}_h = \mathbf{r}_h, \tag{3.12}$$

where $\mathbf{r}_h = \mathbf{g}_h - F_h(\mathbf{u}_n)$.

Equivalent to the FEM approximation of

$$\begin{cases} -\mathrm{div}(K_i \nabla p_i) + \mathbf{b}_i \cdot \nabla p_i + \sum_{j=1}^{l} \underbrace{V_{ij}}_{\partial_j f_i(\cdot, \mathbf{u}_n)} p_j = r_i \\ p_i|_{\partial\Omega} = 0 \end{cases} \quad (i = 1, \dots, l), \tag{3.13}$$

where $r_i = g_i + \mathrm{div}(K_i \nabla u_{n,i}) - \mathbf{b}_i \cdot \nabla u_{n,i} - f_i(x, \mathbf{u}_n)$.

- The corresponding algebraic system:

$$\mathbf{L}_h^{(n)}\mathbf{c} = \mathbf{d}, \tag{3.14}$$

where $\mathbf{c}$ and $\mathbf{d}$ are the coefficient vectors of $\mathbf{p}_h$ and $\mathbf{r}_h$, respectively.

- **Construction of the preconditioner**: for any $u_i|_{\partial\Omega} = 0$ let

$$S_i u_i = -\mathrm{div}(K_i \nabla u_i) + h_i u_i \quad (i = 1, \ldots, l), \tag{3.15}$$

where $h_i \in L^\infty(\Omega)$ and $h_i \geq 0$. Denote by $\mathbf{S}_h$ the stiffness matrix of the operator S given by:

$$S\mathbf{u} = (S_1 u_1, \ldots S_l u_l). \tag{3.16}$$

- Preconditioned system:

$$\mathbf{S}_h^{-1}\mathbf{L}_h^{(n)}\mathbf{c} = \mathbf{f} := \mathbf{S}_h^{-1}\mathbf{d}. \tag{3.17}$$

We have the decomposition

$$\mathbf{S}_h^{-1}\mathbf{L}_h^{(n)} = \mathbf{I} + \mathbf{Q}_{\mathbf{S_h}}^{\mathbf{(n)}},$$

where $\mathbf{Q}_{\mathbf{S_h}}^{\mathbf{(n)}}$ is the corresponding Gram matrix in V_h of the operator $Q_S^{(n)}$ defined implicitly as follows:

$$\langle Q_S^{(n)}\mathbf{v}, \mathbf{z}\rangle_S \equiv \int_\Omega ((\mathbf{b}\cdot\nabla\mathbf{v})\cdot\mathbf{z} + (f'_\xi(x,\mathbf{u}_n) - \mathbf{hI})\mathbf{v}\cdot\mathbf{z}) \quad (\mathbf{v},\mathbf{z}\in \mathrm{H}_0^1(\Omega)^l).$$

Proposition

The operator $Q_S^{(n)} : \mathrm{H}_0^1(\Omega)^l \to \mathrm{H}_0^1(\Omega)^l$ satisfies

$$\|Q_S^{(n)}\mathbf{v}\|_S \leq C_Q\|\mathbf{v}\|_{\mathrm{L}^p(\Omega)} \quad (\mathbf{v}\in \mathrm{H}_0^1(\Omega)^l), \tag{3.18}$$

for some $C_Q > 0$.

Theorem

The GMRES algorithm with $\mathbf{S}_h$-inner product, applied for the $N \times N$ preconditioned system $\mathbf{S}_h^{-1}\mathbf{L}_h^{(n)}\mathbf{c} = \mathbf{S}_h^{-1}\mathbf{d}$, yields

$$\left(\frac{\|r_k\|_{\mathbf{S_h}}}{\|r_0\|_{\mathbf{S_h}}}\right)^{1/k} \leq C\frac{1}{k^\alpha}, \quad \text{where } \alpha = \frac{1}{d} - \frac{1}{2} + \frac{1}{p}. \tag{3.19}$$

Specifically,

$$C = \max\{l\|Q_S^{(n)}\|, R_{l,\alpha} \cdot C_Q(1-\alpha)^{-1}\}.$$

We proved that the estimation of the superlinear convergence rate is independent of the outer Newton iterate $\mathbf{u}_n$ and V_h.

A numerical example: single equation

Let us solve the following PDE numerically:

$$\begin{cases} -\Delta u + \eta u^3 = f, & \text{in } \Omega = [0,1]^2, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3.20}$$

where η and f are defined by

$$\eta(x,y) = ((x-0.5)^2 + (y-0.5)^2)^{-\mu},$$
$$f(x,y) = 10((x-0.5)^2 + (y-0.5)^2)^{-\gamma}, \quad (x,y) \in \Omega.$$

We look for approximate solutions using the DIN plus PCGM technique.

- Given u_n from the DIN iteration, we solve the linearized problem

$$F'(u_n)p_n = -(F(u_n) - f),$$

i.e.,

$$-\Delta p_n + 3\eta u_n^2 p_n = \underbrace{\Delta u_n - \eta u_n^3 + f}_{r_n}.$$

- Apply FEM with stepsize $h = \frac{1}{N+1}$. Algebraic system:

$$(\mathbf{S}_h + \mathbf{D}_h^n)\mathbf{p}_n = \mathbf{d}_n, \tag{3.21}$$

where $\mathbf{d}_n := -\mathbf{G}_h\mathbf{u}_n + h^2(\mathbf{f} - \eta\mathbf{u}_n^3)$.

- Choose $\mathbf{S}_h$ as a preconditioner and apply PCGM.

- Theory holds whenever $\eta u_n^2 \in \mathrm{L}^{p/p-2}(\Omega)$ and $g \in \mathrm{L}^q(\Omega)$, where $\frac{1}{q} + \frac{1}{p} = 1$.
- We define the numbers

$$\delta_k^n = \left(\frac{\|r_k^n\|_{\mathbf{S}_h}}{\|r_0^n\|_{\mathbf{S}_h}} \right)^{\frac{1}{k}} k^\alpha,$$

where k denotes the inner steps of the process and n the outer ones.

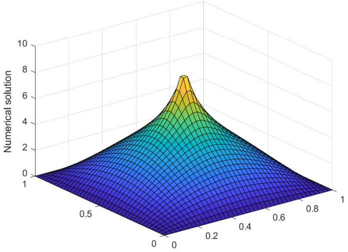
By the previous theorem:

$$\delta_k^n \underbrace{\leq \boldsymbol{C}}_{Expectation},$$

whenever

$$\alpha < \frac{1-\mu}{4}, \quad \gamma < 1 - \alpha.$$

- We check that these numbers are independent of $\mathbf{u}_n$.



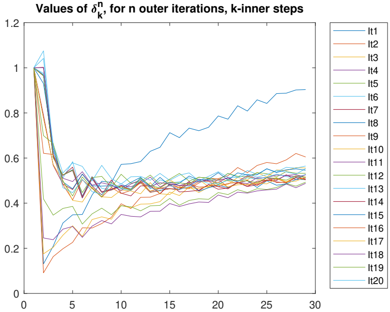


Figure: Numerical solution of (3.20) and behaviour of δ_k^n for $\mu = 0.1$, $\gamma = 0.75$. The right picture is obtained for $\alpha = 0.22$

Numerical solution of a nonlinear system

- We consider the following set of problems:

$$\begin{cases} -\epsilon\Delta u_i + (|\mathbf{u}|^2 + \beta)u_i = g_i, & \text{in } \Omega = [0,1]^2, \\ u_i|_{\partial\Omega} = 0, \quad i = 1, \dots l, \end{cases} \tag{3.22}$$

for $\beta > 0$ and $g_i \in \mathrm{L}^q(\Omega)$, where $\frac{1}{q} + \frac{1}{p} = 1$.

- New choice of preconditioner:

$$S\mathbf{u} = (S_1 u_1, \dots S_l u_l),$$

where

$$S_i u_i = -\epsilon\Delta u_i + \beta u_i \quad (i = 1, \dots, l).$$

- Let $l = 10, \epsilon = 0.1, \beta = 10$ and $g_i(x, y) = ((x - 0.3)^2 + (y - 0.3)^2)^{-\gamma}$.

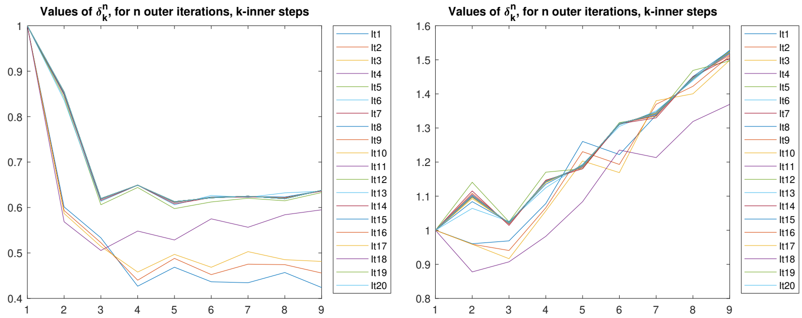


Figure: Behaviour of δ_k^n for $\alpha = 0.24, \gamma = 0.75$ and $\alpha = 0.5, \gamma = 0.9$, respectively.

Conclusions

- Superlinear convergence rate estimation when PCGM is applied to single equations or symmetric systems:

$$\left(\frac{\|e_k\|_{\mathbf{A}_h}}{\|e_0\|_{\mathbf{A}_h}}\right)^{\frac{1}{k}} \leq \boldsymbol{C}k^{-\alpha}, \quad \alpha = \frac{1}{d} - \frac{1}{2} + \frac{1}{p}.$$

- This also holds for the non-symmetric case when replacing PCGM with GMRES.
- We applied these results in the nonlinear case to provide mesh independence estimations of the rate of superlinear convergence for inner iterations of a Newton-Krylov method.

References I

- O. Axelsson, *Iterative Solution Methods*, Cambridge University Press, 1994.
- O. Axelsson and J. Karátson, *Mesh independent superlinear PCG rates via compact-equivalent operators*, SIAM Journal on Numerical Analysis, 45 (2007), pp. 1495–1516.
- O. Axelsson and J. Karátson, *Equivalent operator preconditioning for elliptic problems*, Numerical Algorithms, 50 (2009), pp. 297–380.
- H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, vol. 2, Springer, 2011.
- G. Chávez, G. Turkiyyah, S. Zampini, H. Ltaief, and D. Keyes, *Accelerated cyclic reduction: A distributed-memory fast solver for structured linear systems*, Parallel Computing, 74 (2018), pp. 65–83.
- D. E. Edmunds and H. Triebel, *Entropy numbers and approximation numbers in function spacess*, Proceedings of the London Mathematical Society, 3 (1989), pp. 137–152.

References II

- I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Operator theory: Advances and applications*, Classes of Linear Operators, 49 (1992).
- J. Karatson, *Mesh independent superlinear convergence estimates of the conjugate gradient method for some equivalent self-adjoint operators*, Applications of Mathematics, 50 (2005), pp. 277–290.
- T. Rossi and J. Toivanen, *A parallel fast direct solver for the discrete solution of separable elliptic equations.*, in PPSC, Citeseer, 1997.
- J. Vybíral, *Widths of embeddings in function spaces*, Journal of Complexity, 24 (2008), pp. 545–570.
- R. Winther, *Some superlinear convergence results for the conjugate gradient method*, SIAM Journal on Numerical Analysis, 17 (1980), pp. 14–17.
- Z. Zlatev, *Numerical treatment of large air pollution models*, in Computer Treatment of Large Air Pollution Models, Springer, 1995, pp. 69–109.

Thank you for your attention!

A numerical example

Let us solve the following PDE numerically:

$$\begin{cases} -\Delta u + \eta u = f, & \text{in } \Omega = [0,1]^2, \\ u|_{\partial\Omega} = 0 & \end{cases} \tag{6.23}$$

Here $\eta \in \mathrm{L}^{\frac{p}{p-2}}(\Omega)$ is defined as

$$\eta(x,y) = (x - 0.5)^2 + (y - 0.5)^2)^{-\beta}, \qquad 0 < \beta < \frac{p-2}{p}$$

and

$$f(x,y) = 1.$$

- Apply FEM with stepsize $h = \frac{1}{N+1}$. Algebraic system:

$$(\mathbf{G}_h + \mathbf{D}_h)\mathbf{c} = \mathbf{g}_h. \tag{6.24}$$

- Choose $\mathbf{G}_h$ as a preconditioner and apply PCGM.

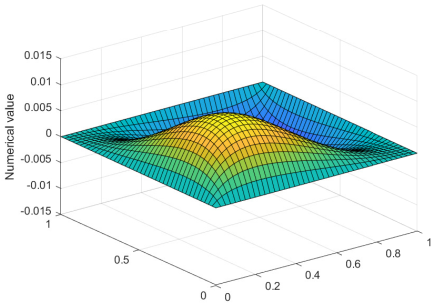


Figure: Graph of the numerical solution with $\beta = 3/4$ and $N = 40$.

- To measure the error of the PCGM, we use the energy norm

$$\|e\|_{\mathbf{A}_h} = \langle \mathbf{A}_h e, e \rangle^{\frac{1}{2}} \qquad (e \in \mathbb{R}^{N^2}),$$

  where $\mathbf{A}_h = \mathbf{G}_h + \mathbf{D}_h$.
- Notice:

$$\|e_k\|_{\mathbf{A}_h} = \|\boldsymbol{A_h}^{-1/2} r_k\|$$

Recall our main result:

Superlinear convergence rate

There exists $\boldsymbol{C} > 0$ such that

$$\left(\frac{\|e_k\|_{\mathbf{A}_h}}{\|e_0\|_{\mathbf{A}_h}}\right)^{\frac{1}{k}} \leq \frac{\boldsymbol{C}}{k^{\alpha}} \to 0, \text{ as } k \to \infty.$$

Here $\alpha = \frac{1}{d} - \frac{1}{2} + \frac{1}{p}$.

To test this result, we study the values of

$$\hat{\delta}_k = \left(\frac{\|r_k\|_{\boldsymbol{G_h}}}{\|r_0\|_{\boldsymbol{G_h}}}\right)^{\frac{1}{k}} k^{\alpha}$$

We performed several runs for different mesh-size and

$$\alpha = 0.12, \beta = 3/4.$$

Notice that the theorem holds when $\alpha < \frac{1-\beta}{2}$.

	$N = 20$	$N = 40$	$N = 80$
1	1.0000	1.0000	1.0000
2	0.5838	0.6236	0.6518
3	0.2865	0.3229	0.3499
4	0.1620	0.1907	0.2129
5	0.1037	0.1239	0.1412
6	0.1320	0.0946	0.0997
7	0.1069	0.1188	0.0999
8	0.1018	0.0921	0.1009
9	0.1001	0.0893	0.0802
10	0.1026	0.0901	0.0781

Table: Values of $\hat{\delta}_k$ for different mesh sizes.

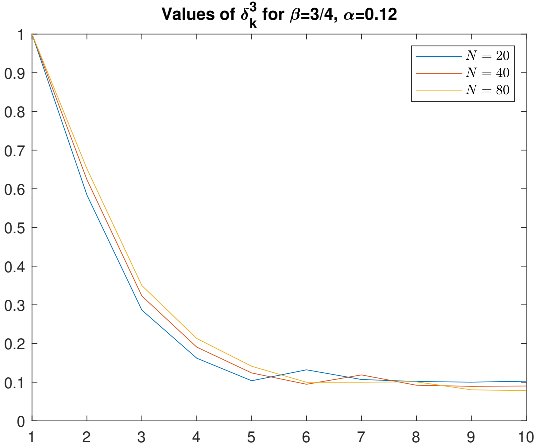


Figure: Graphical representation of Table 1.

Sketch of the proof

- We define the operators

$$Su \equiv -\mathrm{div}(G\nabla u), \quad u \in D \quad \text{and} \quad Qu \equiv \eta u, \quad u \in \mathrm{H}_0^1(\Omega).$$

- Energy space: $H_S = \mathrm{H}_0^1(\Omega)$ with

$$\langle u, v \rangle_S = \int_\Omega G\nabla u \cdot \nabla v. \tag{6.25}$$

It can be proved that there exists a unique operator $Q_S \in \mathcal{B}(H_s)$ such that

$$\langle Q_S u, v \rangle_S = \langle Qu, v \rangle$$

for all $u, v \in H_S$.

Lemma 1

There exists $C > 0$ such that

$$\|Q_S v\|_{H_S} \leq C \|v\|_{L^p(\Omega)}, \quad \forall v \in H_S. \tag{6.26}$$

Altogether, Q_S is compact and self-adjoint in H_S.

Proposition 1

Let $A = I + Q_S$. For any $k = 1, 2, \dots, n$

$$\sum_{j=1}^{k} |\lambda_j(\mathbf{G}_h^{-1}\mathbf{D}_h)| \leq \sum_{j=1}^{k} \lambda_j(Q_S). \tag{6.27}$$

Moreover,

$$\left(\frac{\|e_k\|_{\mathbf{A}_h}}{\|e_0\|_{\mathbf{A}_h}}\right)^{1/k} \leq 2\|A^{-1}\|\frac{1}{k}\sum_{j=1}^{k} \lambda_j(Q_S). \tag{6.28}$$

Now we wish to get a rate estimation from (6.28).

Useful results

1. Let $\lambda_n = \lambda_n(Q_S)$. Since Q_S is a compact self-adjoint operator in H_S, we have the characterization, [7, Ch.6, Th.1.5]:

   $$\lambda_n(Q_S) = \min\{\|Q_S - L_{n-1}\| \,/\, L_{n-1} \in \mathcal{L}(H_S), \mathrm{rank}(L_{n-1}) \leq n-1\}.$$

2. Let $a_n(\mathcal{I})$ denote the approximation numbers of the compact embedding $\mathcal{I}\colon \mathrm{H}_0^1(\Omega) \mapsto \mathrm{L}^p(\Omega)$, defined as

   $$a_n(\mathcal{I}) = \min\{\|\mathcal{I} - L_{n-1}\| / L_{n-1} \in \mathcal{L}(\mathrm{H}_0^1(\Omega), \mathrm{L}^p(\Omega)), \mathrm{rank}(L_{n-1}) \leq n-1\}.$$

3. From [6] we have the estimation

   $$a_n(\mathcal{I}) \leq \hat{C} n^{-\alpha}, \quad \alpha = \frac{1}{d} - \frac{1}{2} + \frac{1}{p},$$

   for some constant $\hat{C} > 0$.

Proposition 2

There exists $C > 0$, such that for all $n \in \mathbb{N}$,

$$\lambda_n(Q_S) \leq C a_n(\mathcal{I}). \tag{6.29}$$

Proposition 3

There exists $C > 0$ such that

$$\frac{1}{k} \sum_{n=1}^{k} \lambda_n(Q_S) \leq C \frac{1}{k^\alpha}.$$

Finally, by (6.28), the theorem is proved. □

- Given u_n from the DIN iteration, we solve the linearized problem

$$F'(u_n)p_n = -(F(u_n) - f).$$

In matrix form:

$$(\mathbf{A}^h + \mathbf{B}_n^h)\mathbf{p}_n = \mathbf{r}_n.$$

Here

$$\mathbf{A}^h = \epsilon \begin{pmatrix} -\Delta_h & & \\ & \ddots & \\ & & -\Delta_h \end{pmatrix} \in \mathbb{R}^{lN^2 \times lN^2},$$

$$\mathbf{B}_n^h = (\mathbf{B}_{t,k})_{i,j} = \left(\int_\Omega \partial_k f_t(u_1^n, \ldots, u_s^n) \psi_i \psi_j \right)_{i,j} \in \mathbb{R}^{lN^2 \times lN^2}.$$

$$(\mathbf{r_n})_i = \left(\epsilon \int_\Omega \nabla u_i^n \cdot \nabla \psi_j + (|\mathbf{u}^n|^2 + \beta) u_i^n \psi_j - \int_\Omega g_i \psi_j \right) \in \mathbb{R}^{lN^2},$$

$i = 1, \ldots, l$ and $j = 1, \ldots, N^2$.