Semi-implicit discretizations for a class of epidemiological models

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Consider the following ODE system with the following special structure:

$$\dot{u} = f(u) = A(u)u \tag{1}$$

where $A : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ with the properties

A1 $a_{ij}(u) \geq 0$ for $i, j = 1, \ldots, d; i \neq j$, while $a_{ii}(u) \leq 0$ for $i = 1, \ldots, d$; $\forall u \geq 0$.

A2 The column-sums add to zero i.e. $\forall u \geq 0$: $\sum_{i=1}^{d} a_{ij}(u) = 0;\ \forall j = 1, \ldots, d.$

(1) is the so-called *Graph-Laplacian form*[1].

Proposition

The solutions of (1) *with initial value* $u_0 \geq 0$ *are*

1. *positive if [A1] holds i.e.* $u(t) \geq 0$ *for all* $t \geq 0$
2. *conservative if [A2] holds i.e. if* $\mathbf{1}^t u_0 = \|u_0\|_1 = N_0$*, then* $\mathbf{1}^t u(t) = N_0$ *for all* $t \geq 0$.

where the order relations are meant component-wise.

- In some cases, we will consider $\dot{u}(t) = A(u)(u)$ with [A1].

- Numerous epidemiological models (and chemical models) can be written in the above form.
- The simplest SIR model:

$$\begin{bmatrix} \dot{S}(t) \\ \dot{I}(t) \\ \dot{R}(t) \end{bmatrix} = \begin{bmatrix} -\alpha I(t) & 0 & 0 \\ \alpha I(t) & -\beta & 0 \\ 0 & \beta & 0 \end{bmatrix} \begin{bmatrix} S(t) \\ I(t) \\ R(t) \end{bmatrix}, \tag{2}$$

where α is the transmission coefficient and β is the recovery rate.

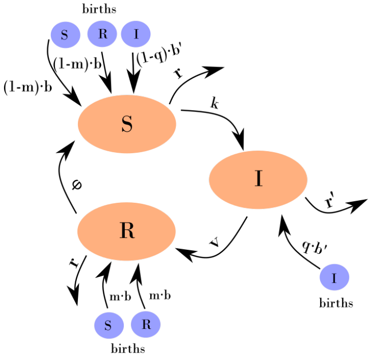
- SEIR model
- We can also consider general force-of infection, not only mass-action type

- A SIR model with vertical transmission and non-permanent immunity with partial vaccination[2]:

$$A(S,I,R) := \begin{pmatrix} -kI + (1-m)b - r & pb' & (1-m)b + \varphi \\ kI & qb' - r' - v & 0 \\ mb & v & mb - r - \varphi \end{pmatrix}.$$

which is conservative if $b = r$ and $b' = r'$.

Parameters	
b	Birth rate of uninfected individuals
b'	Birth rate of infected individuals
r	Death rate of uninfected individuals
r'	Death rate of infected individuals
v	Recovery rate
φ	Rate of immunity loss
$q \in (0,1)$	Rate of vertical transmission
p	$p := 1 - q$
$m \in (0,1)$	Fraction born vaccinated (or vaccine effectiveness)



- Consider that we want to approximate the solutions of the above ODE
- In general, we have to discretize the ODE and solve numerically
- Explicit Euler discretization:

$$y_{n+1} = y_n + hA(y_n)y_n \tag{3}$$

with initial conditions $y_0 = u(0)$, where $h > 0$ is the time-step and y_n approximates $u(t_n)$ where $t_n := hn$ for all $n \in \mathbf{N}$.

  - "Advantages"/good properties:
    - mass conservation: $\mathbf{1}^t y_{n+1} = \mathbf{1}^t y_n = \mathbf{1}^t y_0$, $(\forall n \in \mathbf{N})$
    - regular (no loss of equilibria nor no spurious one)
  - disadvantages/ not-so-good proerties
    - first order consistency
    - only condition positivity (strict conditions in examples)
    - "small" absolute stability region
    - i.e. stability properties same as for the cont. model for only small enough step-sizes
- Advantages and disadvantages of Runge-Kutta methods[3,4,5]

- By using *Nonstandard-finite difference schemes*(NSFD)[6], one can possibly get better qualitative properties
- we call a method NSFD if at least one of the following holds:
  1. stepsize h replaced by $\varphi(h, g(y_n))$ or possibly more general
  2. nonlinear terms replaced by nonlocal discretization: e.g. $y^2 \to y_{n+1}y_n$
- Most NSFD schemes are constructed for a specific model and first order, but not all (e.g. GeCo schemes for production-destruction schemes)[7].

$$\dot{u} = A(u)u$$

- different lonlocal approximations, gives rise to different semi-implicit methods
- method 1. semi-implicit method:

$$y_{n+1} = y_n + hA(y_n)y_{n+1} \tag{4}$$

or equivalently

$$(I - hA(y_n))y_{n+1} = y_n \tag{5}$$

$$y_{n+1} = y_n + hA(y_n)y_{n+1} \tag{6}$$

Proposition

Considering the the discretization (6) *of* (1) *with [A1] and [A2] or with* $\underline{c} = \mathbf{1}^t A(y) \le 0$ *and initial conditions* $y_0 = u(0) \ge 0$*:*

1. *The solutions of* (5) *uniquely exists for all* $n \in \mathbf{N}$
2. *The discretization is regular*
3. *The discretization preserves mass and positivity unconditionally*
4. *The discretization is first order consistent.*

Main idea: $I - hA(y_n)$ is an M-matrix for arbitrary $y_n \ge 0$

Semi-implicit theta method

$$y_{n+1} = y_n + h(1-\theta)A(y_n)y_n + h\theta A(y_n)y_{n+1} \quad (7)$$
$$= y_n + hA(y_n) + \theta A(y_n)(y_{n+1} - y_n) \quad (8)$$

Proposition

Considering the discretization (7) of (1) with $\theta \in [0,1)$ with [A1] and [A2] or with $\underline{c} = \mathbf{1}^t A(y) \leq 0$ and initial conditions $y_0 = u(0) \geq 0$:

1. *The solutions of (5) uniquely exists for all $n \in \mathbf{N}$*
2. *The discretization is regular and first order consistent*
3. *The discretization preserves mass and positivity for*

$$h \leq \frac{1}{1-\theta} h_{EE}$$

where h_{EE} is the maximal time-step for which the explicit Euler discretization (3) preserves the positivity.

semi-implicit method by diagonal splitting

$$y_{n+1} = y_n + h(A(y_n) - diag(A(y_n)))y_n - (-h\,diag(A(y_n))y_{n+1}) \quad (9)$$

or equivalently

$$(I - h\,diag(A(y_n)))y_{n+1} = (I + h(A(y_n) - diag(A(y_n)))y_n \quad (10)$$

Proposition

Considering the the discretization (9) *of* (1) *with [A1] and* **without [A2]** *with initial conditions* $y_0 = u(0) \geq 0$*:*

1. *The solutions of* (10) *uniquely exists for all* $n \in \mathbf{N}$
2. *The discretization is regular*
3. *The discretization preserves positivity unconditionally*
4. *The discretization is first order consistent*
5. *The discretization does not preserves mass, only upto* $O(h)$*.*

working on

- Stability for the equilibria
  - Only Jacobians and results for the special case (DFE for the examples): semi-implicit has the same stability as the implicit Euler.
- Second order methods
  - 4 possibly second order methods based on the diagonally split method with "not-so good" properties.
- Plus - epidemiologically meaningful - conditions on the materix valued function $A(.)$

References I I

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Köszönöm a figyelmet!
Thank you for your attention!