Selected Newton's methods in computational elasto-plasticity

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joint work with O. Axelsson, J. Karátson, M. Béreš, J. Haslinger et al.

Outline and aims of the talk

1. Abstract system of non-linear equations inspired by elasto-plasticity.

   - to explain selected features of elasto-plastic problems within algebraic level
   - specify assumptions for different type of elasto-plastic models

2. Semismooth Newton method, its modifications and convergence analysis.

   - survey of selected Newton-like methods used in elasto-plasticity
   - illustration of convergence results on numerical examples

3. Determination of the limit load in perfect plasticity.

   - important framework for solvability analysis and stability assessment of structures
   - advanced continuation method and related Newton-like solver

4. Brief notes to slope stability assessment.

   - overview of finite element methods on stability analysis
   - illustrative examples from geotechnical practice

1. Abstract system of non-linear equations inspired by elasto-plasticity

Elasto-plastic system of equations

Elasto-plastic problem in terms of displacement after time and space discretization:

$$\text{find } u_h^* \in V_h: \quad \int_\Omega T\left(e(u_h^*)\right) : e(v_h) dx = b(v_h) \quad \forall v_h \in V_h,$$

$$V_h \subset \left\{ v \in H^1(\Omega;\mathbb{R}^3) \mid v = 0 \text{ on } \Gamma_D \right\}, \quad e(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^T\right).$$

Investigated example of the stress-strain operator T – the von Mises model:

$$T(e) = \frac{1}{3}(3\lambda + 2\mu)(tr\, e)I + (1-\alpha)2\mu e^D + \alpha j(2\mu|e^D|)\frac{e^D}{|e^D|}, \qquad j(z) = \begin{cases} z, & z \le \gamma \\ \gamma, & z \ge \gamma \end{cases}$$

j – continuous, piecewise linear scalar function, switch between elasticity and plasticity

$\alpha \in (0,1)$ – hardening parameter, $\alpha = 0$ – linear elasticity, $\alpha = 1$ – elastic-perfectly plasticity

Nonlinear system of equations in $\mathbb{R}^n$:

$$\text{find } u^* \in \mathbb{R}^n: \quad F(u^*) = b, \qquad F: \mathbb{R}^n \to \mathbb{R}^n,\ b \in \mathbb{R}^n$$

$$F(v)^T w \quad := \quad \int_\Omega T(e(v_h)) : e(w_h)\, dx \quad \forall v_h, w_h \in V_h,$$

- Properties of F depend on properties of T.

Basic properties of elasto-plastic functions

$(\mathcal{A}_1)$ F is Lipschitz continuous in $\mathbb{R}^n$.

- F is almost everywhere differentiable in $\mathbb{R}^n$, there exists a generalized derivative of F.
- There exists $F^o \colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $F^o(u) = F'(u)$ for almost all $u \in \mathbb{R}^n$.
- $F^o(u) \in \partial F(u)$ – subdifferential in Clarke's sense

$(\mathcal{A}_2)$ F is strongly semismooth in $\mathbb{R}^n$:

$$\forall u \in \mathbb{R}^n,\ \exists L_u, \epsilon_u > 0 : \quad \|F(v) - F(u) - F^o(v)(v-u)\| \leq L_u \|u - v\|^2 \quad \forall v \in B(u; \epsilon_u),$$

$$F(v) - F(u) = \int_0^1 F^o\big(u + \theta(v-u)\big)(v-u)\, d\theta \qquad \forall u, v \in \mathbb{R}^n.$$

- Continuous piecewise linear functions are strongly semismooth with $L_u = 0$.
- Smooth functions with locally Lipschitz derivatives are strongly semismooth.
- Finite sums, products or compositions of semismooth functions are again semismooth.
- Implicit function theorem for semismooth functions (E-P operators may be implicit!).

Additional properties for associated plasticity

$(\mathcal{A}_3)$ F has a convex potential in $\mathbb{R}^n$ with linear growth at infinity, i.e.,

- $\exists \mathcal{I} \colon \mathbb{R}^n \to \mathbb{R}$ (convex) : $\quad \mathcal{I}'(v) = F(v) \quad \forall v \in \mathbb{R}^n$,
- $\exists\, c_1, c_2 > 0 : \quad \mathcal{I}(v) \geq c_1 \|v\| - c_2 \quad \forall v \in \mathbb{R}^n$.

Consequences:

- F is monotone, i.e. $(F(u) - F(v), u - v) \geq 0$ for any $u, v \in \mathbb{R}^n$.
- $F^o(v)$ is symmetric and positive semidefinite for any $v \in \mathbb{R}^n$.
- Equivalent minimization problem to $F(u^*) = b$:

$$\mathcal{J}(u^*) \leq \mathcal{J}(v) \;\; \forall v \in \mathbb{R}^n, \qquad \mathcal{J}(v) = \mathcal{I}(v) - b^\top v.$$

- Sufficient condition for the existence of u^*: $\|b\| < c_1$,

$$[\mathcal{J}(v) \geq (c_1 - \|b\|)\|v\| - c_2 \to +\infty \quad \text{as} \quad \|v\| \to +\infty \;\; (\textit{coercivity})]$$

- $(\mathcal{A}_3)$ is convenient for E-P models with bounded hardening or perfect plasticity.
- Stronger assumptions are available for E-P models with unbounded hardening.

Additional property for plasticity with hardening

$(\mathcal{A}_4)$ (uniform positive definitness of F^o)

$$\exists\, \beta_1, \beta_2 > 0: \quad \beta_1 \|v\|^2 \leq (F^o(u)v, v) \leq \beta_2 \|v\|^2 \quad \forall u, v \in \mathbb{R}^n$$

Consequences:

- Inverses of $F^o(u)$ are also uniformly positive definite.
- F is strongly monotone, i.e. $(F(u) - F(v), u - v) \geq \beta_1 \|u - v\|^2$ for any $u, v \in \mathbb{R}^n$.
- $\mathcal{J}$ is strictly convex and coercive in $\mathbb{R}^n$.
- There exists a unique solution u^* satisfying $F(u^*) = b$

Remarks:

- $(\mathcal{A}_4)$ will be considered within convergence analysis in Section 2.
- $(\mathcal{A}_4)$ will not be considered within convergence analysis in Section 3.

2. Semismooth Newton method, its modifications and convergence analysis

Semismooth Newton method

Algorithm:

$$F^o(u^k)(u^{k+1} - u^k) = b - F(u^k), \quad k = 0, 1, \ldots, \quad u^0 - \text{ given},$$

Local quadratic convergence under the assumptions $(\mathcal{A}_1) - (\mathcal{A}_4)$:

$$\|u^* - u^{k+1}\| = O(\|u^* - u^k\|^2)$$

Sketch of the proof: if u^k is sufficiently close to u^* then

$$\begin{aligned} u^* - u^{k+1} &= u^* - u^k - F^o(u^k)^{-1}\left[b - F(u^k)\right] \\ &= F^o(u^k)^{-1}\left[F(u^k) - F(u^*) - F^o(u^k)(u^k - u^*)\right] \end{aligned}$$

$$\begin{aligned} \|u^* - u^{k+1}\| &\leq \|F^o(u^k)^{-1}\|\|F(u^k) - F(u^*) - F^o(u^k)(u^k - u^*)\| \\ &\overset{(\mathcal{A}_2, \mathcal{A}_4)}{\leq} \frac{1}{\beta_1} L_{u^*}\|u^* - u^k\|^2. \end{aligned}$$

Remark: This result holds for more general assumptions than $(\mathcal{A}_1) - (\mathcal{A}_4)$, [Qi and Sun 1993]

Quasi-Newton method and its nonsmooth variant

Algorithm: [Faragó, Karátson 2002], [Karátson, Faragó 2003], [Borsos, Karátson 2022]

$$u^{k+1} := u^k + \frac{2}{M_k + m_k} B_k^{-1}(b - F(u^k)) \qquad k = 0, 1, \dots, \quad u^0 - \text{ given},$$

where $B_k \in \mathbb{R}^{n\times n}_{sym}$, $0 < m_{min} \le m_k \le M_k \le M_{max}$ and

$$m_k(B_k v, v) \le (F^o(u^k)v, v) \le M_k(B_k v, v) \quad \forall u, v \in \mathbb{R}^n,\ \forall k \in \mathbb{N}.$$

Original convergence results for smooth operators:

Let $(\mathcal{A}_1) - (\mathcal{A}_4)$ hold and F has a Lipschitz continuous derivative F'. Then

$$\limsup \frac{\|F(u^{k+1})\|_*}{\|F(u^k)\|_*} \le \limsup \frac{M_k - m_k}{M_k + m_k} < 1, \qquad \|v\|_* := (F'(u^*)^{-1} v,\ v)^{1/2}.$$

Remarks:

- only linear convergence, but faster assembling of B_k than $F'(u^k)$
- Examples when the quasi-Newton method is faster than the Newton method: [Borsos, Karátson 2022], in nonlinear elasticity: [Karátson, S., Béreš 2024]
- Recommendation: combination of the quasi-Newton method with deflated CG method

Quasi-Newton method and its nonsmooth variant

Local linear convergence for non-smooth operators I: Let $(\mathcal{A}_1)-(\mathcal{A}_4)$ hold. Then

$$\|u^* - u^{k+1}\| \leq \frac{\beta_2}{\beta_1}\frac{M_k - m_k}{M_k + m_k}\|u^* - u^{k+1}\| + O(\|u^* - u^{k+1}\|^2).$$

criterion: $\frac{\beta_2}{\beta_1}\frac{M_k - m_k}{M_k + m_k} \leq q < 1 \quad \forall k \in \mathbb{R}^n,$ however $\frac{\beta_2}{\beta_1} > 1$

Sketch of the proof: if u^k is sufficiently close to u^* then

$$u^* - u^{k+1} = \left(I - \frac{2}{M_k + m_k} B_k^{-1} F^o(u_k)\right)(u^* - u^k) + \\ + \frac{2}{M_k + m_k} B_k^{-1}\left[F(u^k) - F(u^*) - F^o(u^k)(u^k - u^*)\right],$$

$$\|u^* - u^{k+1}\| \overset{(\mathcal{A}_2,\mathcal{A}_4)}{\leq} \left\|I - \frac{2}{M_k + m_k} B_k^{-1} F^o(u_k)\right\| \|u^* - u^k\| + O(\|u^* - u^k\|^2),$$

$$\left\|I - \frac{2}{M_k + m_k} B_k^{-1} F^o(u_k)\right\| \leq \left\|F^o(u_k)^{-1} - \frac{2}{M_k + m_k} B_k^{-1}\right\| \|F^o(u_k)\| \overset{(\mathcal{A}_4)}{\leq} \frac{\beta_2}{\beta_1}\frac{M_k - m_k}{M_k + m_k}.$$

Quasi-Newton method and its nonsmooth variant

Local linear convergence for non-smooth operators II: Let $(\mathcal{A}_1)-(\mathcal{A}_4)$ hold. Then

$$\|u^* - u^{k+1}\|_{u*} \leq \sqrt{\frac{\gamma_{2,k}}{\gamma_{1,k}}} \frac{M_k - m_k}{M_k + m_k} \|u^* - u^{k+1}\|_{u*} + O(\|u^* - u^{k+1}\|^2_{u*}),$$

where

$$\|v\|_{u^*} := \sqrt{(F^o(u^*)v, v)}, \quad \|v\|_{u^k} = \sqrt{(F^o(u^k)v, v)} \quad \forall u, v \in \mathbb{R}^n,$$

$$\gamma_{1,k}\|v\|^2_{u^*} \leq \|v\|^2_{u^k} \leq \gamma_{2,k}\|v\|^2_{u^*} \quad \forall u, v \in \mathbb{R}^n,\ \forall k \in \mathbb{N},\ \gamma_{1,k} \geq \gamma_{min} > 0$$

criterion: $\sqrt{\frac{\gamma_{2,k}}{\gamma_{1,k}}} \frac{M_k - m_k}{M_k + m_k} \leq q < 1 \quad \forall k \in \mathbb{N},$ realistic assumption: $\sqrt{\frac{\gamma_{2,k}}{\gamma_{1,k}}} \approx 1$

Sketch of the proof: if u^k is sufficiently close to u^* then

$$\|u^* - u^{k+1}\|_{u^k} \overset{(\mathcal{A}_2,\mathcal{A}_4)}{\leq} \left\| I - \frac{2}{M_k + m_k} B_k^{-1} F^o(u_k) \right\|_{u^k} \|u^* - u^k\|_{u^k} + O(\|u^* - u^k\|^2_{u^k}),$$

$$\left\| I - \frac{2}{M_k + m_k} B_k^{-1} F^o(u_k) \right\|_{u^k} \leq \frac{M_k - m_k}{M_k + m_k}, \quad \frac{\|u^* - u^{k+1}\|_{u^k}}{\|u^* - u^{k+1}\|_{u^*}} \geq \sqrt{\gamma_{1,k}}, \quad \frac{\|u^* - u^k\|_{u^k}}{\|u^* - u^k\|_{u^*}} \leq \sqrt{\gamma_{2,k}}$$

Examples of the preconditioners B_k

Quasi-Newton 1: $B_k = K_{elast}$

- elastic stiffness matrix with fixed material parameters
- advantage: a constant matrix with a simple assembling
- disadvantage: poor approximation of $F^o(u^k)$

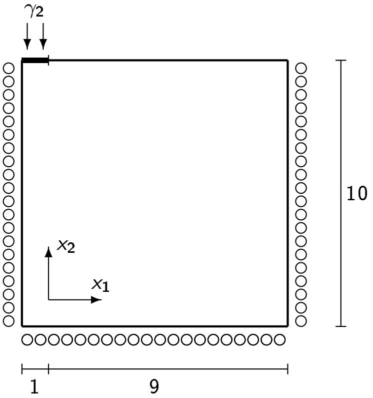
Quasi-Newton 2: $B_k = K_{elast,k}$

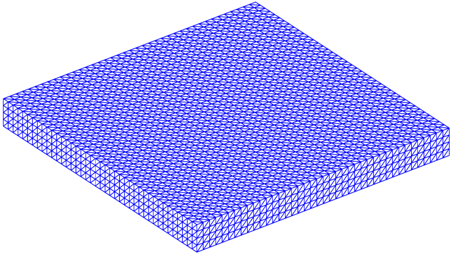
- elastic stiffness matrix with variable material parameters
- advantage: better approximation of $F^o(u^k)$
- disadvantage: assembling in each iteration

Smoothing Newton method: [Qi, Sun 2002] $B_k = F'_\epsilon(u^k)$

- F_ϵ is a smooth approximation of F
- advantage: M_k and m_k are close to one as $\epsilon \to 0$
- disadvantage: assembling of $F'_\epsilon(u^k)$ is not faster than assembling of $F^o(u^k)$
- The operator F'_ϵ will be used later, within the continuation Newton method.

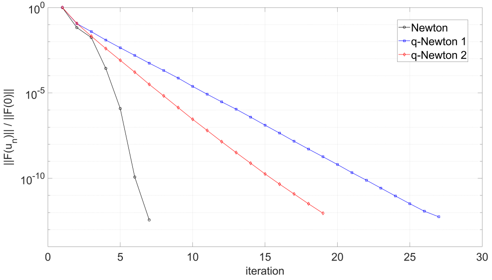
Numerical example in 3D – strip footing



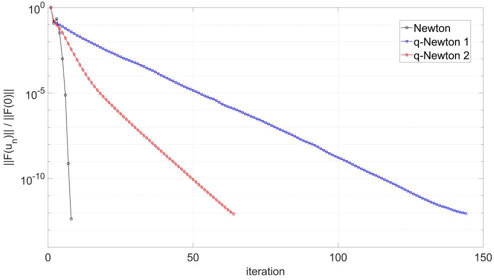


- 2 investigated values of the hardening: $\alpha = 0.5$ and $\alpha = 0.9$ (stronger nonlinearity)
- 3 investigated meshes with 38 400, 307 200 and 1 036 800 elements
- comparison of the Newton, Quasi-Newton 1 and Quasi-Newton 2 methods
- similar results for smooth version of the E-P problem, see [Karátson, S., Béreš 2024]

Comparison of iteration numbers

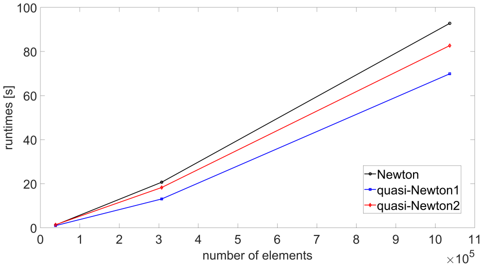


- $\alpha = 0.5$
- finest mesh, DCG
- Newton: 7 iterations
- q-Newton1: 27 it.
- q-Newton2: 19 it.

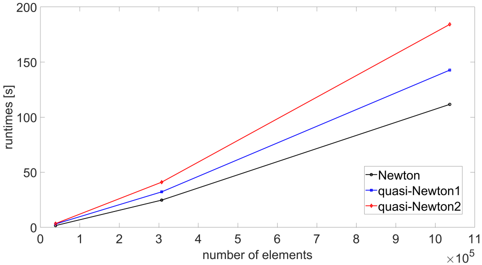


- $\alpha = 0.9$
- finest mesh, DCG
- Newton: 8 iterations
- q-Newton1: 144 it.
- q-Newton2: 64 it.

Comparison of computational times



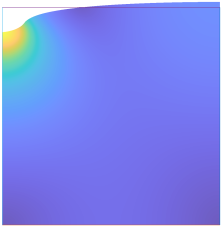
- $\alpha = 0.5$
- 3 meshes
- Newton - slowest
- q-N1 - fastest

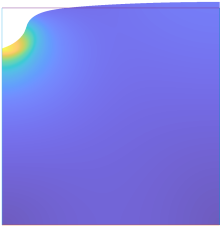


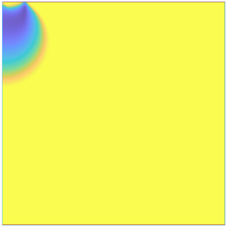
- $\alpha = 0.9$
- 3 meshes
- Newton - fastest
- q-N2 -slowest

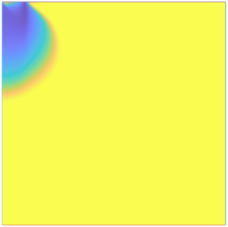
Comparison of physical arrays

$\alpha = 0.5$ $\alpha = 0.9$









Semismooth Newton method with damping

Algorithm: (used in elasto-plasticity in [S. 2012])

$$u^{k+1} := u^k + \alpha_k \, s^k \qquad k = 0, 1, \ldots, \quad u^0 - \text{ given},$$
$$F^o(u^k)s^k = b - F(u^k)$$
$$\alpha_k = \arg \min_{\omega \in [0,1]} \mathcal{J}(u^k + \omega s^k), \qquad \mathcal{J}'(v) = F(v) - b^\top v$$

Remarks:

- Newton' methods without damping sometimes do not converge in elasto-plasticity
- damping enables to investigate global convergence
- optimization framework simplifies convergence analysis
- alternative line-search based on the Armijo rule: *choose* α_k *satisfying*

$$\mathcal{J}(u^k + \alpha_k s^k) - \mathcal{J}(u^k) \leq -\varrho \alpha_k (F^o(u^k)s^k, s^k), \quad \varrho \in (0,1)$$

- similar convergence analysis for the Armijo line search

Global convergence of the damped method

Algorithm: $u^{k+1} := u^k + \alpha_k\, s^k$, $F^o(u^k)s^k = b - F(u^k)$, $\alpha_k = \arg\min_{\beta\in[0,1]} \mathcal{J}(u^k + \beta s^k)$

Key estimates derived under the assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_3)$ **and** $(\mathcal{A}_4)$:

$$(\mathcal{J}'(u^k), s^k) \leq -\beta_1 \|s^k\|^2,$$

$$\alpha_k \geq \frac{\beta_1}{\beta_2} > 0, \quad \text{if } s^k \neq 0,$$

$$\mathcal{J}(u^{k+1}) - \mathcal{J}(u^k) \leq -\frac{1}{2}\beta_1\alpha_k^2 \|s^k\|^2 \leq -\frac{\beta_1^3}{2\beta_2^2}\|s^k\|^2,$$

$$\sum_{k=0}^{+\infty} \|s^k\|^2 \leq \frac{2\beta_2^2(\mathcal{J}(u^0) - \mathcal{J}(u^*))}{\beta_1^3} \quad \Longrightarrow \quad s^k \to 0,$$

$$\sum_{k=0}^{+\infty} \|u^* - u^k\|^2 \leq \frac{2\beta_2^4(\mathcal{J}(u^0) - \mathcal{J}(u^*))}{\beta_1^5} \quad \Longrightarrow \quad u^k \to u^*.$$

Remarks:

- The global convergence result can be extended to infinite dimensional Hilbert spaces.
- Semismoothness of F is not necessary for the global convergence.

Superlinear convergence of the damped method

Algorithm: $u^{k+1} := u^k + \alpha_k s^k$, $F^o(u^k)s^k = b - F(u^k)$, $\alpha_k = \arg\min_{\beta\in[0,1]} \mathcal{J}(u^k + \beta s^k)$

Key results under the assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_4)$:

$$\lim_{k\to+\infty} \alpha_k = 1$$

$$\|u^* - u^{k+1}\| = (1-\alpha_k)\|u^* - u^k\| + O(\|u^* - u^k\|^2) = o(\|u^* - u^k\|)$$

Remark:

- Semismoothness of F is crucial for local superlinear convergence of the damped method.
- Superlinear convergence of the damped Newton was illustrated on numerical examples.
- Numbers of iteration only slightly depend on mesh density, see [S. 2012].

Continuation Newton method [Axelsson, S. 2015]

Severely nonlinear system of equations with a nonsmooth function:

$$\text{find } u^* \in \mathbb{R}^n: \quad F(u^*) = b \quad F\colon \mathbb{R}^n \to \mathbb{R}^n,\ b \in \mathbb{R}^n$$

Load-based continuation method:

$$F(0) = 0, \quad F(\hat{u}(t)) = tb, \quad 0 \le t \le 1, \quad \hat{u}(1) = u^*.$$

Smooth approximation of F:

$$\{F_\epsilon\}_{\epsilon \in (0,\epsilon_0)} - \text{ smooth}: \quad \lim_{\epsilon \to 0} F_\epsilon(v) = F(v), \qquad \lim_{\epsilon \to 0} F'_\epsilon(v) = F^o(v) \ \ \forall v \in \mathbb{R}^n$$

One-step smoothing Newton method: $0 = t_0 < t_1 < \ldots < t_N = 1,\ \tau_k := t_{k+1} - t_k$

$$F'_\epsilon(u^k)(u^{k+1} - u^k) = t_{k+1} b - F(u^k), \quad k = 0, 1, \ldots N-1, \quad u^0 = 0,$$

Aim: find assumptions guaranteeing that u^k is close to $\hat{u}(t_k)$ for any $k = 1, \ldots N$

Assumptions and their consequences

$$(\mathcal{A}_5)\quad \exists M>0:\quad \|F_\epsilon(u)-F(u)-(F_\epsilon(v)-F(v))\|\leq M\epsilon\|u-v\|\quad \forall u,v\in\mathbb{R}^n,\ \forall\epsilon\in(0,\epsilon_0).$$

$$(\mathcal{A}_6)\quad \exists L>0:\quad \|F_\epsilon(v)-F_\epsilon(u)-F'_\epsilon(v)(v-u)\|\leq \frac{L}{2\epsilon}\|u-v\|^2\quad \forall u,v\in\mathbb{R}^n,\ \forall\epsilon\in(0,\epsilon_0).$$

$$(\mathcal{A}_7)\quad \exists q>0:\quad \begin{cases} [F(u)-F(v)]^T(u-v)\geq q\|u-v\|^2 & \forall u,v\in\mathbb{R}^n,\\ [F_\epsilon(u)-F_\epsilon(v)]^T(u-v)\geq q\|u-v\|^2 & \forall u,v\in\mathbb{R}^n,\ \forall\epsilon\in(0,\epsilon_0).\end{cases}$$

- Hence: $\|\hat{u}(t_{k+1})-\hat{u}(t_k)\|\leq \frac{1}{q}\|F(\hat{u}(t_{k+1}))-F(\hat{u}(t_k))\|=\frac{\|b\|}{q}\tau_k\quad \forall k=0,1,\dots N-1$
- Hence: $\|[F'_\epsilon(u^k)]^{-1}\|\leq\frac{1}{q}\quad \forall k=1,2\dots N$
- If $(\mathcal{A}_4)$ hold then $q=\beta_1$.

Convergence of the algorithm

Let the assumptions $(\mathcal{A}_5)$, $(\mathcal{A}_6)$, $(\mathcal{A}_7)$ hold with the constants M, L and q. Let

$$\epsilon \leq \frac{q}{4M}, \quad \tau_k \leq \frac{q^2\epsilon}{4L\|b\|} \quad \forall k = 0, 1, \dots, N-1.$$

Then

$$\|\hat{u}(t_k) - u^k\| \leq \frac{\|b\|}{q} \max_{0\leq l\leq k-1} \tau_l\,, \quad k = 0, 1 \dots .$$

Sketch of the proof:

The result can be shown by mathematical induction

$$\begin{aligned}
\hat{u}(t_{k+1}) - u^{k+1} \quad &= \quad -(F'_\epsilon(u^k))^{-1}\left[F(\hat{u}(t_{k+1})) - F_\epsilon(\hat{u}(t_{k+1})) - (F(u^k) - F_\epsilon(u^k))\right] \\
&\quad +(F'_\epsilon(u^k))^{-1}\left[F_\epsilon(u^k) - F_\epsilon(\hat{u}(t_{k+1})) - F'_\epsilon(u^k)(u^k - \hat{u}(t_{k+1}))\right].
\end{aligned}$$

Hence:

$$\|\hat{u}(t_{k+1}) - u^{k+1}\| \leq \frac{1}{q}\left[M\epsilon\|\hat{u}(t_{k+1}) - u^k\| + \frac{L}{2\epsilon}\|\hat{u}(t_{k+1}) - u^k\|^2\right],$$

$$\|\hat{u}(t_{k+1}) - u^{k+1}\| \leq \frac{\|b\|}{2q}\tau_k + \left(\frac{1}{4} + \frac{L}{q\epsilon}\|\hat{u}(t_k) - u^k\|\right)\|\hat{u}(t_k) - u^k\|.$$

Regularization of the stress-strain operator

Algebraic notation (recalling):

$$F(v)^T w \quad := \quad \int_\Omega T(e(v_h)) : e(w_h)\,\mathrm{d}x \quad \forall v_h, w_h \in V_h,$$

$$F_\epsilon(v)^T w \quad := \quad \int_\Omega T_\epsilon(e(v_h)) : e(w_h)\,\mathrm{d}x \quad \forall v_h, w_h \in V_h.$$

Stress-strain operator and its smooth approximation:

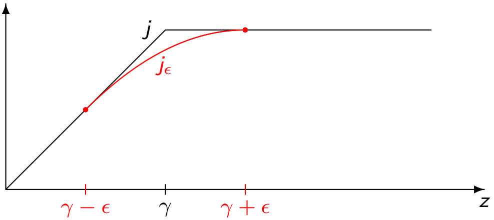
$$T(e) = \frac{1}{3}(3\lambda + 2\mu)(tr\ e)I + (1-\alpha)2\mu e^D + \alpha j(2\mu|e^D|)\frac{e^D}{|e^D|},$$

$$T_\epsilon(e) = \frac{1}{3}(3\lambda + 2\mu)(tr\ e)I + (1-\alpha)2\mu e^D + \alpha j_\epsilon(2\mu|e^D|)\frac{e^D}{|e^D|},$$

j, j_ϵ – nonlinear scalar function and its smooth approximation

The function j and its smooth approximation j_ϵ

$$j(z) = \begin{cases} z, & z \le \gamma \\ \gamma, & z \ge \gamma \end{cases}, \qquad j_\epsilon(z) := \begin{cases} z, & z \le \gamma - \epsilon \\ \gamma - \frac{1}{4\epsilon}(z-\gamma-\epsilon)^2, & z \in [\gamma-\epsilon, \gamma+\epsilon] \\ \gamma, & z \ge \gamma + \epsilon \end{cases}$$



Relationships between j and j_ϵ implying the assumptions $(\mathcal{A}_5)$–$(\mathcal{A}_7)$:

$$\begin{aligned} |j(z) - j_\epsilon(z)| &\le \frac{\epsilon}{4} \quad \forall z \in \mathbb{R},\ \forall \epsilon \in (0,\gamma), \\ |j(z_1) - j(z_2) - j_\epsilon(z_1) + j_\epsilon(z_2)| &\le \frac{\epsilon}{2}|z_1 - z_2| \quad \forall z_1, z_2 \in \mathbb{R},\ \forall \epsilon > 0, \\ \left| j_\epsilon(z_2) - j_\epsilon(z_1) - j_\epsilon'(z_2)(z_2 - z_1) \right| &\le \frac{1}{4\epsilon}(z_1 - z_2)^2 \quad \forall z_1, z_2 \in \mathbb{R},\ \forall \epsilon > 0 \end{aligned}$$

Assumptions $(\mathcal{A}_5)$–$(\mathcal{A}_7)$ for the E-P operators

$$(1-\alpha)\|v\|_e^2 \leq v^T K'_\epsilon(w)v \leq \|v\|_e^2 \quad \forall v, w \in \mathbb{R}^n,\ \forall \epsilon \in (0, \epsilon_0),$$

$$v^T(F(w+v) - F(w)) \geq (1-\alpha)\|v\|_e^2 \quad \forall v, w \in \mathbb{R}^n,$$

$$\|F(v) - F(w) - K_\epsilon(v) + K_\epsilon(w)\|_* \leq M\epsilon\|v-w\|_e \quad \forall v, w \in \mathbb{R}^n,\ \forall \epsilon \in (0, \epsilon_0),$$

$$\|K_\epsilon(v) - K_\epsilon(u) - K'_\epsilon(v)(v-u)\|_* \leq \frac{L}{2\epsilon}\|v-w\|_{e,L^4}^2 \quad \forall u, w \in S,\ \forall \epsilon \in (0, \epsilon_0),$$

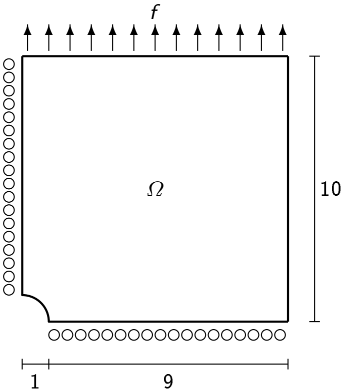
where

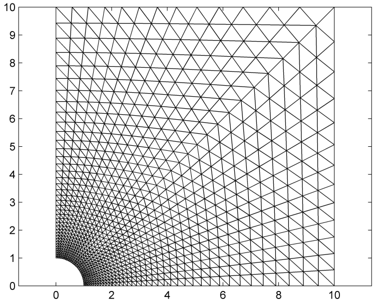
$$\|v\|_e,\ \|v\|_{e,L^4} \text{ – energy norms}, \quad \|v\|_* := \sup_{w\in\mathbb{R}^n,\ \|w\|_e=1} |v^T w| \quad \forall v \in \mathbb{R}^n.$$

Remarks:

- The estimates are dependent on the hardening parameter α.
- The estimates are independent of the discretization parameter.
- The convergence result could be extended to the functional setting, unlike $\epsilon \to 0$.

Numerical example





$f = (0, 1000)$, $E = 206900$ (Young's modulus), $\nu = 0.29$ (Poisson's ratio),

$1 - \alpha = 4.2 * 10^{-4} << 1$, $\gamma = 450\sqrt{2/3}$ (yield stress), $\epsilon = 1 << \gamma$ (regularization),

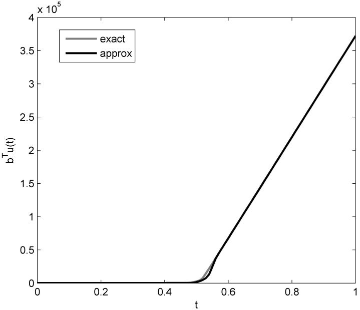
Numerical results – comparison of solution paths

$$F(\hat{u}(t_k)) = t_k b, \quad u^k = u^{k-1} + (F'_\epsilon(u^{k-1}))^{-1}[t_k b - F(u^{k-1})]$$

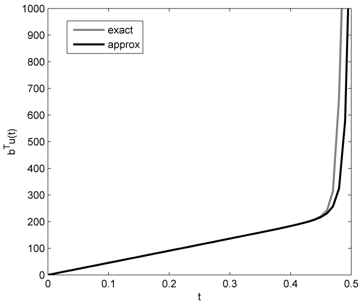
Comparison of exact solution path $\{\hat{u}(t)\}_{t\in[0,1]}$ with its approximation $\{u^k\}_{k=0}^N$:

- the quantity $b^\top \hat{u}(t)$ is used for the visualization, where b is r.h.s
- the curves almost coincide with the exception of $t \in (0.45, 0.55)$
- linear elastic branch for $t \leq 0.45$, hardening branch of the curve for $t \in (0.55, 1)$

$t \in [0,1]$



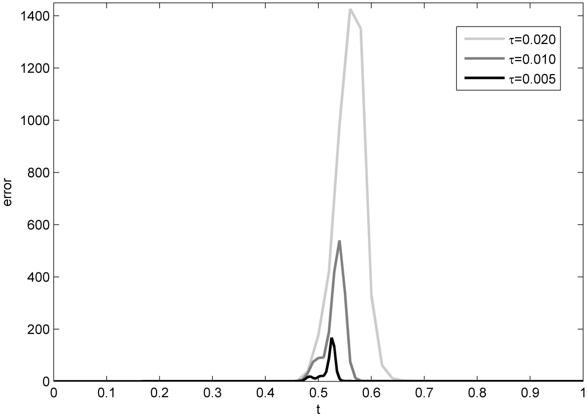
Detail of the curve for $t \in [0, 0.5)$



Numerical results – convergence of the algorithm

Dependence of $\|\hat{u}(t_k) - u^k\|_e$ **on** $\tau = t_k - t_{k-1} = const.$:

- error measure is minimal for $t \in [0, 0.45]$ and $t \in [0.55, 1]$
- error measure tends to zero as $\tau \to 0$



3. Determination of the limit load in perfect plasticity by Newton-like methods

Elastic-perfectly plastic problem and limit load

About the elastic-perfectly plastic problem:

- $T(e) = \frac{1}{3}(3\lambda + 2\mu)(tr\, e)I + (1-\alpha)2\mu e^D + \alpha j(2\mu|e^D|)\frac{e^D}{|e^D|}, \quad \alpha = 1.$
- Assumptions $(\mathcal{A}_1) - (\mathcal{A}_3)$ hold (F is semismooth and has a potential with linear growth).
- Assumption $(\mathcal{A}_4)$ does not hold (F is not strongly monotone).
- Problem $F(u^*) = b$ has the solution u^* only for sufficiently small b.

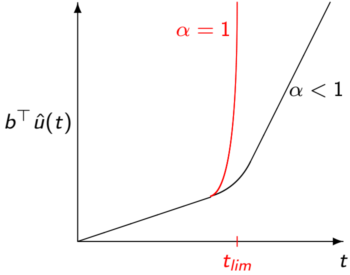
Problem parametrization and the limit load:

- $F(\hat{u}(t)) = tb, \quad t \geq 0$, t is a scalar load factor
- t_{lim} – limit load factor = supremum of t for which $\hat{u}(t)$ exists.
- t_{lim} – important for safety assessment of structures.
- Solution exists for any $t \in [0, t_{lim})$, but is unbounded in vicinity of t_{lim}.
- Continuation over t is not too numerically stable.

Advanced continuation technique for finding t_{lim}

Energy-based control the loading path:

- Idea: use the dependence between t and $b^\top \hat{u}(t)$ and its inverse
- [S., Haslinger, Hlaváček, Čermák 2015], [S., Haslinger, Reddy, Repin 2021]



Advanced continuation method and limit analysis

Nonlinear system with additional equation:

$$\text{find } \bar{u}(\omega) \in \mathbb{R}^n,\ t(\omega) \geq 0: \qquad F(\bar{u}(\omega)) = t(\omega)b,\ b^T\bar{u}(\omega) = \omega.$$

- ω is a given parameter, t is additional unknown
- $\bar{u}(\omega) = \hat{u}(t(\omega)), \quad \omega \to +\infty \quad \Rightarrow \quad t(\omega) \to t_{lim},\ \|\bar{u}(\omega)\| \to \infty,\ \|\bar{u}(\omega)/\omega\| \to c < \infty$

Transformed problem and its optimization form: $v(\omega) = \bar{u}(\omega)/\omega$

$$\text{find } v(\omega) \in \mathbb{R}^n,\ t(\omega) \geq 0: \qquad F(\omega v(\omega)) = t(\omega)b,\ b^T v(\omega) = 1.$$

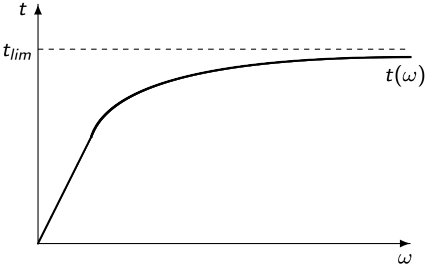
$$\mathcal{I}_\omega(v(\omega)) = \min_{\substack{v\in\mathbb{R}^n \\ b^\top v = 1}} \mathcal{I}_\omega(v), \qquad \mathcal{I}_\omega(v) = \frac{1}{\omega}\mathcal{I}(\omega v),\ \mathcal{I}'(v) = F(v)$$

Limit analysis problem: $\omega \to +\infty$

$$t_{lim} = \inf_{\substack{v\in\mathbb{R}^n \\ b^\top v = 1}} \mathcal{I}_\infty(v), \qquad \mathcal{I}_\infty = \lim_{\omega\to+\infty} \mathcal{I}_\omega$$

- $\mathcal{I}_\infty$ is convex, 1-positively homogeneous, not finite-valued everywhere
- On a functional level, problem is defined on BD spaces, hidden constraint: $\operatorname{div} v = 0$

Continuation strategy for determining t_{lim}



1. Generate (adaptively) a sequence $0 < \omega_1 < \omega_2 < \ldots < \omega_N$
2. For any ω belonging to the sequence, find $v(\omega) \in \mathbb{R}^n$, $t(\omega) \geq 0$:

$$F(\omega v(\omega)) = t(\omega)b, \quad b^T v(\omega) = 1.$$

3. $t_{lim} \approx t(\omega_N)$

Remarks:

- The increment of ω is increased if the increment of t is too small.
- The solver for fixed ω is initiated using solutions from previous 2 steps.

Newton-like method for constraint optimization

The problem for given $\omega > 0$:

$$F_\omega(v^*) = t^* b,\ b^T v^* = 1 \quad \text{or} \quad \mathcal{I}_\omega(v^*) = \min_{\substack{v \in \mathbb{R}^n \\ b^\top v = 1}} \mathcal{I}_\omega(v), \qquad F_\omega(v) = F(\omega v)$$

Semismooth Newton method as sequential quadratic programming:

$$u^{k+1} := u^k + s^k \qquad k = 0, 1, \dots, \quad u^0 - \text{ given},\ b^\top u^0 = 1$$

$$s^k = \arg \min_{\substack{s \in \mathbb{R}^n \\ b^\top s = 0}} \left[\frac{1}{2} (F^o_\omega(u^k) s, s) + (F_\omega(u^k), s) \right]$$

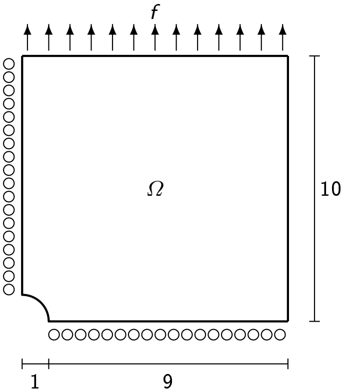
$$\text{or} \quad s^k = h^k + \delta_k g^k, \quad F^o_\omega(u^k) g^k = b, \quad F^o_\omega(u^k) h^k = -F_\omega(u^k), \quad \delta_k = -\frac{b^\top h^k}{b^\top g^k}$$

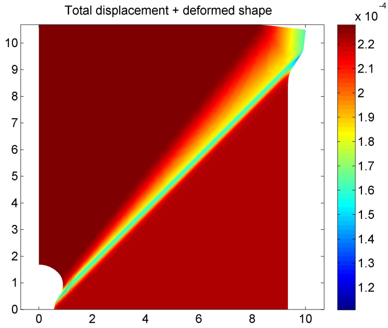
Crucial estimates implying local quadratic convergence:

$$\left(F^o_\omega(u^k)(u^{k+1} - u^*), u^{k+1} - u^* \right) \le \left(F_\omega(u^k) - F_\omega(u^*) - F^o_\omega(u^k)(u^k - u^*), u^* - u^{k+1} \right)$$

- The algorithm can also be used for contact problems of E-P bodies.
- We have also used damped and regularized versions of this algorithm.

Solution corresponding to limit load factor t_{lim}

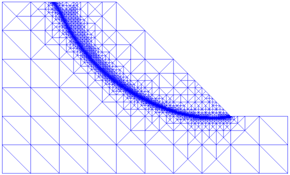


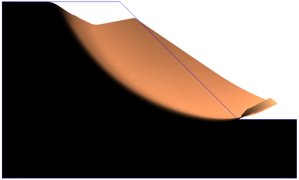


- solution for t_{lim} represents plastic collapse
- failure zone – zone with discontinuity
- P1 elements lead to locking phenomena [Repin, S., Haslinger 2018]

Mesh adaptive solution concept

- Apply the continuation over ω only on the coarsest mesh.
- Fix sufficiently large ω and use it for all finer meshed.
- Refine elements where higher strains appear.
- interpolate solution from the coarser mesh and use it for the initialization on a finer mesh.
- Use damped semismooth Newton method for finer meshes and the fixed ω.





4. Brief notes to slope stability assessment.

Limit vs. shear strength reduction analysis

Limit analysis:

- Factor of safety represents the limit load factor.
- Robust method supported by mathematical theory.
- Not so conventional in slope stability.

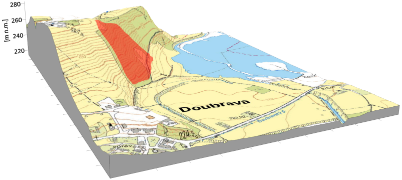
Shear strength reduction method:

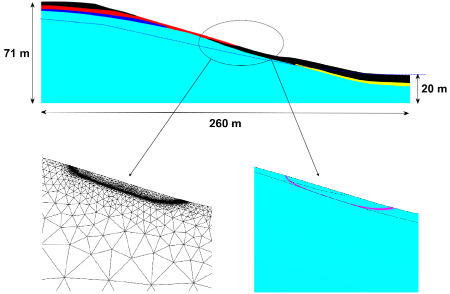
- Conventional method in slope stability, implemented in many commercial codes.
- Strength parameters are reduced by a scalar factor up to the critical state. The safety factor represents the critical value.
- Iterative limit load methods can be applied [Tschuchnigg, Schweiger, Sloan 2015]
- Relationship between these two methods [S., Hrubešová, Michalec, Tschuchnigg 2021]

Basic elasto-plastic models used in geotechnics:

- Mohr-Coulomb or Drucker-Pragers yield criteria.
- They distinguish different behavior in tension and compression.
- Limit analysis for geotechnical models leads to conic optimization.

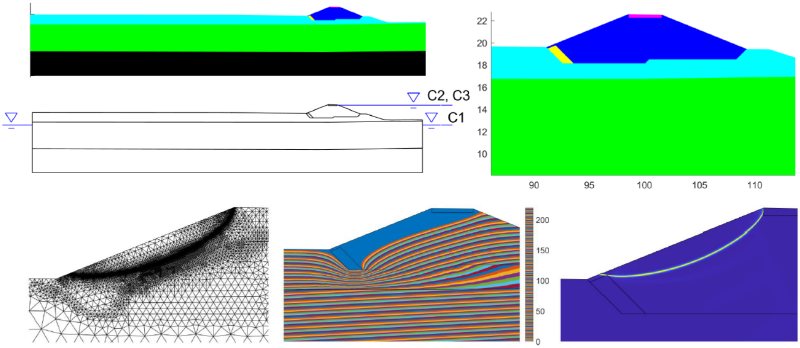
Application I – slope stability





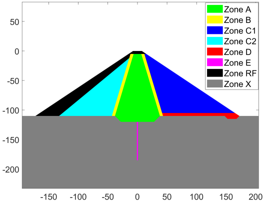
- case study of a real slope in locality Doubrava (north-east part of Czechia)
- unstable slope with observed landslides, and various soil layers
- numerical results confirmed that $FoS \approx 1$, mesh adaptivity was used
- [S., Hrubešová, Michalec, Tschuchnigg 2021]

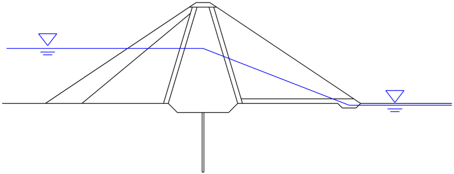
Application II – river embankment

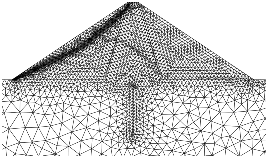


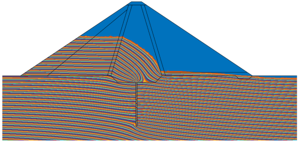
- case studies of a real river embankment in Lužec (near Prague)
- influence of pore pressure – unconfined seepage problem, phreatic surface
- mesh adaptive solution for porous flow and mechanical problems
- [S., Tschuchnigg, Hrubešová, Michalec 2023]

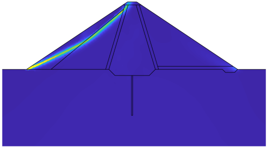
Application III – embankment dam











- benchmark problem with high embankment dam (more than 100 meters)
- sizes of the computational domain: 1200 x 500 meters
- one-sided coupling of porous flow and mechanical models
- [S., Tschuchnigg, Hrubešová, Michalec 2023]

Conclusion

Summary:

- Selected Newton-like methods used in computational plasticity.
- Semismooth variants of the methods.
- Convergence analysis and numerical examples.
- Limit load and stability assessment.

Other related topics:

- Verification of the semismoothness for various E-P operators:
  [S. 2014], [S., Cermak, Kruis et al. 2016], [S., Cermak, Ligursky 2017]
- Development of in house codes in Matlab:
  [S., Cermak, Ligursky 2017], [Cermak, S., Valdman 2019], [Karátson, S., Béreš 2024]
- Duality and a posteriori error analysis for limit loads:
  [Repin, S., Haslinger 2018], [Haslinger, Repin, S. 2019], [S., Haslinger, Reddy, Repin 2021]
- Contact of E-P bodies and strain-gradient plasticity:
  [S., Haslinger, Hlaváček, Cermak 2015], [Reddy, S. 2020], [Reddy, S. 2024]

References on semismooth Newton methods

Semismooth and smoothing Newton methods (in general):

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Semismooth Newton method in elasto-plasticity:

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Thank you for your attention!