Stokes problem, related inequalities, constants and representations

Alkalmazott Analízis Szeminárium, BME, MI

Sándor Zsuppán

Berzsenyi Dániel Evangélikus (Líceum) Gimnázium és Kollégium, Sopron

Budapest,2017.03.16.

Outline

Research topics and publications

- Stokes problem, related norm inequalities and eigenvalue problems via conformal mapping:
  - On the spectrum of the Schur complement of the Stokes operator via conformal mapping, Methods and Applications of Analysis, Vol.11, No.1 (2004), 133–154.
  - On connections between the Stokes-Schur and the Friedrichs operator, with applications to the inf-sup problem, Annales Univ. Sci. Budapest., 48.(2005), 151–171.
  - On the domain dependence of the inf-sup and related constants via conformal mapping, J. Math. Anal. Appl. 382 (2011), 856–863.
- Representations of Stokes' and Naviers functions via harmonic potentials
  - On representations of Stokes flows and of the solutions of Navier's equation for linear elasticity, Analysis 28 (2008), 219–237.
- Further publications
  - On the Stokes problem, Thesis, Eötvös Loránd Univ. Budapest, 2008. (Thesis supervisor: Dr. Gisbert Stoyan)
  - On the Friedrichs-Velte and related constants of the union of overlapping domains, Dimenziók (2016), doi:10.20312/dim.2016.01

History

1898- E.& F. Cosserat: Original formulation of the Cosserat eigenvalue problem

1924 Lichtenstein: Boundary integral formulation for the Cosserat problem

1937 Friedrichs: Original formulation of the Friedrichs inequality for complex analytic and conjugate harmonic functions (2D)

1980- Putinar-Shapiro: Friedrichs operator on the domains Bergman space, quadrature domains

1995 Lin-Rochberg: compactness of Friedrichs operator via conformal mapping

1972 Babuška-Aziz: inequality $|u|_1 \leq C_\Omega \| \operatorname{div} u \|_0$, $u \in \ker \operatorname{div}^\perp \subset H_0^1(\Omega)^d$

1983 Horgan-Payne: equivalence of Babuška-Aziz and Friedrichs, estimation for 2D star-shaped domains (in 3D by Payne 2007) $\rightarrow$ revised later by Costabel-Dauge

1997 Crouzeix: Schur complement operator and a decomposition of $H_0^1(\Omega)^d$

1996- Velte: decomposition of $H_0^1(\Omega)^d$, related inequalities and constants in 2D and 3D

1999- Stoyan: continuous and discrete Crouzeix-Velte decompositions and inf-sup constants

2013- Costabel-Dauge: equivalence of Babuška-Aziz and Friedrichs inequalities, revised Horgan-Payne estimation, continuity results w.r.t. domain and spaces

1934 Papkovich-Neuber potentials for Stokes flows

1990 Kratz-Peyerimhoff: numerical method for Stokes with conformal mapping

1991- Kratz-Lindae: representations for Stokes flows in 2D and 3D

1993 Hou-Manouzi: div-curl formulation of 2D-Stokes problem bases on the Kratz representation

1994 Galdi: Babuška-Aziz constant estimate for union of star-shaped domains

2000 Kessler: Korn constant estimate of union domains (improving Dafermos' result 1968)

Stokes problem

Stokes problem:

Given $\vec{f}$, find $\vec{u}$ and p such that

$$-\Delta\vec{u} + \operatorname{grad} p = \vec{f} \qquad \text{on } \Omega \tag{1a}$$

$$\operatorname{div}\vec{u} = 0 \qquad \text{on } \Omega \tag{1b}$$

$$\vec{u} = 0 \qquad \text{on } \partial\Omega \tag{1c}$$

Variational formulation

Given $f \in H^{-1}(\Omega)$ solve for $\vec{u} \in H_0^1(\Omega)^d$ and $p \in L_{2,0}(\Omega)$:

$$\langle \nabla\vec{u}, \nabla\vec{v}\rangle - \langle p, \operatorname{div}\vec{v}\rangle = \langle f, v\rangle \qquad \forall\vec{v} \in H_0^1(\Omega)^d \tag{2}$$

$$\langle \operatorname{div}\vec{u}, q\rangle = 0 \qquad \forall q \in L_{2,0}(\Omega) \tag{3}$$

- velocity function $\vec{u} \in (C^2(\Omega) \cap C(\overline{\Omega}))^d$ – classical solution; $\vec{u} \in H_0^1(\Omega)^d$ – weak solution
- pressure function $p \in C^1(\Omega)$ – classical solution; $p \in L_2(\Omega)$ – weak solution; unique up to a constant, normalizations $p(x_0) = 0$ for a $x_0 \in \Omega$ or $\int_\Omega p = 0$

Stable solvability of the Stokes problem

inf-sup condition, LBB-condition

The domain $\Omega \subseteq \mathbb{R}^d$ satisfies the inf-sup condition with the inf-sup constant β_Ω if

$$\inf_{0\neq p\in L_{2,0}(\Omega)} \sup_{0\neq u\in H_0^1(\Omega)^d} \frac{\langle \operatorname{div} u, p\rangle}{|u|_1 \|p\|_0} = \beta_\Omega > 0. \tag{4}$$

- Here: $\langle \cdot, \cdot \rangle$ is the inner product of $L_2(\Omega)$, $\|\cdot\|_0^2 = \langle \cdot, \cdot \rangle$,
- $|u|_1 = \|\nabla u\|_0$ is the seminorm in $H_0^1(\Omega)^d$
- given $p \in L_{2,0}(\Omega)$ define $u(p) = -\Delta^{-1}\nabla p \in H_0^1(\Omega)^d$ the solution of

$$\langle \nabla u, \nabla v\rangle = \langle \operatorname{div} v, p\rangle, \quad \forall v \in H_0^1(\Omega)^d$$

$$\frac{\langle \operatorname{div} v, p\rangle}{|v|_1} = \frac{\langle \nabla u(p), \nabla v\rangle}{|v|_1} \leq \frac{|u(p)|_1 \cdot |v|_1}{|v|_1} = |u(p)|_1 = \frac{\langle \nabla u(p), \nabla u(p)\rangle}{|u(p)|_1} = \frac{\langle \operatorname{div} u(p), p\rangle}{|u(p)|_1}$$

hence the sup (4) is a max: $\sup\limits_{0\neq v\in H_0^1(\Omega)^d} \frac{\langle \operatorname{div} v, p\rangle}{|v|_1} = \frac{\langle \operatorname{div} u(p), p\rangle}{|u(p)|_1}$

- this is in fact $\sup\limits_{0\neq v\in H_0^1(\Omega)^d} \frac{\langle v, \nabla p\rangle}{|v|_1} = \|\nabla p\|_{-1} = |u(p)|_1 = \sup\limits_{0\neq v\in H_0^1(\Omega)^d} \frac{\langle \operatorname{div} v, p\rangle}{|v|_1}$

Stability for the Stokes problem

Pressure stability for Stokes problem:

Let Ω satisfy $\beta_\Omega > 0$. For $\vec{f} \in L_2(\Omega)^d$ there is a weak solution $(\vec{u}, p)$ of the Stokes problem (1), and

$$\|\nabla \vec{u}\|_0 \leq C_P \|\vec{f}\|_0, \tag{5}$$

$$\|p\|_0 \leq \frac{2C_P}{\beta_\Omega} \|\vec{f}\|_0, \tag{6}$$

where $C_P < \infty$ is the constant in the Poincaré inequality

$$\|g\|_0 \leq C_P \|\nabla g\|_0 \text{ for every } g \in H_0^1(\Omega)^d.$$

- also velocity stability $|\vec{u}|_1 \leq |f|_{-1}$ and
- pressure stability $\|p\|_0 \leq \frac{1}{\beta_\Omega} |f|_{-1}$

Equivalent formulations of the inf-sup condition

inf-sup condition, LBB-condition

For a domain $\Omega \subseteq \mathbb{R}^d$ the inf-sup constant β_Ω is

$$\inf_{0\neq p\in L_{2,0}(\Omega)} \sup_{0\neq \vec{u}\in H_0^1(\Omega)^d} \frac{\langle \mathrm{div}\, \vec{u}, p\rangle}{|\vec{u}|_1 \|p\|_0} = \beta_\Omega .$$

1. $\forall p \in L_{2,0}(\Omega) : \sup\limits_{0\neq u\in H_0^1(\Omega)^d} \frac{\langle \mathrm{div}\, u, p\rangle}{|u|_1} \geq \beta_\Omega \|p\|_0$
2. $\forall p \in L_{2,0}(\Omega) : \sup\limits_{0\neq u\in H_0^1(\Omega)^d} \frac{\langle u, \nabla p\rangle}{|u|_1} \geq \beta_\Omega \|p\|_0$
3. $\forall p \in L_{2,0}(\Omega) : \|\nabla p\|_{-1} \geq \beta_\Omega \|p\|_0$ with the norm $\|\nabla p\|_{-1} = \sup\limits_{0\neq u\in H_0^1(\Omega)^d} \frac{\langle u, \nabla p\rangle}{|u|_1}$ of the dual space $\mathbf{H}^{-1}(\Omega)$ of $H_0^1(\Omega)^d$
4. $\nabla : L_{2,0}(\Omega) \to \mathbf{H}^{-1}(\Omega)$ is injective, has closed range and left inverse of norm at most β_Ω^{-1}
5. $\mathrm{div} : H_0^1(\Omega)^d \to L_{2,0}(\Omega)$ is surjective, has right inverse of norm $\leq \beta_\Omega^{-1}$
6. $\forall p \in L_{2,0}(\Omega) \setminus \{0\}$, $\exists u \in H_0^1(\Omega)^d \setminus \{\mathbf{0}\}$: $\langle \mathrm{div}\, u, p\rangle \geq \beta_\Omega \|p\|_0 |u|_1$

Babuška-Aziz inequality

Babuška-Aziz inequality:

The domain $\Omega \subseteq \mathbb{R}^d$ is defined to satisfy the Babuška-Aziz inequality if there is some $0 < C_\Omega < \infty$ such that for every $p \in L_{2,0}(\Omega)$ there exists $u \in H_0^1(\Omega)^d$ such that

$$\operatorname{div} u = p \text{ and } |u|_1^2 \leq C_\Omega \|p\|_0^2. \tag{7}$$

- Nečas (1967): $\|\nabla p\|_{-1}^2 \leq C_\Omega \|p\|_0^2$ for Lipschitz domains
- The least positive C_Ω in (7) is the Babuška-Aziz constant of the domain.
- $\operatorname{div} : H_0^1(\Omega)^d \to L_{2,0}(\Omega)$ is surjective, has right inverse of norm $\leq C_\Omega$
- Connection to the inf-sup constant: $C_\Omega = \beta_\Omega^{-2}$.

Theorem (Babuška-Aziz, Payne-Weinberger)

For every bounded Lipschitz domain in $\mathbb{R}^d$ there holds $C_\Omega < \infty$.

Theorem (Acosta-Durán-Muschietti, 2006)

For every bounded John domain in $\mathbb{R}^d$ there holds $C_\Omega < \infty$.

Friedrichs inequality

Friedrichs' inequality (Friedrichs, 1937; Shapiro, 1980)

Let Ω be a bounded plane domain satisfying an interior cone condition and $w_0 \in \Omega$. For some finite constants $\Gamma_\Omega \geq 1$ and $\Gamma_{\Omega,w_0} \geq 1$ there hold

$$\|u\|_0^2 \leq \Gamma_\Omega \|v\|_0^2 \qquad (8a)$$

$$\|u\|_0^2 \leq \Gamma_{\Omega,w_0} \|v\|_0^2 \qquad (8b)$$

for the real and imaginary parts u and v of a square integrable analytic function subject to one of the normalizations $\int_\Omega u \, \mathrm{d}A = 0$ or $u(w_0) = 0$.

- u and v are conjugate harmonic: $\nabla u = \nabla^\perp v$ (Cauchy-Riemann equation)
- another form $\left|\int_\Omega f^2 dA\right| \leq \frac{\Gamma_\Omega - 1}{\Gamma_\Omega + 1} \int_\Omega |f|^2 dA$ ($f = u + i \cdot v$)
- u and v are interchangeable: if v is normalized then $\|v\|_0^2 \leq \Gamma_\Omega \|u\|_0^2$
- comparable constants: $\Gamma_\Omega \leq \Gamma_{\Omega,w_0} \leq \frac{|\Omega|}{\pi \cdot d(w_0, \partial\Omega)^2} \Gamma_\Omega$ ($\rightarrow$ generalizable for subdomains)
- exact values are known in a very few cases (disc, ellipse, quadrature domains)
- useful upper estimation for star-shaped domains by Horgan-Payne (1983), revised by Costabel-Dauge (2013)

Velte inequality

Velte inequality (Velte, 1998)

Let Ω be a bounded simply-connected spatial domain with C^2 boundary. Then there are optimal constants $\Gamma_\Omega \geq 1$ and $\tilde{\Gamma}_\Omega \geq 1$ depending only on the shape of Ω such that for any pair u and v conjugate in the sense of the Moisil-Teodorescu equations the inequalities

$$\|u\|_0^2 \leq \Gamma_\Omega \|v\|_0^2, \text{ provided } \int_\Omega u \, dV = 0; \text{ and} \quad (9)$$

$$\|v\|_0^2 \leq \tilde{\Gamma}_\Omega \|u\|_0^2, \text{ provided } v \cdot n = 0 \text{ on } \partial\Omega \quad (10)$$

hold, where n denotes the outer unit normal to $\partial\Omega$.

- Moisil-Teodorescu equations: $\mathrm{rot}\, v = -\nabla u$ and $\mathrm{div}\, v = 0$ instead of Cauchy-Riemann (but setting $u = u(x_1, x_2, 0)$ and $v = (0, 0, v_3(x_1, x_2, 0))$ specialzes M-T to C-R)
- u and v are not interchangeable: u is scalar and v is vector (another normalizations)
- exact values of the constants are known in a very few cases: sphere, ellipsoids
- upper estimation for Γ_Ω of star-shaped domains by Payne (2007)

Connection between Friedrichs and Babuška-Aziz

Theorem (Costabel-Dauge, 2013)

Let $\Omega \subset \mathbb{R}^2$ be any bounded domain. Then the Babuška-Aziz constant C_Ω is finite iff the Friedrichs constant Γ_Ω is finite, and

$$C_\Omega = \Gamma_\Omega + 1$$

Outline of the proof:

1. prove $\Gamma_\Omega \leq C_\Omega - 1$: set $u \in L_{2,0}(\Omega)$, $\operatorname{div} w = u$ and $|w|_1^2 \leq C_\Omega \|u\|_{0,\Omega}^2$
   use $|w|_1^2 = \|\operatorname{div} w\|_0^2 + \|\operatorname{rot} w\|_0^2$ to have $\|\operatorname{rot} w\|_0^2 \leq (C_\Omega - 1)\|u\|_0^2$
   use duality and Cauchy-Riemann:
   $\|u\|_0^2 = \langle u, \operatorname{div} w\rangle = \langle -\nabla u, w\rangle = \left\langle -\nabla^\perp v, w\right\rangle = -\langle v, \operatorname{rot} w\rangle$
   use Cauchy-Schwarz: $\|u\|_0^2 \leq \|v\|_0 \cdot \|\operatorname{rot} w\|_0 \leq \sqrt{(C_\Omega - 1)}\|u\|_0 \cdot \|v\|_0$
2. prove $C_\Omega \leq \Gamma_\Omega + 1$: set $u \in L_{2,0}(\Omega)$
   define $w \in H_0^1(\Omega)^2$ by $\langle \nabla w, \nabla \tilde{w}\rangle = \langle u, \operatorname{div} \tilde{w}\rangle, \forall \tilde{w} \in H_0^1(\Omega)^2$ (e.g. $\Delta w = \nabla u$)
   $u - \operatorname{div} w \in L_{2,0}(\Omega)$ and $\operatorname{rot} w \in L_{2,0}(\Omega)$ are conjugate harmonic by $\Delta = \nabla \operatorname{div} + \nabla^\perp \operatorname{rot}$

- corresponding constants for any domain $\frac{1}{\beta_\Omega^2} = C_\Omega = \Gamma_\Omega + 1$
- first $C_\Omega = \Gamma_\Omega + 1$ for smooth domains by Horgan-Payne (1983)
- connection to the Korn constant $K_\Omega = 2C_\Omega$ for smooth domains also by Horgan-Payne (1983)

Connection between Velte and Babuška-Aziz

Theorem (Velte, 1998)

Let $\Omega \subset \mathbb{R}^3$ be a simply-connected spatial domain with C^2-boundary. Then the Babuška-Aziz constants C_Ω and $\tilde{C}_\Omega$ are finite iff the Velte constants Γ_Ω and $\tilde{\Gamma}_\Omega$ are finite, and there hold

$$C_\Omega = \Gamma_\Omega + 1 \text{ and } \tilde{C}_\Omega = \tilde{\Gamma}_\Omega + 1.$$

- the Babuška-Aziz constants here are the optimal constants in the norm inequalities

$$\begin{aligned} |w|_1^2 &\leq C_\Omega \|\operatorname{div} w\|_0^2, \text{ for every } w \in \ker \operatorname{div}^\perp \subseteq H_0^1(\Omega)^3 \\ |w|_1^2 &\leq \tilde{C}_\Omega \|\operatorname{rot} w\|_0^2, \text{ for every } w \in \ker \operatorname{rot}^\perp \subseteq H_0^1(\Omega)^3 \end{aligned}$$

- the 2D Costabel-Dauge proof remains valid for any simply-connected 3D domains and for both Velte-constants (Zsuppán, 2013)
- unified framework for Babuška-Aziz and Friedrichs-Velte constants using differential forms and the exterior derivative by Costabel, 2015
- exact values or estimations for one of the constants applies for the others

Exact values and estimates of the constants

Cosserat, 1898-

$\Gamma_\Omega = d - 1$, where Ω is a d-dimensional ball

Friedrichs, 1937

$\Gamma_\Omega = \frac{a^2}{b^2} \geq 1$, where Ω is an ellipse with semiaxes a and b

Bogovskii, 1979; Galdi, 1994

$\Omega \subset \mathbb{R}^d$ star-shaped w.r.t. a ball $B \subset \Omega$ for some constant γ_d depending only on the dimension d

$$\beta_\Omega \geq \gamma_2 \frac{\mathrm{diam}(B)}{\mathrm{diam}(\Omega)} \text{ for } d = 2, \text{ and } \beta_\Omega \geq \gamma_d \left(\frac{\mathrm{diam}(B)}{\mathrm{diam}(\Omega)}\right)^{d+1}, \text{ for } d \geq 3$$

If Ω is the finite union of overlapping star-shaped domains, then $\beta_\Omega > 0$.

Chizhonkov-Olshanskii, 2000

- if $\Omega = (0;1) \times (0;\epsilon)$ $(0 < \epsilon < 1)$, then $\frac{\epsilon}{\sqrt{60}} \leq \beta_\Omega \leq \frac{\pi\epsilon}{\sqrt{12}}$
- if $\Omega = \{x \in \mathbb{R}^2 \mid 1 \leq |x| \leq 1 + \epsilon\}$, then $\beta_\Omega^2 = \frac{1}{2}\left(1 - \sqrt{\frac{1}{\log(1+\epsilon)} \cdot \frac{(1+\epsilon)^2 - 1}{(1+\epsilon)^2 + 1}}\right)$

Schur complement operator

Schur complement operator

Define the Schur complement operator $\mathcal{S}$ of the Stokes system by

$$\mathcal{S} = \operatorname{div} \Delta^{-1} \nabla : L_{2,0}(\Omega) \xrightarrow{\nabla} \mathbf{H}^{-1}(\Omega) \xrightarrow{\Delta^{-1}} \mathbf{H}_0^1(\Omega) \xrightarrow{\operatorname{div}} L_{2,0}(\Omega) \tag{11}$$

- that is: $\mathcal{S}p = \operatorname{div} w(p)$, where $w(p) \in \mathbf{H}_0^1(\Omega)$ is the weak solution of $\Delta w(p) = \nabla p$:

$$\langle \nabla w(p), v \rangle = -\langle \nabla p, v \rangle = \langle p, \operatorname{div} v \rangle, \text{ for every } v \in \mathbf{H}_0^1(\Omega).$$

- $\mathcal{S}$ is symmetric and positive: $\langle \mathcal{S}p, q \rangle = \langle p, \operatorname{div} w(q) \rangle = \langle -\nabla p, w(q) \rangle = \langle \nabla w(p), \nabla w(q) \rangle$
- $-\Delta : \mathbf{H}_0^1(\Omega) \to \mathbf{H}^{-1}(\Omega)$ is an isometry, hence for $p \in L_{2,0}(\Omega)$:

$$\frac{\langle \mathcal{S}p, p \rangle}{\langle p, p \rangle} = \frac{\left\langle \operatorname{div} \Delta^{-1} \nabla p, p \right\rangle}{\langle p, p \rangle} = \frac{\left\langle -\Delta^{-1} \nabla p, \nabla p \right\rangle}{\langle p, p \rangle} = \frac{|\nabla p|_{-1}^2}{\|p\|_0^2} \geq \beta_\Omega^2$$

- $\|\mathcal{S}\| \leq 1$ in operator norm because $|w|_1^2 = \|\operatorname{div} w\|_0^2 + \|\operatorname{rot} w\|_0^2$, hence $Spectrum(\mathcal{S}) \subseteq [0; 1]$

Schur complement operator II

Theorem (Crouzeix, 1997)

If Ω is bounded of class C^3 then $\mathcal{S} - \frac{1}{2}\mathcal{I} : b^2(\Omega) \to b^2(\Omega)$ is compact, where $L_{2,0}(\Omega) = b^2(\Omega) \oplus \Delta H_0^2(\Omega)$, with the harmonic Bergman space $b^2(\Omega)$.
If $\Omega \subset \mathbb{R}^2$ has a corner then $\mathcal{S} - \frac{1}{2}\mathcal{I} : b^2(\Omega) \to b^2(\Omega)$ is not compact.

- norm identity for $u \in \mathbf{H}_0^1(\Omega)$: $|u|_1^2 = \|\operatorname{div} u\|_0^2 + \|\operatorname{rot} u\|_0^2$ implies decomposition $\mathbf{H}_0^1(\Omega) = \ker \operatorname{div} \oplus \ker \operatorname{rot} \oplus W$ (Crouzeix-Velte decomposition)
- if $p \in \Delta H_0^2(\Omega)$, then $\mathcal{S}p = p$, and for the least positive eigenvalue of $\mathcal{S}|_{b^2(\Omega)}$ is β_Ω^2
- if $\Omega \subset \mathbb{R}^2$, then the spectrum of $\mathcal{S}|_{b^2(\Omega)}$ is the Intervall $[\beta_\Omega^2, 1 - \beta_\Omega^2]$
- $\mathcal{S}$ is involved in the error reduction of the pressure approximation (Uzawa algorithm)
- each corner with angle ω of the polygonal Ω adds an intervall $\left[\frac{1}{2}\left(1 - \frac{\sin\omega}{\omega}\right), \frac{1}{2}\left(1 + \frac{\sin\omega}{\omega}\right)\right]$ to the essential spectrum of $\mathcal{S}$, hence $\beta_\Omega \le \min \sqrt{\frac{1}{2}\left(1 - \frac{\sin\omega}{\omega}\right)}$ (Costabel-Dauge, 2015)

Friedrichs operator of a planar domain

Friedrichs operator (Shapiro, 1993)

For $\Omega \subseteq \mathbb{C}$ set $AL_2(\Omega)$ the subspace of complex analytic $L_2(\Omega)$-functions. Define

$$\mathcal{F} = \mathcal{P} \circ \mathcal{C} : AL_2(\Omega) \to AL_2(\Omega), \tag{12}$$

where $\mathcal{P}$ is the projection of $L_2(\Omega)$ onto $AL_2(\Omega)$, and $\mathcal{C}(f) = \bar{f}$.

- if $K(w, \omega)$ is the reproducing kernel of $AL_2(\Omega)$ then

$$\mathcal{F}(f)(w) = \int_\Omega K(w, \omega)\overline{f(\omega)}\,\mathrm{d}A(\omega)$$

- if Ω is bounded, the $\mathcal{F}(1) = 1$
- $(f, \mathcal{F}(g)) = \overline{(\mathcal{F}(f), g)} = \int_\Omega fg\,\mathrm{d}A \to (f, \mathcal{F}^2(f)) = (\mathcal{F}(f), \mathcal{F}(f)) \to \|\mathcal{F}^2\| \leq 1$
- $\mathcal{F}^2$ is $\mathbb{C}$-linear, self-adjoint, positive and $\|\mathcal{F}^2|_{AL_{2,0}(\Omega)}\| < 1$ if Ω fulfills the Friedrichs inequality

Friedrichs operator of a planar domain II

Theorem (Lin-Rochberg, 1995)

Let Ω a proper simply connected subdomain of $\mathbb{C}$, and $g : \mathbb{D} \to \Omega$ its associated conformal map. $\mathcal{F}$ is compact iff $\mathcal{P}_{\mathbb{D}}\left(\frac{\overline{g'}}{g'}\right)$ is in the little Bloch space $B_0(\mathbb{D})$.

- $f \in B_0(\mathbb{D})$ if $\lim_{|z|\to 1-}(1-|z|^2)f'(z) = 0$.
- pull back by conformal mapping $K(w,\omega) = \frac{R'(w)\overline{R'(\omega)}}{\pi\left(1-R(w)\overline{R(\omega)}\right)^2}$ with $R = g^{-1}$:

$$\mathcal{F}_{\mathbb{D}}(p)(z) = \int_{\mathbb{D}} \frac{1}{(1-z\zeta)^2} \frac{\overline{g'(\zeta)}}{\overline{g'(\zeta)}}\overline{p(\zeta)} \frac{\mathrm{d}A(\zeta)}{\pi}$$

  where $p = (f \circ g)g'$ and $\mathcal{F}_{\mathbb{D}}(p) = g'\mathcal{F}(f) \circ g$
- sufficient criterion: if $\arg(g') \in C(\overline{\mathbb{D}})(\leftrightarrow \partial\Omega$ is C^1 Jordan curve), then $\mathcal{F}$ is compact
- if Ω has a corner, then $\mathcal{F}$ is not compact (Friedrichs, 1937)

Schur complement and Friedrichs operator

Theorem (Zsuppán, 2005)

If $\Omega \subset \mathbb{C}$ is a bounded domain with sufficiently smooth boundary, then

$$2\mathcal{S} = \mathcal{I} - \mathcal{C} \circ \mathcal{F}$$

- theorem announced by Costabel-Dauge, 1999–2000
- Crouzeix studied $\mathcal{S} - \frac{1}{2}\mathcal{I}$ which is in fact $-\frac{1}{2}\mathcal{C} \circ \mathcal{F}$ on the Bergman space of a planar domain
- my proofs are based on explicit construction of the Dirichlet problem on Ω with special boundary data
  - for simply connected Ω, such that $H^1(\Omega)$ has traces in $L_2(\partial\Omega)$ based on the following by Putinar-Shapiro (2000):

$$\left.\begin{array}{rcl} \Delta u & = & 0 \,|_{\Omega} \\ u(\zeta) & = & \overline{\zeta} f(\zeta) \,|_{\zeta \in \partial\Omega} \end{array}\right\} \to u(z) = \overline{G(z)} + h(z) : \left\{ \begin{array}{c} G'(z) = \mathcal{F}(f)(z) \\ h(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{\zeta} f(\zeta) - \overline{G(\zeta)}}{\zeta - z} \,\mathrm{d}\zeta \end{array}\right.$$

  - for multiply connected Ω, that $\partial\Omega$ is C^2 based on the correspondence between Bergman-kernel and Green function by Garabedian (1951):

$$K(w, \omega) = \partial_w \partial_{\overline{\omega}} G(w, \omega)$$

(if this equality holds on some domain, then this is enough for a proof)

Usage of conformal mapping

Schur complement operator for special domains, Zimmer (1996)

Set $\Omega = g(\mathbb{D})$ with $g(z) = \sum_{m=0}^{M} a_m z^m$ and $g'(z)^{-1} = \sum_{\ell=0}^{M} b_\ell z^\ell$. We have

$$\mathcal{S}(p_{R,I})(z) = \frac{1}{2}\left(p_{R,I}(z) \pm \mathrm{Re}\,\frac{q(z)}{g'(z)}\right), \text{ for } p(z) = p_R(z) + ip_I(z) = \frac{1}{g'(z)}\sum_{n=0}^{M-1} p_n z^n$$

$$\mathcal{S}(p_{R,I})(z) = \frac{1}{2} p_{R,I}(z), \text{ for } p(z) = z^n, n \geq M$$

where $q(z) = \sum_{n=0}^{M-1} q_n z^n$ and $q_k = \sum_{\ell=0}^{M-1} \mathcal{M}_{k,\ell}\overline{p_\ell}$.

- The entries of the matrix $\mathcal{M}$ are composed from the mapping coefficients:

$$\mathcal{M}_{k,\ell} = \begin{cases} (k+1)\sum_{m=k+1+\ell}^{M} a_m \overline{b_{m-k-1-\ell}}, & \text{for } 0 \leq k+\ell \leq M-1 \\ 0, & \text{otherwise} \end{cases}$$

- $p \mapsto \mathcal{M}\mathcal{C}p : \ell_{(2,-1)} \to \ell_{(2,-1)}$ is the matrix formulation of $p \mapsto \mathcal{F}_{\mathbb{D}}p : AL_2(\mathbb{D}) \to AL_2(\mathbb{D})$, where $\mathcal{F}_{\mathbb{D}}p = g' \cdot (\mathcal{F}(p \circ g^{-1}) \circ g)$ is the pullback of $\mathcal{F}$ by g (Lin-Rochberg, 1995)
- Inf-sup and related constants of such domains are computable from the eigenvalue problem of a finite matrix.

Matrix representation of $\mathcal{F}$ and $\mathcal{S}$ via conformal maps

Matrix representation of $\mathcal{F}$ via conformal maps, Zsuppán (2005)

If $\Omega = g(\mathbb{D})$ with $g(z) = \sum_{m=0}^{\infty} a_m z^m$ then $\mathcal{F} : AL_2(\Omega) \to AL_2(\Omega)$ correspondes to $\mathcal{M} \circ \mathcal{C} : \ell_{(2,-1)} \to \ell_{(2,-1)}$ with the infinite matrix

$$\mathcal{M} = \left(s_{m,k}\right)_{m,k=0}^{\infty}, \text{ where } s_{m,k} = (m+1)s_{m+k} \text{ and } s_n = \sum_{j=0}^{\infty} a_{n+j+1}\overline{b_j} = \frac{1}{\pi}\int_{\mathbb{D}} \bar{z}^n \frac{g'(z)}{\overline{g'(z)}}\,\mathrm{dA}(z),$$

if one has $a \in \ell_{(2,\alpha)}$ and $b \in \ell_{(2,-\alpha)}$ for the coefficient vectors a and b of g and $\frac{1}{g'}$, respectively.

- $\ell_{(2,\pm\alpha)} = \left\{ p = (p_0, p_1, \dots) \mid \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)}\right)^{\pm 1} |p_n|^2 < \infty \right\}$
- $|s_n| \leq \frac{2}{n+2}$ for every $n \geq 0$, hence every entry of $\mathcal{M}$ is at most 2
- if $|\Omega| < \infty$ then at least one of s_n is not zero
- $s_n = 0$ for every $n \geq M$ iff g is a polynomial of order M
- $\mathcal{M}$ has finite rank iff g is fractional rational function and $\operatorname{codim}\ker\mathcal{F}$ is the order of the denominator of the Schwartz function of Ω
- $\mathcal{I} - 2\mathcal{S} = \mathcal{C} \circ \mathcal{F} = \mathcal{C} \circ \mathcal{M} \circ \mathcal{C}$
- offers the possibility to approximate g by a polynomial of a fractional rational function in order to approximate the inf-sup constant of Ω

Quadrature domains

Friedrichs operator of a quadrature domain, Putinar-Shapiro (2000)

Let be Ω a quadrature domain, i.e. there are nodes $a_i \in \Omega$ and weights $c_{i,j} \in \mathbb{C}$ such that

$$\int_\Omega f(z)\,\mathrm{d}A(z) = \sum_{i=1}^{n} \sum_{j=0}^{m_i} c_{i,j} f^{(j)}(a_i), \text{ for every } f \in AL_2(\Omega),$$

then the Friedrichs operator of Ω can be described by a finite matrix composed with help of the nodes and weights of the quadrature identity.

- The order of the quadrature domain Ω is $\mathcal{O}(\Omega) = \sum_{i=1}^{n} m_i$. There are quadrature domains for every given order.
- The disc is the only quadrature domain of order 1 w.r.t. its center.
- There are null quadrature domains (half plane, exterior of a parabola) for which $\mathcal{F} \equiv 0$.
- Ellipses are quadrature domains in a generalized sense w.r.t. the line segments connecting its foci.
- Simply connected quadrature domains are conformal maps of the unit disc by rational mapping functions (incl. polynomials)

Continuity of $\mathcal{F}$ w.r.t. the domain via conformal map

Continuity of $\mathcal{F}$ w.r.t. the domain, Zsuppán (2005)

If $\tilde{\Omega} = g(\Omega)$ with conformal map g, then we have in operator norm

$$\|\tilde{\mathcal{F}} - \mathcal{F}\| \leq 2 \sup_{\partial\Omega} |\sin(\arg g')|$$

- Also: $\|\tilde{\mathcal{S}} - \mathcal{S}\| \leq \sup_{\partial\Omega} |\sin(\arg g')|$
- the matrix representant $\mathcal{M}$ and its entries depend also continuously on the domain in terms of g':

$$|\tilde{s}_n - s_n| \leq \frac{4}{n+2} \sup_{\partial\Omega} |\sin(\arg g')|$$

A negative continuity result via conformal maps

Exact constants for epitrochoids, Zsuppán (2004)

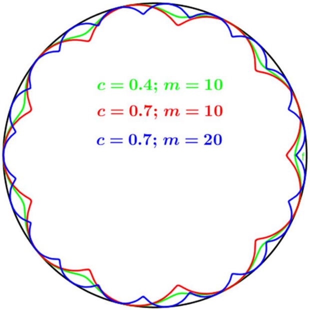
If $\Omega_{m,c} = g_{m,c}(\mathbb{D})$ by the polynomial map $g_{m,c}(z) = \frac{z - \frac{c}{m} z^m}{1 + \frac{|c|}{m}}$, $(m \geq 1,\ |c| \leq 1)$, then

$$\beta_{\Omega_{m,c}} = \begin{cases} \frac{1}{2}\sqrt{1 - \frac{m+1}{2}\frac{|c|}{m}} & \text{for } m \text{ odd} \\ \frac{1}{2}\sqrt{1 - \sqrt{\frac{m}{2}\left(\frac{m}{2}+1\right)}\frac{|c|}{m}} & \text{for } m \text{ even} \end{cases} \tag{13}$$

- the inf-sup constant is not unconditionally a continuous domain functional: $\lim_{m\to\infty} \Omega_{m,c} = \mathbb{D}$ but

$$\lim_{m\to\infty} \beta_{\Omega_{m,c}} = \frac{1}{2}\sqrt{1 - \frac{|c|}{2}} < \frac{1}{2} = \beta_{\mathbb{D}}$$

- for continuity one must have more than $|\Omega_m - \mathbb{D}| \to 0$ (boundary shape involved)
- $m = 4,\ c = 1$ the inf-sup constant $\beta^2_{\Omega_{4,1}} = \frac{1}{2} - \frac{\sqrt{6}}{8}$ is a double eigenvalue of the Schur complement operator



Lichtenstein integral equation and conformal mapping

Lichtenstein integral equation (1924)

If u satisfies the Lamé equation $\Delta u + \lambda \nabla \operatorname{div} u = 0$ in $\Omega \subset \mathbb{R}^d$ and $u = u_0$ on $\partial\Omega$ then for $\lambda \neq -1$ there follows for $\theta = \operatorname{div} u$ satisfies the boundary integral equation

$$\frac{2+\lambda}{\lambda}\theta(x) + \int_{\partial\Omega} L(x,y)\theta(y)\mathrm{d}s(y) = \frac{2}{\lambda}\operatorname{div}(Hu_0)(x), \text{ for } x \in \partial\Omega$$

with the weakly singular kernel $L(x,y) = (x-y)\cdot\nabla_x\partial_{n(y)}G(x,y) \sim (1-d)\partial_{n(y)}G(x,y)$, where G is the Green function of Ω and H denotes harmonic extension.

- Idea of Lichtenstein: $\Delta U = \nabla P$ and $\Delta P = 0$ implies $\Delta\left(U - \frac{1}{2}xP\right) = 0$

In order to use conformal mapping set

$$\Omega = g(\mathbb{D}), w = g(z), u(z) = U(g(z)), p(z) = P(g(z))$$

and transform the eigenvalue problem of $\mathcal{S}$ to the unit disc:

$$\left.\begin{array}{rcl} \Delta_w U(w) & = & \nabla_w P(w),\ w \in \Omega \\ \operatorname{div}_w U(w) & = & \lambda P(w),\ w \in \Omega \\ U(w) & = & 0,\ w \in \partial\Omega \end{array}\right\} \iff \left\{\begin{array}{rcl} \Delta_z u(z) & = & g'\nabla_z p(z),\ z \in \mathbb{D} \\ 2\operatorname{Re}\left(\frac{1}{g'(z)}\partial_z u(z)\right) & = & \lambda p(z),\ z \in \mathbb{D} \\ u(z) & = & 0,\ z \in \partial\mathbb{D} \end{array}\right.$$

For $\lambda \neq 1$ follows $\Delta_z p = 0$ and so $p(z) = \int_{\partial\mathbb{D}} \frac{1-|z|^2}{|z-\zeta|^2} p(\zeta)\frac{|\mathrm{d}\zeta|}{2\pi}$ for $z \in \mathbb{D}$.

Observe $\Delta_z u = g'\nabla_z p \iff \Delta_z\left(u - \frac{1}{2}gp\right) = 0$ and use $u(z) = 0$ on $\partial\mathbb{D}$:

$$u(z) = \frac{1}{2}\int_{\partial\mathbb{D}} \frac{1-|z|^2}{|z-\zeta|^2}(g(z) - g(\zeta))p(\zeta)\frac{|\mathrm{d}\zeta|}{2\pi} \text{ for } z \in \mathbb{D}$$

Lichtenstein integral equation and conformal mapping II

Substitute the formula for $u(z)$ into the divergence equation:

$$(1-\lambda)p(z) = \int_{\partial\mathbb{D}} \mathrm{Re}\left\{ \frac{g(z)-g(\zeta)}{g'(z)(z-\zeta)} \cdot \frac{\zeta}{\zeta - z} \right\} p(\zeta) \frac{|\mathrm{d}\zeta|}{2\pi} \text{ for } z \in \mathbb{D}$$

Suppose g'' is continuous on $\overline{\mathbb{D}}$ and take the limit $z \to \partial\mathbb{D}$:

$$\int_{\partial\mathbb{D}} L(z,\zeta) p(\zeta) \frac{|d\zeta|}{2\pi} = \left(\frac{1}{2} - \lambda\right) p(z), \text{ for } z \in \partial\mathbb{D}$$

The kernel of this boundary integral problem is continuous on $\partial\mathbb{D} \times \partial\mathbb{D}$:

$$L(z,\zeta) = \begin{cases} \mathrm{Re}\left\{ \left[\frac{g(\zeta)-g(z)}{g'(z)(\zeta - z)} - 1 \right] \cdot \frac{\zeta}{\zeta - z} \right\} & \text{for} \quad z \neq \zeta \\ \mathrm{Re}\left\{ \frac{z g''(z)}{2 g'(z)} \right\} & \text{for} \quad z = \zeta \end{cases}$$

Examples:

- if $g(z) = z$, then $L(z,\zeta) \equiv 0$
- if $g(z) = z + az^2$, then $L(z,\zeta) = \mathrm{Re}\, \frac{a\zeta}{1+2az}$
- if $g(z) = z + \sum_{k=2}^{n} a_n z^n$, then $L(z,\zeta) = \mathrm{Re}\, \frac{1}{g'(z)} \sum_{m=2}^{n} a_m \sum_{k=0}^{m-2} (k+1) z^k \zeta^{m-1-k}$ (eigenvalue problem of a finite matrix)
- Similar idea used by Kratz (1991) for explicit solution of the inhomogeneous Stokes problem
- if Ω has corners then g' is not bounded in $\overline{\mathbb{D}}$ and $L(z,\zeta)$ has a singularity on $\partial\mathbb{D} \times \partial\mathbb{D}$

Estimations of the constants via conformal mapping

Theorem (Zsuppán, 2008)

Let g such that $0 < \inf_{\partial\Omega} |g'|$ and $\sup_{\partial\Omega} |g'| < \infty$ then

$$\left(\frac{\inf_{\partial\Omega} |g'|}{\sup_{\partial\Omega} |g'|}\right)^2 \Gamma_\Omega \leq \Gamma_{g(\Omega)} \leq \left(\frac{\sup_{\partial\Omega} |g'|}{\inf_{\partial\Omega} |g'|}\right)^2 \Gamma_\Omega.$$

- $\frac{\inf_{\partial\Omega} |g'|}{\sup_{\partial\Omega} |g'|}\beta_\Omega \leq \beta_{g(\Omega)} \leq \frac{\sup_{\partial\Omega} |g'|}{\inf_{\partial\Omega} |g'|}\beta_\Omega$ using $\beta_\Omega = \frac{1}{\sqrt{1+\Gamma_\Omega}}$ (Horgan-Payne, 1983)
- if $\Omega = \mathbb{D}$, then $\frac{1}{\sqrt{2}}\frac{\inf_{\partial\mathbb{D}} |g'|}{\sup_{\partial\mathbb{D}} |g'|} \leq \beta_{g(\mathbb{D})} \leq \frac{1}{\sqrt{2}}$
- if $\partial\Omega$ is a Lyapunov curve, then $\log\left|\frac{g'(e^{i\theta_1})}{g'(e^{i\theta_2})}\right| \leq K|\theta_1 - \theta_2|^\alpha$ for some $K > 0$, $0 < \alpha < 1$ (Kellogg, 1931) and hence $\beta_\Omega \geq \frac{1}{\sqrt{2}} e^{-K(2\pi)^\alpha}$, where $K \leq \left(\int_{\partial\mathbb{D}} \left|\mathrm{Im}\,\frac{zg''(z)}{g'(z)}\right|^{\frac{1}{1-\alpha}} |\mathrm{d}z|\right)^{1-\alpha}$
- if $g(\mathbb{D})$ is star-shaped w.r.t. a disc D centered in 0, then $\frac{zg'(z)}{g(z)} = h(z) = \int_{\partial\mathbb{D}} \frac{1+xz}{1-xz}\,\mathrm{d}\mu(x)$ for some measure μ on $\partial\mathbb{D}$ with $\int_{\partial\mathbb{D}} \mathrm{d}\mu(x) = 1$, hence

$$\Gamma_{\Omega,0} \leq \frac{\int_\Omega u^2(w)\,\mathrm{d}A(w)}{\int_\Omega v^2(w)\,\mathrm{d}A(w)} \leq \frac{\sup_{\partial\mathbb{D}} |g|^2}{\inf_{\partial\mathbb{D}} |g|^2} \cdot \frac{\int_\mathbb{D} u^2(w(z))|h(z)|^2\,\mathrm{d}A(z)}{\int_\mathbb{D} v^2(w(z))|h(z)|^2\,\mathrm{d}A(z)} \overset{???}{\leq} c_\mu \left(\frac{\max_{\partial\mathbb{D}} |g(z)|}{\min_{\partial\mathbb{D}} |g(z)|}\right)^2$$

Continuity of the inf-sup constant w.r.t. the domain

Sufficient condition for continuity of the inf-sup constant w.r.t. the domain via conformal mapping (Zsuppán, 2008)

If the conformal map g of Ω fulfills $|g' - 1| \leq \varepsilon < 1$ on $\overline{\Omega}$, then

$$|\beta_{g(\Omega)} - \beta_\Omega| \leq \frac{\sqrt{2}\varepsilon}{1-\varepsilon}.$$

- if $g' \to 1$ on $\overline{\Omega}$ in the sup-norm, then $\beta_{g(\Omega)} \to \beta_\Omega$
- as example (13) for $g_{m,c}(z) = z - \frac{c}{m} z^m$ shows, $g(z) \to z$ in the sup-norm and $g'(z) \to 1$ in some L_p-norm are not sufficient for convergence

Sufficient condition for continuity of the inf-sup constant w.r.t. the domain, Bernardi et.al. (2015)

Set $\varepsilon > 0$. Let Ω and $\tilde{\Omega}$ be $\mathbb{R}^d$-Lipschitz domains and $F : \tilde{\Omega} \to \Omega$ a diffeomorphism with

$$F \in W^{1,\infty}(\tilde{\Omega})^d, \|\nabla F - \mathcal{I}\|_{L^\infty(\tilde{\Omega})} \leq \varepsilon \text{ and } F^{-1} \in W^{1,\infty}(\Omega)^d, \|\nabla F^{-1} - \mathcal{I}\|_{L^\infty(\Omega)} \leq \varepsilon$$

then there follows $|\beta_{\tilde{\Omega}} - \beta_\Omega| \leq c(d)\varepsilon$ with some constant $c(d)$ depending only on the dimension.

- if $\Omega_N = F_N(\Omega)$ with $\|\nabla(F_N - \mathrm{Id})\|_{L^\infty} \to 0$ then $\lim_{N\to\infty} \beta_{\Omega_N} = \beta_\Omega$
- continuity for polygonal approximations if corner points coincide

Discrete inf-sup constant

Definition of discrete inf-sup constant

Set the subspaces $M_n \subset L_{2,0}(\Omega)$ and $X_n \subset H_0^1(\Omega)^d$ and define the discrete inf-sup constant by

$$\beta_n = \inf_{p \in M_n} \sup_{u \in X_n} \frac{\langle \mathrm{div}\, u, p \rangle_\Omega}{|u|_{1,\Omega} \|p\|_{0,\Omega}}.$$

- $\forall n : \beta_n \geq \beta_\infty > 0$ ensures stability of Galerkin approx. methods for the Stokes equation

Upper semicontinuity w.r.t. functions spaces, Bernardi et.al. (2015)

Let $(M_n)_n$ asymptotically dense in $L_{2,0}(\Omega)$ then

$$\limsup_{n \to \infty} \beta_n \leq \beta_\Omega.$$

- Consequently: for inner approximations $(\Omega_n)_n$, $\Omega_n \subset \Omega$, $|\Omega \setminus \Omega_n| \to 0$ there follows $\limsup_{n\to\infty} \beta_n \leq \beta_\Omega$.
- Gaultier & Lezaun (1996): let $\{p_k\}_{k \geq 1}$ be a basis of (separable) $L_{2,0}(\Omega)$ and set $M_n = \mathrm{span}\,\{p_k\}_{1 \leq k \leq n}$; if

  $$X_n = \mathrm{span}\,\{u_k\}_{1 \leq k \leq n} \subset \mathbf{H}_0^1(\Omega) \text{ with } \textbf{exact} \text{ solution } u_k = \Delta^{-1} \nabla p_k,$$

  then $\lim_{n \to \infty} \beta_n = \beta_\Omega$

Approximation of the constants via conformal mapping

Define for $g(z) = \sum_{k=0}^{\infty} g_k z^k$ the n-th Cesáro mean of order $\alpha \geq 0$ by

$$\sigma_n^\alpha(g)(z) = \sum_{k=0}^{n} \frac{\binom{n+\alpha-k}{n-k}}{\binom{n+\alpha}{n}} g_k z^k$$

Approximation of convex mappings by Cesáro means, Greiner-Ruscheweyh (1994)

For a convex conformal mapping g of $\mathbb{D}$ the Cesáro mean of order $\alpha \geq 1$ are univalent and fulfill

$$g_{\frac{n}{n+\alpha+1}} \prec \sigma_n^\alpha(g) \prec g$$

for every $n \in \mathbb{N}$, where $g_\rho(z) = g(\rho z)$ for some $0 < \rho < 1$ and for all $z \in \mathbb{D}$.

The domains $\sigma_n^\alpha(g)(\mathbb{D})$ are inner approximations of $g(\mathbb{D})$.

Task:

Calculate the inf-sup constants of $\sigma_n^\alpha(g)(\mathbb{D})$ from the eigenvalue problem of a finite matrix in order to approximate the inf-sup constant of the convex domain $g(\mathbb{D})$.

The Horgan-Payne estimation

Horgan-Payne estimate for the Friedrichs constant, 1983

Let be $\Omega = \{(r,\theta) \mid 0 \le r \le f(\theta) \le 1 \text{ for } 0 \le \theta < 2\pi\}$ star-shaped w.r.t. the origin and define $P(\alpha,\theta) = \frac{1}{\alpha f^2(\theta)}\left(1 + \frac{f'^2(\theta)}{f^2(\theta) - \alpha f^4(\theta)}\right)$.

$$\Gamma_\Omega \le \Gamma_{\Omega,0} \le \inf_{0 \le \theta < 2\pi} \sup_{0 < \alpha < \frac{1}{f^2(\theta)}} P(\alpha,\theta). \tag{14}$$

The estimation applies originally for $\Gamma_{\Omega,0}$ but we also have $\Gamma_\Omega \le \Gamma_{\Omega,0}$.

Stoyan, 1999

Define Horgan-Payne angle $\gamma(x) = \arccos \frac{x \cdot n(x)}{|x| \cdot |n(x)|} \in \left[0; \frac{\pi}{2}\right)$ for $x \in \partial\Omega$ and $n(x)$ the outer normal vector at x, then

$$\beta_\Omega^2 \ge \frac{1 - \sin\gamma}{2}, \text{ where } \gamma = \max_{x \in \partial\Omega} \gamma(x)$$

- if $\Omega =$ regular polygon, then $\gamma = \frac{\pi}{n}$ and $\beta_\Omega^2 \ge \frac{1}{2}\left(1 - \sin\frac{\pi}{n}\right)$
- if $\Omega = (0;1) \times (0;\epsilon)$ is a rectangle, then $\gamma = \frac{\pi}{2} - \arctan\epsilon$

Payne's estimation for the Velte constant

Velte inequality

For conjugate harmonic scalar u and vector v:

$$\begin{aligned} \|u\|_0^2 &\leq \Gamma_\Omega \|v\|_0^2, \text{ provided } \int_\Omega u \, \mathrm{d}V = 0 \\ \|u\|_0^2 &\leq \Gamma_{\Omega,w_0} \|v\|_0^2, \text{ provided } u(w_0) = 0. \end{aligned}$$

Estimate for the Velte constant – Payne, 2007

Set $\Omega = \{(r,\theta,\varphi) \mid r \leq r_0(\theta,\varphi)\}$ star-shaped w.r.t. a ball of radius $a < \min_{(\theta,\varphi)} r_0(\theta,\varphi)$ centered in the origin. Define $Q = \max_{(\theta,\varphi)} \left[\left(\frac{\partial_\theta r_0}{r_0} \right)^2 + \left(\frac{\partial_\varphi r_0}{r_0 \sin\theta} \right)^2 \right]^{\frac{1}{2}}$. Then there follows

$$\Gamma_\Omega \leq \Gamma_{\Omega,0} \leq 2 \cdot \left(\frac{Q}{3} + \sqrt{\left(\frac{Q}{3}\right)^2 + Q + 1} \right)^2 \cdot \max_{\partial\Omega} \left(\frac{r_0}{a} \right)^3$$

For the δ-decentered unit ball $(0 \leq \delta < 1)$: $r_0(\theta,\varphi) = \delta \sin\theta\cos\varphi + \sqrt{1 - \delta^2 + \delta^2 \sin^2\theta \cos^2\varphi}$

$Q = \max_{(\theta,\varphi)} \left(\frac{\delta^2 - \delta^2 \sin^2\theta\cos^2\varphi}{1-\delta^2+\delta^2\sin^2\theta\cos^2\varphi} \right)^{\frac{1}{2}} = \frac{\delta}{\sqrt{1-\delta^2}}$, $\Gamma_{\Omega,0} \leq 2 \cdot \left(\frac{\delta}{3\sqrt{1-\delta^2}} + \sqrt{\frac{\delta^2}{9(1-\delta^2)} + \frac{\delta}{\sqrt{1-\delta^2}} + 1} \right)^2 \left(\frac{1+\delta}{1-\delta} \right)^3$

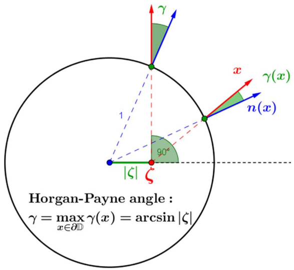
$\Gamma_{\Omega,0} \leq \mathcal{O}(1-\delta)^{-4}$ hence $\beta_\Omega \geq \mathcal{O}\left((1-\delta)^2\right)$ better as Bogovskii and Galdi: $\beta_\Omega \geq \gamma_3 (1-\delta)^4$

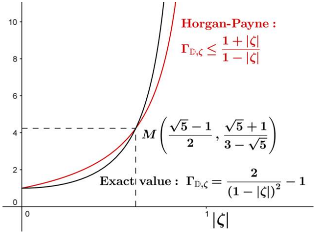
The Horgan-Payne estimation for the decentered disc

Friedrichs constant of the decentered disc, Zsuppán (2011)

Set $\zeta \in \mathbb{D}$ the center of polar coordinates for the disc, then

$$\Gamma_{\mathbb{D},\zeta} = \frac{2}{(1-|\zeta|^2)^2} - 1$$





Horgan-Payne estimation gives $\Gamma_{\mathbb{D},\zeta} \le \inf_{0\le\theta<2\pi} \sup_{0<\alpha<\frac{1}{f^2(\theta)}} P(\alpha,\theta) = \frac{1+|\zeta|}{1-|\zeta|}$

However, for $\frac{\sqrt{5}-1}{2} < |\zeta| < 1$ we have $\frac{2}{(1-|\zeta|^2)^2} - 1 = \Gamma_{\mathbb{D},\zeta} > \frac{1+|\zeta|}{1-|\zeta|}$

The estimation is invalid for the ζ-decentered disc if ζ lies near the boundary.

The Horgan-Payne estimation revisited

Revised Horgan-Payne estimation for the Friedrichs constant, Costabel-Dauge (2013)

If $\Omega = \{(r,\theta) \mid 0 \le r \le f(\theta) \le 1 \text{ for } 0 \le \theta < 2\pi\}$ is star-shaped w.r.t. the origin, then

$$\Gamma_\Omega \le (\Gamma_{\Omega,0} \le) \inf_{0<\alpha<1} \sup_{0\le\theta<2\pi} \frac{1}{\alpha f^2(\theta)} \left(1 + \frac{f'^2(\theta)}{f^2(\theta) - \alpha f^4(\theta)}\right)$$

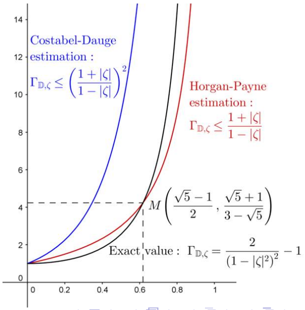
Horgan-Payne estimation: $\Gamma_\Omega \le (\Gamma_{\Omega,0} \le) \inf_{0\le\theta<2\pi} \sup_{0<\alpha<\frac{1}{f^2(\theta)}} \frac{1}{\alpha f^2(\theta)} \left(1 + \frac{f'^2(\theta)}{f^2(\theta)-\alpha f^4(\theta)}\right)$.

Counter examples are symmetric domains with a narrow pass between the parts.

The original estimation remains valid for some domains: rectangles, circumscribed polygons.

- For the decentered disc the Costabel-Dauge estimation holds for every $0 \le |\zeta| < 1$:
  $\Gamma_{\mathbb{D},\zeta} = \frac{2}{(1-|\zeta|^2)^2} - 1 \le \left(\frac{1+|\zeta|}{1-|\zeta|}\right)^2$.
- Zsuppán (2013): if $\partial\Omega$ is C^2 and $\omega \in \Omega$, then there are positive constant m and M such that

$$\frac{m}{\text{dist}(\omega,\partial\Omega)^2} \le \Gamma_{\Omega,\omega} \le \frac{M}{\text{dist}(\omega,\partial\Omega)^2}$$



Estimate for the union of overlapping domains

Friedrichs-Velte constant of the union domains, Zsuppán (2013)

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded domain of the form $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_s = \Omega_1 \cap \Omega_2$ and $|\Omega_s| > 0$. If the Friedrichs-Velte constants of Ω_1 and Ω_2 are finite, then Γ_Ω is also finite, and

$$\Gamma_\Omega \leq \frac{|\Omega_1|}{|\Omega_s|}\Gamma_{\Omega_1} + \frac{|\Omega_2|}{|\Omega_s|}\Gamma_{\Omega_2}.$$

Steps of the proof:

- define the Friedrichs-Velte constant $\Gamma_{\Omega,A}$ of Ω w.r.t. a subdomain $A \subseteq \Omega$:

$$\|u\|_\Omega^2 \leq \Gamma_{\Omega,A}\|v\|_\Omega^2 \text{ provided } \langle u \rangle_A = 0$$

  (instead of $\langle u \rangle_\Omega = \frac{1}{|\Omega|}\int_\Omega u = 0$).

- the constants are comparable: $\Gamma_\Omega \leq \Gamma_{\Omega,A} \leq \frac{|\Omega|}{|A|}\Gamma_\Omega$ because $\|u - \langle u \rangle_\Omega\|_0^2 \leq \|u\|_0^2 \leq \frac{|\Omega|}{|A|}\|u - \langle u \rangle_\Omega\|_0^2$

- $\int_{\Omega_1 \cup \Omega_2} u^2 = \int_{\Omega_1} u^2 + \int_{\Omega_2} u^2 - \int_{\Omega_s} u^2$ implies $\Gamma_{\Omega_1 \cup \Omega_2, \Omega_s} \leq \Gamma_{\Omega_1,\Omega_s} + \Gamma_{\Omega_2,\Omega_s}$

Comparison to other estimates for union domains

Babuška-Aziz constant of the union of overlapping domains, Zsuppán (2013)

If $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ is bounded of the form $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_s = \Omega_1 \cap \Omega_2$ and $|\Omega_s| > 0$, then

$$C_\Omega \leq \max_{j=1,2} \frac{|\Omega_j|}{|\Omega_s|} C_{\Omega_j}.$$

- proof based on a decomposition of $f \in L_{2,0}(\Omega)$ as $f = f_1 + f_2$ with $f_j \in L_{2,0}(\Omega_j)$, $j = 1, 2$ by Galdi (1994), where it is utilized for unions of finitely many star-shaped domains
- $\Gamma_\Omega = C_\Omega + 1$ implies $\Gamma_\Omega \leq \max_{j=1,2} \left(\frac{|\Omega_j|}{|\Omega_s|} \Gamma_{\Omega_j} + \frac{|\Omega_j \setminus \Omega_s|}{|\Omega_s|} \right)$ which is better than $\Gamma_\Omega \leq \frac{|\Omega_1|}{|\Omega_s|} \Gamma_{\Omega_1} + \frac{|\Omega_2|}{|\Omega_s|} \Gamma_{\Omega_2}$ unless $|\Omega_1 \setminus \Omega_s| > |\Omega_2|$ and $\Gamma_{\Omega_2} < \frac{|\Omega_1 \setminus \Omega_s|}{|\Omega_2|}$.

Korn constant of the union of overlapping domains, Kessler (2000)

If $\Omega \subset \mathbb{R}^2$ is bounded of the form $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_s = \Omega_1 \cap \Omega_2$ and $|\Omega_s| > 0$, then

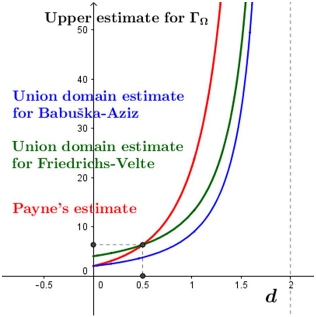
$$K_\Omega \leq \min_{j=1,2} \left\{ K_{\Omega_j} + \frac{|\Omega_j|}{|\Omega_s|} \left(\sqrt{K_{\Omega_1}} + \sqrt{K_{\Omega_2}} \right)^2 \right\}.$$

- $\Gamma_\Omega \leq \frac{|\Omega_1|}{|\Omega_s|} \Gamma_{\Omega_1} + \frac{|\Omega_2|}{|\Omega_s|} \Gamma_{\Omega_2}$ and $K_\Omega = 2(1 + \Gamma_\Omega)$ implies $K_\Omega \leq \frac{|\Omega_1|}{|\Omega_s|} \Gamma_{\Omega_1} + \frac{|\Omega_2|}{|\Omega_s|} \Gamma_{\Omega_2} - \frac{|\Omega|}{|\Omega_s|}$, which is better for example if $|\Omega| > 8|\Omega_1|$ and Ω_1 is similar to Ω_2

A 3D example for comparison of the estimates

Let Ω be the union of two unit balls, distance of their centers is d, $0 \leq d < 2$.

- $\Gamma_\Omega \leq \frac{|\Omega_1|}{|\Omega_s|}\Gamma_{\Omega_1} + \frac{|\Omega_2|}{|\Omega_s|}\Gamma_{\Omega_2}$ gives:
$$\Gamma_\Omega \leq \frac{64}{(2-d)^2(4+d)}$$
- $C_\Omega \leq \max_{j=1,2} \frac{|\Omega_j|}{|\Omega_s|} C_{\Omega_j}$ and $\Gamma_\Omega = 1 + C_\Omega$ gives:
$$\Gamma_\Omega \leq \frac{48}{(2-d)^2(4+d)} - 1$$
- Payne's estimate w.r.t. the center of the line segment connecting the ball centers gives:



$$\Gamma_\Omega \leq \frac{2}{9}\left(\frac{2+d}{2-d}\right)^{\frac{3}{2}}\left(\frac{d}{\sqrt{4-d^2}} + \sqrt{9 + \frac{9d}{\sqrt{4-d^2}} + \frac{d^2}{4-d^2}}\right)^2$$

Union type estimates are better in this case than the direct estimate.

Horgan-Payne type estimate with arbitrary weight function

Set Ω star-shaped w.r.t. the origin; $\partial\Omega : r = f(\varphi)$, $0 < f(\varphi) \leq 1$ for $0 \leq \varphi < 2\pi$.
Follow the proof of Horgan-Payne estimate: let H, G be conjugate harmonic: $\partial_\rho H = \frac{1}{\rho}\partial_\theta G$.

$$H(r,\theta) - H(\mathbf{0}) = \int_0^r \partial_\rho H(\rho,\theta)\,\mathrm{d}\rho = \int_0^r \frac{1}{\rho}\partial_\theta G(\rho,\theta)\,\mathrm{d}\rho$$

Let be $0 < \psi_0 < \psi(r,\theta)$ a weight function in Ω and integrate over Ω:

$$\begin{aligned}\int_0^{2\pi}\int_0^{f(\theta)} \left(H(r,\theta) - H(\mathbf{0})\right)\psi(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi}\int_0^{f(\theta)} \psi(r,\theta)\left(\int_0^r \frac{1}{\rho}\partial_\theta G(\rho,\theta)\,\mathrm{d}\rho\right) r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_0^{f(\theta)} \partial_\theta G(\rho,\theta)\left(\frac{1}{\rho^2}\int_\rho^{f(\theta)} r\psi(r,\theta)\,\mathrm{d}r\right)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta\end{aligned}$$

Set $\Psi(\rho,\theta) = \frac{1}{\rho^2}\int_\rho^{f(\theta)} r\psi(r,\theta)\,\mathrm{d}r$, consider $\Psi\mid_{\partial\Omega} = \lim_{\rho\to f(\theta)} \Psi(\rho,\theta) = 0$ and integrate by parts:

$$\int_0^{2\pi}\int_0^{f(\theta)} \left(H(r,\theta) - H(\mathbf{0})\right)\psi(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta = -\int_0^{2\pi}\int_0^{f(\theta)} \partial_\theta \Psi(\rho,\theta) G(\rho,\theta)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

Horgan-Payne type estimate with arbitrary weight function

Set $H = h^2 - g^2$ and $G = 2hg$ with h and g conjugate harmonic and $h(\mathbf{0}) = 0$:

$$\int_\Omega (h^2 - g^2)\psi \,\mathrm{d}A \le -\int_\Omega 2hg\partial_\theta \Psi \,\mathrm{d}A,$$

where $\partial_\theta \Psi(\rho,\theta) = \frac{1}{\rho^2} f'(\theta) f(\theta)\psi(f(\theta),\theta) + \frac{1}{\rho^2} \int\limits_\rho^{f(\theta)} r\partial_\theta \psi(r,\theta)\,\mathrm{d}r.$

Using the estimate $2hg\partial_\theta\Psi \le (\psi-\alpha)h^2 + \frac{(\partial_\theta \Psi)^2}{\psi-\alpha} g^2$ for some $\alpha < \psi_0$, we have

$$\int_\Omega (h^2 - g^2)\psi \,\mathrm{d}A \le \int_\Omega (\psi-\alpha)h^2 + \frac{(\partial_\theta\Psi)^2}{\psi-\alpha} g^2 \,\mathrm{d}A$$

There follows with optimizing on α:

$$\int_\Omega h^2 \,\mathrm{d}A \le \inf_{\alpha<\psi_0} \sup_{0\le \frac{\rho}{f(\theta)}, \frac{\theta}{2\pi} < 1} \left(\frac{\psi(\rho,\theta)}{\alpha} + \frac{(\partial_\theta \Psi(\rho,\theta))^2}{\alpha(\psi(\rho,\theta)-\alpha)} \right) \int_\Omega g^2 \,\mathrm{d}A$$

1. in the original Horgan-Payne estimate $\psi(\rho,\theta) = \frac{1}{f(\theta)^2}$ and $\Psi(\rho,\theta) = \frac{f'(\theta)}{f(\theta)^3}$ (no dependece on ρ, hence no problem with $\sup\limits_{0\le\rho\le f(\theta)}$)
2. Is there a better choice than $\psi(\rho,\theta) = \frac{1}{f(\theta)^2}$?
3. Bacause of normalization $h(\mathbf{0}) = 0$ still estimates only $\Gamma_{\Omega,0}$ instead of directly estimating Γ_Ω

How to estimate directly Γ_Ω?

Use known inequalities utilizing the normalization $\langle u\rangle_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}A = 0$:

Poincaré-Friedrichs inequalities:

Set $\Omega \subset \mathbb{R}^n$ a domain and let $u \in H^1(\Omega)$ satisfy $\frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}A = 0$ then

- $\|u\|_0 \le P_\Omega \|\nabla u\|_0$
- $\|u\|_0 \le P_{\Omega,w}\|w\cdot\nabla u\|_0$ for some fixed vector field w on Ω
- $\|u\|_0 \le P_{\Omega,\delta}\|d_\Omega^\delta \nabla u\|_0$, where $0 \le \delta \le 1$ and d_Ω is the boundary distance function of Ω

Set $\Omega \subset \mathbb{R}^2$ and let u and v be conjugate harmonic satisfying $\langle u\rangle_\Omega = 0$.
Use also $\Delta\left(\frac{1}{2}u^2\right) = u\Delta u + |\nabla u|^2 = |\nabla u|^2$ and the Green theorem:

$$\begin{aligned}\|w\cdot\nabla u\|_0^2 &= \|w\cdot\nabla^\perp v\|_0^2 = \|-w^\perp\cdot\nabla v\|_0^2 \le \int_\Omega |w|^2|\nabla v|^2 = \int_\Omega |w|^2\Delta\left(\frac{1}{2}v^2\right)\\ &= \frac{1}{2}\int_{\partial\Omega}\left[|w|^2\frac{\partial v^2}{\partial n} - v^2\frac{\partial |w|^2}{\partial n}\right] + \frac{1}{2}\int_\Omega v^2\Delta\left(|w|^2\right)\end{aligned}$$

If one can choose w such that $\Delta\left(|w|^2\right) = 1$ in Ω, $|w|^2 = 0$ and $\frac{\partial |w|^2}{\partial n} > 0$ on $\partial\Omega$ then

$$\|u\|_0^2 \le P_{\Omega,w}^2\|w\cdot\nabla u\|_0^2 \le \frac{1}{2}P_{\Omega,w}^2\|v\|_0^2 \Longrightarrow \Gamma_\Omega \le \frac{1}{2}P_{\Omega,w}^2$$

and estimates for $P_{\Omega,w}$ imply estimates for Γ_Ω.

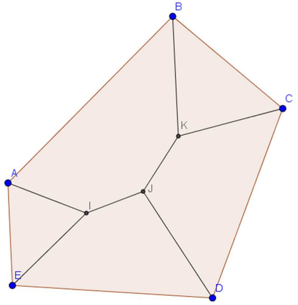
How to estimate directly Γ_Ω?

Righthand side of the improved Poincaré-Friedrichs inequality with exponent $\delta = 1$:

$$\begin{aligned} \|d_\Omega \nabla u\|_0^2 &= \|d_\Omega \nabla^\perp v\|_0^2 = \|d_\Omega \nabla v\|_0^2 = \int_\Omega d_\Omega^2 \Delta \left(\frac{1}{2} v^2\right) = \\ &= \int_{\partial\Omega} d_\Omega v \left(d_\Omega \frac{\partial v}{\partial n} - v \frac{\partial d_\Omega}{\partial n} \right) + \frac{1}{2} \int_\Omega v^2 \Delta \left(d_\Omega^2\right) \end{aligned}$$

Problem: $\Delta\left(d_\Omega^2\right) = 2|\nabla d_\Omega|^2 + 2\Delta d_\Omega = 2 + 2\Delta d_\Omega$ is not bounded in whole Ω, for example for the unit disc $d_\mathbb{D} = 1 - r$ and $\Delta d_\mathbb{D} = 2 - \frac{1}{r}$.

- Simplify the question for bounded convex polygons: $\Delta d_\Omega = 0$ a.e. in Ω.
- $\Omega = \bigcup_j \Omega_j$: partition along the mother body of the polygon [Gustaffson 1998]
- Use the above on every part Ω_j, sum up and consider that some boundary terms vanish:
  $\|d_\Omega \nabla u\|_0^2 = \int_\Omega v^2 - \int_{MB(\Omega)} d_\Omega v^2 \frac{\partial d_\Omega}{\partial n}$
- $\frac{\partial d_\Omega}{\partial n} > 0$ on the line segments of the MB
- Use the improved Poincaré-Friedrichs inequality:
  $\|u\|_0 \le P_{\Omega,1} \|v\|_0 \Rightarrow \Gamma_\Omega \le P_{\Omega,1}^2$



Inf-sup for differential forms

Denote Λ^ℓ, $0 \le \ell \le n$ the exterior algebra of $\mathbb{R}^n$ and denote $L_2(\Omega, \Lambda^\ell)$ the set of differential forms with square integrable coefficients:

- $d : \mathscr{C}_0^\infty(\Omega, \Lambda^\ell) \to \mathscr{C}_0^\infty(\Omega, \Lambda^{\ell+1})$ the exterior derivative on smooth forms with compact support in Ω
- $\delta : \mathscr{C}_0^\infty(\Omega, \Lambda^\ell) \to \mathscr{C}_0^\infty(\Omega, \Lambda^{\ell-1})$ the coderivative: formal adjoint of d w.r.t. the scalar product of $L_2(\Omega, \Lambda^\ell)$
- $\Delta = d\delta + \delta d$ is the Hodge-Laplacian and set $|u|_1^2 = \langle \Delta u, u \rangle$ (seminorm)
- $|u|_1^2 = \|du\|^2 + \|\delta u\|^2$ ($\to$ Crouzeix-Velte decomposition for differential forms?)
- Consider the following extensions of d and δ:

$$\begin{array}{ccccc} & & L_2(\Omega, \Lambda^{\ell+1}) & & \\ & \overset{\underline{d}}{\nearrow} & & \overset{\underline{\delta}}{\searrow} & \\ H_0^1(\Omega, \Lambda^\ell) & & & & H^{-1}(\Omega, \Lambda^\ell) \\ & \overset{\overline{\delta}}{\searrow} & L_2(\Omega, \Lambda^{\ell-1}) & \overset{\overline{d}}{\nearrow} & \end{array}$$

- Set $M = (\ker \overline{d})^\perp = \left\{ u \in L_2(\Omega, \Lambda^{\ell-1}) \mid \forall v \in L_2(\Omega, \Lambda^{\ell-1}) : \overline{d} v = 0 \Rightarrow \langle u, v \rangle = 0 \right\}$
- Set $M^* = (\ker \underline{\delta})^\perp = \left\{ u \in L_2(\Omega, \Lambda^{\ell+1}) \mid \forall v \in L_2(\Omega, \Lambda^{\ell+1}) : \underline{\delta} v = 0 \Rightarrow \langle u, v \rangle = 0 \right\}$

Inf-sup condition for differential forms, Costabel (2015)

Define the domain $\Omega \subset \mathbb{R}^n$ satisfying the inf-sup condition if $\inf\limits_{p \in M} \sup\limits_{u \in H_0^1(\Omega, \Lambda^\ell)^n} \dfrac{\langle p, \overline{\delta} u \rangle}{|u|_1 \|p\|} = \beta_{\Omega,\ell} > 0$

Babuška-Aziz and Friedrichs-Velte for differential forms

$$\begin{array}{ccccc} & & L_2(\Omega,\Lambda^{\ell+1}) & & \\ & \overset{\underline{d}}{\nearrow} & & \overset{\underline{\delta}}{\searrow} & \\ H_0^1(\Omega,\Lambda^\ell) & & & & H^{-1}(\Omega,\Lambda^\ell) \\ & \overset{\overline{\delta}}{\searrow} & L_2(\Omega,\Lambda^{\ell-1}) & \overset{\overline{d}}{\nearrow} & \end{array}$$

Babuška-Aziz inequality for differential forms, Costabel (2015)

For any $q \in M$ there exists $v \in H_0^1(\Omega,\Lambda^\ell)$ such that $\overline{\delta} v = q$ and $|v|_1^2 \leq C_{\Omega,\ell}\|q\|^2$.

- $\overline{\delta}$ maps $H_0^1(\Omega,\Lambda^\ell)$ on a dense subspace of M
- $C_{\Omega,\ell} < \infty$ iff $\overline{\delta} H_0^1(\Omega,\Lambda^\ell) = M$

Friedrichs-Velte inequality for differential forms, Costabel (2015)

For any $h \in M$, $g \in L_2(\Omega,\Lambda^{\ell+1})$ satisfying $\overline{d}h = \underline{\delta}g$ there follows $\|h\| \leq \Gamma_{\Omega,\ell}\|g\|$

- such a g with minimal norm satisfies $g \in M^* = (\ker \underline{\delta})^\perp$
- de Rham complex: $d \circ d = 0$ and $\delta \circ \delta = 0$, hence $h \in M \Rightarrow \delta h = 0$ and $g \in M^* \Rightarrow dg = 0$
- for solutions of $\delta h = 0$, $dh = \delta g$ and $dg = 0$ follow $\Delta h = d\delta h + \delta dh = 0 + \delta\delta g = 0$ and $\Delta g = d\delta g + \delta dg = ddh + 0 = 0$, hence h and g are both harmonic

Equivalence of Babuška-Aziz and Friedrichs-Velte for differential forms

Costabel (2015)

$\Gamma_{\Omega,\ell} < \infty$ iff $C_{\Omega,\ell} < \infty$ and $C_{\Omega,\ell} = \Gamma_{\Omega,\ell} + 1$. The constants are finite if Ω is Lipschitz.

- Special case $n \geq 2$ and $\ell = 1$: inf-sup for the div

$L_2(\Omega, \Lambda^2)$ (2-)vectors

$\underline{d} = \mathrm{rot}$ $\nearrow$ $\qquad$ $\underline{\delta} = \mathrm{rot}^*$ $\searrow$

$H_0^1(\Omega, \Lambda^1)$ $\qquad$ (1-)vectors $\qquad$ $H^{-1}(\Omega, \Lambda^1)$

$\overline{\delta} = -\,\mathrm{div}$ $\searrow$ $\qquad$ $\overline{d} = \mathrm{grad}$ $\nearrow$

$L_2(\Omega, \Lambda^0)$ scalars

- Special case $n = 3$ and $\ell = 2$: inf-sup for the rot

$L_2(\Omega, \Lambda^3)$ scalar densities

$\underline{d} = \mathrm{div}$ $\nearrow$ $\qquad$ $\underline{\delta} = -\,\mathrm{grad}$ $\searrow$

$H_0^1(\Omega, \Lambda^2)$ $\qquad$ (2-)vectors $\qquad$ $H^{-1}(\Omega, \Lambda^2)$

$\overline{\delta} = \mathrm{rot}$ $\searrow$ $\qquad$ $\overline{d} = \mathrm{rot}$ $\nearrow$

$L_2(\Omega, \Lambda^1)$ (1-)vectors

- possible utilization for clique complexes of graphs, (discrete level)?

Representation of Stokes flows via harmonic potentials

Stokes function

Define $u \in C^2(\Omega)^d$ $(d = 2, 3)$ be a Stokes function with the pressure $p \in C^1(\Omega)$ if

$$\Delta u = \nabla p \text{ and } \operatorname{div} u = 0.$$

Papkovich-Neuber representation (1934)

Any Stokes function u and pressure p can be written in the form

$$u(x) = -\nabla H_0(x) - \nabla\left(x \cdot H(x)\right) + 2H(x) \text{ and } p(x) = -2 \operatorname{div} H(x)$$

with the (non-unique) harmonic scalar and vector potentials, H_0 and H, respectively.

Kratz representation 2D (1991)

$u : \Omega \to \mathbb{R}^2$ is a Stokes function with pressure $p : \Omega \to \mathbb{R}$ iff there exists a harmonic function $\vec{h}$ in Ω such that

$$u(x) = -\frac{1}{2}\left(x \operatorname{div} h(x) + x^{\perp} \operatorname{rot} h(x)\right) + h(x) \text{ and } p(x) = -2 \operatorname{div} h(x).$$

This harmonic function h is unique, and we have $h(x) = u(x) - \frac{1}{4}\left(p(x)x - x^{\perp} \operatorname{rot} u(x)\right)$.

Numerical method based on this representation by Hou & Manouzi (1993).

Representation of Stokes flows via harmonic potentials, II

Kratz representation for star-shaped domains 3D (1991)

Assume $\Omega \subset \mathbb{R}^3$ is star-shaped w.r.t the origin. $u : \Omega \to \mathbb{R}^3$ is a Stokes function with pressure $p : \Omega \to \mathbb{R}$ iff there exists a harmonic function h in Ω such that

$$\begin{aligned} u(x) &= -\frac{1}{2}\left(x \operatorname{div} h(x) + x \times \operatorname{rot} h(x)\right) + \frac{3}{2} h(x) \\ p(x) &= -2 \operatorname{div} h(x) \end{aligned}$$

This harmonic function h is unique, and we have

$$\begin{aligned} h(x) &= \frac{2}{3} u(x) - \frac{1}{6}\left(p(x) x - x \times \operatorname{rot} u(x)\right) + \frac{2}{3} x \times \nabla \phi(x) \\ \phi(x) &= -\frac{1}{4} \int_0^1 t^4 x \cdot \operatorname{rot} u(tx) dt. \end{aligned}$$

The function $\phi(x)$ solves the equation

$$4\phi(x) + x \cdot \nabla \phi(x) = -\frac{1}{4} x \cdot \operatorname{rot} u(x) \text{ for } x \in \Omega$$

Representation of Stokes flows via harmonic potentials, III

Kratz-Lindae representation for star-shaped domains 3D (1992)

Assume $\Omega \subset \mathbb{R}^3$ is star-shaped w.r.t the origin. $u : \Omega \to \mathbb{R}^3$ is a Stokes function with pressure $p : \Omega \to \mathbb{R}$ iff there exist scalar harmonic functions Ψ, Φ and φ in Ω such that

$$u = \nabla\varphi + \mathrm{rot}\left(r^2 x \times \nabla\Psi\right) + x \times \nabla\Phi, \text{ and } p = -6\Psi - 10x\cdot\nabla\Psi - 4x\cdot\nabla\left(x\cdot\nabla\Psi\right),$$

where $r = |\vec{x}|$. These harmonic functions are uniquely determined by u and p:

$$\begin{aligned}
\Psi(x) &= \frac{1}{2}\int_0^1 \{\sqrt{\tau} - 1\} p(\tau x) d\tau \\
\varphi(x) &= \int_0^1 x \cdot \tilde{v}(\tau x) d\tau \text{ with } \tilde{v} = u - \mathrm{rot}\left(x \times \nabla\left(r^2\Psi\right)\right) \\
\Phi(x) &= -\int_0^1 w(\tau x) d\tau \text{ with } v^* = \tilde{v} - \nabla\varphi \text{ and } w(x) = x \cdot \int_0^1 \mathrm{rot}\, v^*(\tau x) d\tau
\end{aligned}$$

- four harmonic potentials represent three components of u and p

Equivalence of representations for 3D star-shaped domains

Equivalent representations for star-shaped domains, Zsuppán (2008)

The Kratz- and Kratz-Lindae representations for Stokes functions with harmonic potentials in 3D star-shaped domains are equivalent.

Outline of the proof:

- derivation a similar representation for the solutions of the Moisil-Teodorescu equations $\mathrm{rot}\, q = -\nabla p$ and $\mathrm{div}\, q = 0$: $q = \nabla\varphi + x \times \nabla\phi$ and $p = \phi + x \cdot \nabla\phi$, where $\phi(x) = \int_0^1 \frac{1}{\tau^2} e^{1-\frac{1}{\tau}} p(\tau x) d\tau$ and $\varphi(x) = \int_0^1 x \cdot q(\tau x) d\tau$.
- given Ψ, Φ and φ, define $\phi(\vec{x}) = 3\int_0^1 \Phi(t\vec{x})\, dt$ and

$$h(x) = \frac{2}{3}\nabla\varphi(x) + (\Psi(x) + x \cdot \nabla\Psi(x))\, x + x \times \left(\frac{2}{3}\nabla\phi(x) + x \times \nabla\Psi(x)\right).$$

- given harmonic vector potential h, $\mathrm{rot}\, h$ and $\mathrm{div}\, h$ satisfy the Moisil-Teodorescu equation, hence use the corresponding representation

For the Papkovich-Neuber potentials there follows:

$$H_0(x) = -\varphi(x) \text{ and } H(x) = (\Psi(x) + 2x \cdot \nabla\Psi(x))\, x - r^2\nabla\Psi(x) + \frac{1}{2}x \times \nabla\Phi(x)$$

Generalizations of the representation formulae

Navier (Lamé) equation in linear elasticity: $\Delta u = \nabla p$ and $\operatorname{div} u = \nu p$ for some constant ν.

Representation of 3D Navier functions by harmonic potentials, Zsuppán (2008)

Assume $\Omega \subset \mathbb{R}^3$ is star-shaped w.r.t the origin, $\nu \neq \frac{3}{4}, 1$. $u : \Omega \to \mathbb{R}^3$ is a Navier function with pressure $p : \Omega \to \mathbb{R}$ iff there exists a harmonic function h in Ω such that

$$u(x) = -\frac{1}{2}\left(x \operatorname{div} \vec{h}(x) + x \times \operatorname{rot} h(x)\right) + \left(\frac{3}{2} - 2\nu\right) h(x).$$

h is unique: $h = \frac{2}{3-4\nu}\left(u - \frac{1}{4}\left(px - \frac{1}{1-\nu} x \times \operatorname{rot} u\right) + x \times \nabla\phi\right)$ with
$\phi(x) = -\frac{1}{4(1-\nu)} \int_0^1 t^{4(1-\nu)} x \cdot \operatorname{rot} u(tx) dt$.

Representation of 2D Navier functions by harmonic potentials, Zsuppán (2008)

Assume $\Omega \subset \mathbb{R}^2$ and $\nu \neq \frac{1}{2}, 1$. $u : \Omega \to \mathbb{R}^2$ is a Navier function with pressure $p : \Omega \to \mathbb{R}$ iff there exists a harmonic function h in Ω such that

$$u(x) = -\frac{1}{2}\left(x \operatorname{div} h(x) + x^{\perp} \operatorname{rot} h(x)\right) + (1 - 2\nu) h(x) \text{ and } p(x) = -2 \operatorname{div} h(x)$$

h is unique: $h = \frac{1}{1-2\nu}\left(u - \frac{1}{4}\left(px - \frac{1}{1-\nu} x^{\perp} \operatorname{rot} u\right)\right)$.

For $h(x_1, x_2, 0) = (h_1, h_2, 0)$ the 3D formula with ν specializes to the 2D formula with $\nu - \frac{1}{4}$.

Generalizations of the representation formulae, II

Use the Kratz representation for the Stokes functions and the identities

$$\begin{aligned} x\,\mathrm{div}\,h + x \times \mathrm{rot}\,h &= h + \nabla(x\cdot h) + \mathrm{rot}(x \times h) \text{ in 3D} \\ x\,\mathrm{div}\,h + x \times \mathrm{rot}\,h &= \nabla(x\cdot h) - \nabla^{\perp}(x^{\perp}\cdot h) \text{ in 2D} \end{aligned}$$

Representation of 3D Stokes functions in arbitrary domains, Zsuppán (2008)

Set $\Omega \subseteq \mathbb{R}^3$ be a domain. (u,p) is a Stokes pair on Ω iff there are harmonic functions w and ψ on Ω such that

$$u = -\frac{1}{2}\left(\nabla(x\cdot w) + \mathrm{rot}\,(x \times w + \psi x)\right) + w \text{ and } p = -2\,\mathrm{div}\,w.$$

These not unique harmonic functions are $w = \frac{2}{3}u - \frac{1}{6}\left(px - x \times \mathrm{rot}\,u\right)$ and $\psi = -\frac{1}{6}x\cdot \mathrm{rot}\,u$.

- if the 3D domain is star-shaped w.r.t. the origin then $u = -\frac{1}{2}\left(\nabla(x\cdot h) + \mathrm{rot}(x\times h)\right) + h$ and $p = -2\,\mathrm{div}\,h$, and h is unique: $h = \frac{2}{3}u - \frac{1}{6}\left(px - x \times \mathrm{rot}\,u\right) + \frac{2}{3}x \times \nabla\phi$ with $\phi(x) = -\frac{1}{4}\int_0^1 t^4 x\cdot \mathrm{rot}\,u(tx)dt$
- for a 2D domain $u = -\frac{1}{2}\left(\nabla(x\cdot h) + \nabla^{\perp}(-x^{\perp}\cdot h)\right) + h$ and $p = -2\,\mathrm{div}\,h$ and h is unique without assuming the domain star-shaped
- if $u = u(x_1, x_2, 0) = (u_1, u_2, 0)$, then $\mathrm{rot}\,u = (0, 0, \partial_1 u_2 - \partial_2 u_1)$ and $\psi \equiv 0$ and the 3D representation specializes to the 2D representation

Possible unification of the 2D and 3D representations

From exterior calculus:

- $\wedge : \Lambda^k \times \Lambda^n \to \Lambda^{k+m}$ denotes the exterior (wedge) product
- $\lrcorner : \Lambda^k \times \Lambda^n \to \Lambda^{m-k}$ denotes the interior product (contraction)
  contraction of $u = dx_{j_1} \wedge \cdots \wedge dx_{j_k}$ with a vector $a = (a_1, \ldots, a_n)$ identified with a 1-form is defined as $a \lrcorner u = \sum_{\ell=1}^{k} (-1)^{\ell-1} a_{j_\ell} dx_{j_1} \wedge \cdots \wedge \widehat{dx_{j_\ell}} \wedge \cdots \wedge dx_{j_k}$.
- connection to 3D calculus

in $\mathbb{R}^3$	u scalar 0-form :	$a \wedge u = ua$	$a \lrcorner u = 0$
	u vector 1-form (field intensity):	$a \wedge u = a \times u$	$a \lrcorner u = a \cdot u$
	u vector 2-form (flux) :	$a \wedge u = a \cdot u$	$a \lrcorner u = -a \times u$
	u scalar 3-form (density):	$a \wedge u = 0$	$a \lrcorner u = ua$

- connection to 2D calculus

in $\mathbb{R}^2$	u scalar 0-form :	$a \wedge u = ua$	$a \lrcorner u = 0$
	u vector 1-form :	$a \wedge u = a^\perp \cdot u$	$a \lrcorner u = a \cdot u$
	u scalar 2-form (density):	$a \wedge u = 0$	$a \lrcorner u = -ua^\perp$

Possible unification of the 2D and 3D representations II

- set $x = x_i dx^i$ the radial 1-form, $h = h_i dx^i$ a 1-form and $\phi = \phi dx^1 \wedge dx^2 \wedge dx^3$ a 3-form (i.e. $\phi = 0$ in 2D):
- representation formula for the Stokes function u (1-form):

$$\begin{aligned} u &= -\frac{1}{2}\left(d\left(x \lrcorner h\right) + \delta\left(x \wedge h + x \lrcorner \phi\right)\right) + h \\ &= \begin{cases} \text{2D:} & -\frac{1}{2}\left(\nabla(x \cdot h) + \nabla^{\perp}(-x^{\perp} \cdot h)\right) + h \\ \text{3D:} & -\frac{1}{2}\left(\nabla(x \cdot h) + \mathrm{rot}(x \times h + \phi x)\right) + h \end{cases} \end{aligned}$$

- for the harmonic potential h (n-dimensional 1-form):

$$\begin{aligned} h &= \frac{2}{n} u - \frac{1}{2n}\left(x \wedge p + x \lrcorner du\right) + \frac{2}{n} x \lrcorner \delta\phi \\ &= \begin{cases} \text{2D:} & h = u - \frac{1}{4}\left(px - x^{\perp}\, \mathrm{rot}\, u\right) \\ \text{3D:} & h = \frac{2}{3} u - \frac{1}{6}\left(px - x \times \mathrm{rot}\, u\right) + \frac{2}{3} x \times \nabla\phi \end{cases} \end{aligned}$$

- for the auxiliary harmonic function ϕ (3-form):
  - $\mathbb{R}^3$: if u is a 1-form then $\phi = -\frac{1}{4}\int_0^1 x \lrcorner du(tx)\,\mathrm{d}t$ is a 3-form hence $\delta(x \lrcorner \phi)$ and $x \lrcorner \delta\phi$ are a 1-forms
  - $\mathbb{R}^2$: if u is a 1-form then $\phi = -\frac{1}{4}\int_0^1 x \lrcorner du(tx)\,\mathrm{d}t$ is a 1-form and $x \lrcorner \phi$ is a 0-form and hence $\delta(x \lrcorner \phi)$ is zero (is not involved in the 2D formula)

Possible further work

- resume the work utilizing conformal maps for the approximation of the constants,
- improve on the given estimations for the constants of union domains...
- ... and generalize them for differential forms,
- unify the 2D and 3D representation formulae for Stokes flows with harmonic potentials using differential forms
- investigate the abstract setting for inf-sup derived by Costabel (2015) for graphs

Köszönöm a figyelmet!

References I

- Bernardi C., Costabel M., Dauge M. , Girault V.: Continuity properties of the inf-sup constant for the divergence. SIAM J. Math. Anal. 48(2) (2016), 1250-1271.
- Babuška I., Aziz A.K., Survey lectures on the mathematical foundations of the finite element method, in The mathematical foundations of the finite element method with applications to partial differential equations (Proc. Sympos., Univ. Maryland, Baltimore, Md., 1972), Academic Press, New York, 1972, pp. 1359.
- E.& F. Cosserat, Sur le équation de la theorié de l'élasticité, C.R. Acad. Sci. (Paris) 126 (1898), 1089–1901.
- Costabel M., Dauge M.: On the inequalities of Babuka-Aziz, Friedrichs and Horgan-Payne. (2013) Arch. Rational Mech. and Anal. 217 (2015) No 3, 873-898.
- Crouzeix M., On an operator related to the convergence of Uzawa's algorithm for the Stokes equation. In: Computational Science for the 21st Century (J. Périaux et al., eds.). New York: Wiley 1997, 242–249.
- Dafermos C. M., Some Remarks on Korn's Inequality. Zeitschrift für Angewandte Mathematik und Physik 19 (1968), 913–920.
- Friedrichs K., On certain inequalities and characteristic value problems for analytic functions and for functions of two variables. Trans. AMS 41 (1937), 321–364.
- Galdi, G. P.: An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Linearized Steady Problems, Band 1., Springer-Verlag (1994)
- Garabedian P.R., A partial differential equation arising in conformal mapping, Pacific J. Math. 1(1951), 485–524.

References II

- Gustaffson B., On mother bodies of convex polyhedra, SIAM J. MATH. ANAL. 29 (1998), No. 5, 1106–1117.
- Horgan C.O. and Payne L.E., On inequalities of Korn, Friedrichs and Babuška-Aziz. Achive Rat. Mech. Anal. 82 (1983), 165–179.
- Hou L.S. and Manouzi H., A div-curl formulation of the Stokes boundary value problem based on a harmonic representation formula, Appl. Math. Lett. Vol. 6, No. 6 (1993), 23–27.
- Kessler M., Die Ladyzhenskaya-Konstante in der numerischen Behandlung von Strömungsproblemen, Thesis, Bayerische Julius-Maximilians-Universität, Würzburg (2000)
- Kozhevnikov A., A history of the Cosserat spectrum, in: The Mazya Anniversary Collection: Volume 1: On Mazyas work in functional analysis, partial differential equations and applications, Operator Theory: Advances and Applications 109 (1999), Birkhuser Basel, pp.223–234.
- Kratz W. and Peyerimhoff A., A numerical algorithm for the Stokes problem based on an integral equation for the pressure via conformal mappings, Numerische Mathematik 58 (1990), 255–272.
- Kratz W., On the Representation of Stokes Flows, SIAM Journal on Mathematical Analysis 22(2) (1991), 414–423.
- Kratz W. and Lindae A., A representation formula for three-dimensional Stokes flows, Zeitschrift für Analysis und ihre Anwendungen 11(3) (1992), 371–375.
- Lichtenstein L., Über die erste Randwertaufgabe der Elastizitätstheorie, Math Z. 20 (1924), 21–28.
- Lin P., Rochberg R., On the Friedrichs operator, Proc. Amer. Math. Soc. 123 (1995), 3335–3342.

References III

- Maz'ya V., Mikhlin S., The Cosserat spectrum of the equations of the theory of elasticity, Vestnik Leningrad Univ. Math., 22 (1967), 58 – 63.
- Neuber H., Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätstheorie, Zeitschrift für Angew. Math. Mech.14(1934), 203–212.
- Payne L.E., A bound for the optimal constant in an inequality of Ladyzhenskaya and Solonnikov, IMA Journal of Applied Mathematics 72 (2007), 563–569.
- Putinar M., Shapiro, H.S., The Friedrichs operator of a planar domain, Oper. Theory Adv. Appl.,113 , Birkhäuser Verlag, Basel (2000), 303–330.
- Putinar M., Shapiro, H.S., The Friedrichs operator of a planar domain II, Oper. Theory Adv. Appl.,127 , Birkhäuser Verlag, Basel (2001), 519–551.
- Shapiro H.S., On some Fourier and distribution-theoretic methods in approximation theory, in vol. Approximation Theory. III, Academic Press, San Diego (1980), 87–124.
- Stoyan G., Towards discrete Velte decompositions and narrow bounds for inf-sup constants, Computers & Maths. with Appls., 38, 7-8, (1999), 243–261.
- Velte W., On inequalities of Friedrichs and Babuška-Aziz, Meccanica 31 (1996), 589–596.
- Velte W., On inequalities of Friedrichs and Babuška-Aziz in dimension three, J. for Anal. and its Appls, Vol.17, No.4 (1998), 843–857.
- Zimmer S., Rand-Druckkorrektur für die Stokes-Gleichung, Thesis, Techn. Univ. München, 1996.