POLYNOMIAL APPROXIMATION AND INTERPOLATION ON THE REAL LINE WITH RESPECT TO GENERAL CLASSES OF WEIGHTS

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Dedicated to Professor P. L. Butzer on his 70th birthday

Abstract. Results on weighted polynomial approximation and interpolation with respect to Freud weights are extended to a more general class of weights. Among others, weights having zeros (Freud–Jacobi type weights) are considered, and a system of nodes is constructed for which the weighted Lebesgue constant of Lagrange interpolation is of optimal order.
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Recently, there has been a considerable interest in different aspects of polynomial approximation (orthogonal polynomials, interpolation) with respect to Freud and Erdős weights on the real line. In this paper we extend some of these results for a more general class of weights. We will consider weights which have finitely many zeros on the real line, and prove density theorems for polynomial approximation in the corresponding space of functions. Also, we will construct systems of nodes of interpolation where the weighted Lebesgue constant is of optimal order. Allowing roots for the weight opens the possibility of considering spaces of piecewise continuous (unbounded) functions. As far as we know, it was D. S. Lubinsky and E. B. Saff [1, Theorems 3.4–3.5] who considered such weights (from different aspects).

Our starting point is the following result, attributed to Akhiezer, Babenko, Carleson and Dzrbasjan (see D. S. Lubinsky [2]).

Theorem A. *Let $w = e^{-Q}$ where Q is even on $\mathbf{R}$, $Q(e^x)$ is convex on $(0, \infty)$, and let*

$$C_w(\mathbf{R}) := \{f|\, f \in C(\mathbf{R}),\ \lim_{|x|\to\infty} (f(x)w(x)) = 0\}. \tag{1}$$

For an $f \in C_w(\mathbf{R})$ define the best polynomial approximation

$$E_n(f)_w := \inf_{p\in\Pi_n} ||w(f - p)||$$

where $||\cdot||$ is the supremum norm over $\mathbf{R}$, and Π_n is the set of polynomials of degree at most n. Then

$$\lim_{n\to\infty} E_n(f)_w = 0 \quad for\ all \quad f \in C_w(\mathbf{R}) \tag{2}$$

if and only if

$$\int_0^\infty \frac{Q(x)}{1 + x^2} dx = \infty.$$

Our first result generalizes the "if" part of this theorem for a wider class of weights defined below.

Definition 1. The set of weight-functions $w(x) = e^{-Q(x)} \in \mathcal{W}_1$ is defined by the following conditions. Let $-\infty < t_1 < \ldots < t_s < \infty$ be arbitrary fixed real numbers, and let $Q(x)$ satisfy the following properties:

(i) $0 < Q \in C(\mathbf{R} \setminus \cup_{i=1}^s \{t_i\})$, $\lim_{x\to t_i} Q(x) = \infty \quad (i = 1, \ldots, s)$,

(ii) $\limsup\limits_{x\to\infty} |Q(x) - Q(-x)| < \infty$,

(iii) $Q(e^x)$ is convex for x large, and

(iv) $\int\limits_{t_s+1}^{\infty} \frac{Q(x)dx}{1+x^2} = \infty$.

This class of weights $\mathcal{W}_1$ is more general than those considered in the above cited theorem, since $w \in \mathcal{W}_1$ vanishes at t_i $(i = 1, \ldots, s)$. Also, condition (ii) permits a certain asymmetry of the weight at $\pm\infty$. Finally, (iii) requires convexity only for large x.

Here are two characteristic examples for weights in $\mathcal{W}_1$:

$$w(x) = e^{-|x|^\alpha} \prod_{i=1}^{s} |x - t_i|^{\alpha_i} |\log|x - t_i||^{\beta_i} \tag{3}$$

$$(\alpha \geq 1, \alpha_i \geq 0,\ \beta_i \in \mathbf{R},\ \beta_i < 0 \, \text{if} \, \alpha_i = 0,\ i = 1, \ldots, s),$$

and

$$w(x) = \exp\left(-|x|^\alpha - \sum_{i=1}^{s} \frac{b_i}{|x - t_i|^{\alpha_i}}\right) \quad (\alpha \geq 1,\ b_i, \alpha_i > 0,\ i = 1, \ldots, s). \tag{4}$$

Now let (compare (1))

$$C_w(\mathbf{R}) := \{f |\, f \in C(\mathbf{R} \setminus \cup_{i=1}^{s} \{t_i\}),\ \lim_{x \to t_i} (w(x)f(x)) = 0,\ i = 0, 1, \ldots, s+1\},$$

where $t_0 = -t_{s+1} = -\infty$. Hence $C_w(\mathbf{R})$ contains functions which are unbounded at the t_i's.

Theorem 1. *We have* (2) *for all* $w \in \mathcal{W}_1$ *and* $f \in C_w(\mathbf{R})$.

Proof. Let $f \in C_w(\mathbf{R})$ and $\varepsilon > 0$ be arbitrary. Then there exists $\delta = \delta(\varepsilon) > 0$ such that

$$w(x)|f(x)| < \varepsilon \quad \text{if} \quad x \in I_i := (t_i - \delta, t_i + \delta),\ i = 1, \ldots, s. \tag{5}$$

Let $l_i(x)$ be the linear function which interpolates $f(x)$ at $t_i \pm \delta$, $i = 1, \ldots, s$, and let

$$f_\varepsilon(x) := \begin{cases} f(x) & \text{if } x \notin \cup_{i=1}^{s} I_i, \\ \min\{f(x), l_i(x)\} & \text{if } x \in I_i,\ i = 1, \ldots, s. \end{cases} \tag{6}$$

Now let $\max(|t_1|, |t_s|) < a < b$ be such that the continuous function

$$\tilde{Q}(x) := \begin{cases} \min_{|x| \leq a} Q(x) & \text{if } 0 \leq x \leq a, \\ \text{linear} & \text{if } a \leq x \leq b, \\ Q(x) & \text{if } x \geq b \end{cases}$$

has the property that $\tilde{Q}(e^x)$ is convex for all $x \geq 0$. Such a and b exist by the properties of Q. But then, extending $\tilde{Q}$ to $(-\infty, 0)$ as an even function, the resulting weightfunction $\tilde{w}(x) := e^{-\tilde{Q}(x)}$ evidently satisfies the conditions of Theorem A. Besides, by definition

$$\tilde{Q}(x) \leq Q(x) + M, \quad \text{where} \quad M := \sup_{x \geq a} |Q(x) - Q(-x)|. \tag{7}$$

Hence and by (6), $f_\varepsilon(x) \in C_{\tilde{w}}(\mathbf{R})$, and by Theorem A there exists a polynomial $p(x)$ such that

$$||\tilde{w}(x)[f_\varepsilon(x) - p(x)]|| < \varepsilon. \tag{8}$$

Hence and by (7)

$$w(x)|f(x)-p(x)| \le w(x)[|f(x)-f_\varepsilon(x)| + |f_\varepsilon(x)-p(x)|] \le$$

$$\le w(x)|f(x)-f_\varepsilon(x)| + e^M \tilde{w}(x)|f_\varepsilon(x)-p(x)| \le w(x)|f(x)-f_\varepsilon(x)| + e^M\varepsilon.$$

Here by (6), the first term is zero if $x \notin \cup_{i=1}^s I_i$. Now by (5) and (6)

$$w(x)|f(x)-f_\varepsilon(x)| \le w(x)|f(x)| + w(x)|f_\varepsilon(x)| \le$$

$$\le 2w(x)|f(x)| \le 2\varepsilon \qquad (x \in \cup_{i=1}^s I_i).$$

Collecting our estimates we get

$$w(x)|f(x)-p(x)| \le (2+e^M)\varepsilon \qquad (x \in \mathbf{R}),$$

which proves the theorem.

Now we define a subset $\mathcal{W}_2$ of $\mathcal{W}_1$.

Definition 2. We shall say that $w(x) = v(x)e^{-Q(x)} \in \mathcal{W}_2$ if the following conditions hold:
(a) Q is even, continuous in $\mathbf{R}$, $0 < Q' \in C(0,\infty)$, and there exist $1 < A \le B < \infty$ such that

$$A \le \frac{(xQ')'}{Q'} \le B \qquad (x \ge 0); \tag{9}$$

(b) $v(x) \ge 0$ is continuous in $\mathbf{R}$, $v(x) > 0$ if $x \in \mathbf{R} \setminus \cup_{i=1}^s \{t_i\}$, and there exist integers $m_i \ge 0$ $(i - 1,\dots,s)$ and constants $c_1, c_2, c_3 \ge 0$ such that[1]

$$c_1 \left|\frac{x-t_i}{y-t_i}\right|^{m_i+1} \le \frac{v(x)}{v(y)} \le c_2 \left|\frac{x-t_i}{y-t_i}\right|^{m_i} \tag{10}$$

$$(|x-t_i| \le |y-t_i| \le c_3,\ i = 1,\dots,s).$$

(c) $v(x)$ is twice differentiable for large $|x|$, and [2]

$$v(x) \sim v(-x) \qquad (x \to \infty), \tag{11}$$

$$\left|\left(\frac{v'(x)}{v(x)}\right)'\right| = o(|x|^{A-2}) \qquad (|x| \to \infty). \tag{12}$$

It is easy to see that $\mathcal{W}_2 \subset \mathcal{W}_1$. Namely, with the notation

$$\tilde{Q}(x) := Q(x) - \log v(x),$$

we can show that (i), (ii) with $\tilde{Q}$ instead of Q follow from (b), (a)-(11), respectively, while (iii) follows from the fact that $\tilde{Q}(e^x)$ is convex for large x. Namely,

[1] In what follows, $c_1, \dots$ will denote positive constants possibly depending on the weights but independent of n.
[2] $\sim$ means that the ratio of the two sides remains between two positive constants as $x \to \infty$.

$$Q'(x) \geq Q'(1)\min(x^{A-1}, x^{B-1}) \qquad (x \geq 0) \tag{13}$$

(cf. G. Criscuolo, B. DellaVecchia, D. S. Lubinsky and G. Mastroianni [3], Lemma 4.1(a)-(b)), and since (12) evidently implies

$$\frac{|v'(x)|}{v(x)} = o(|x|^{A-1}) \qquad (|x| \to \infty), \tag{14}$$

we get from (9)

$$(x\tilde{Q}'(x))' = (xQ'(x))' - \left(\frac{xv'(x)}{v(x)}\right)' \geq AQ'(x) - o(x^{A-1}) \geq cx^{A-1} > 0 \qquad (x \geq 1)$$

with some constant $c > 0$. Finally, (iv) follows from

$$\tilde{Q}(x) \geq cx^A - o(x^A) \qquad (x \text{ large})$$

(see (13) and (14)).

A characteristic example for weight in $\mathcal{W}_2$ is the function (3). The only difficulty in checking this is to choose the m_i's in (10):

$$m_i = \begin{cases} [\alpha_i] & \text{if } \alpha_i \text{ is not an integer,} \\ \alpha_i & \text{if } \alpha_i \text{ is an integer and } \beta_i < 0, \\ \alpha_i - 1 & \text{if } \alpha_i \text{ is an integer and } \beta_i \geq 0. \end{cases}$$

However, it is easy to see that the function in (4) is not in $\mathcal{W}_2$, since (10) does not hold because of the non-polynomial decrease of the weight near the singularities. The weights with $Q(x) = |x|^\alpha$ and

$$v(x) = \begin{cases} -x & \text{if } -1 \leq x \leq 0, \\ x^2 & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } |x| \geq 1 \end{cases}$$

or

$$v(x) = |x|\left(|x| + \left|\sin\frac{1}{x}\right|\right)$$

are also not in $\mathcal{W}_2$; the first because of the asymmetry and the second because of the oscillation at the singularity (again, the critical condition (10) does not fulfil). Nevertheless, these weights are easily seen to be in $\mathcal{W}_1$.

Since Theorem 1 ensures the density of polynomials for weights in the class $\mathcal{W}_\infty$, it makes sense to look for systems of nodes of interpolation for which the weighted Lebesgue constant

$$\lambda_w := \left\| w(x) \sum_{k=1}^{n} \frac{|l_k(x)|}{w(x_k)} \right\|$$

(cf. e.g. Szabados [4]) is optimal in order for $w \in \mathcal{W}_2$.

THEOREM 2. *For any $w \in \mathcal{W}_2$, there exists a system of nodes $\{x_k\}_{k=1}^n \subset \mathbf{R}$ such that*

$$\lambda_w = O(\log n).$$

This order of magnitude of the Lebesgue constant is probably optimal, but we do not address this problem here. Theorem 2 is a generalization of Theorem 1, (7) from [4].

PROOF. Let $a > \max_{1 \leq i \leq s} |t_i|$ and

$$V(x) := \begin{cases} \sqrt{\frac{v(x)v(-x)}{u(x)u(-x)}} & \text{if } |x| \geq a, \\ V(a)e^{\alpha(|x|^B - a^B) + \beta(|x|^B - a^B)^2} & \text{if } |x| < a, \end{cases} \tag{15}$$

where

$$u(x) := \prod_{i=1}^{s} |x - t_i|^{m_i} \tag{16}$$

and

$$\alpha := \frac{1}{Ba^{B-1}} \frac{V'}{V}(a), \quad \beta := \frac{1}{2B^2 a^{2B-2}} \left(\frac{V'}{V}\right)'(a) - \frac{B-1}{2B^2 a^{2B-1}} \frac{V'}{V}(a). \tag{17}$$

(Here $V'(a)$ and $V''(a)$ are meant to be right derivatives calculated from the first part of the definition of $V(x)$ in (15).)

First we show that $w_1(x) := e^{-Q_1(x)}$, where

$$Q_1(x) := Q(x) - \log V(x), \tag{18}$$

is a Freud weight, i.e. it satisfies (a) of Definition 2 (with Q_1 instead of Q). $\bar{Q}$ is, by definition, even and continuous in $\mathbf{R}$. An easy calculation shows that the values of α and β in (17) are defined such that $V(x)$ is twice differentiable at $x = \pm a$, and thus by (c) of Definition 2, for all $x \in \mathbf{R}$. Further, we obtain from (15), (16) and (14)

$$\left|\frac{V'}{V}(x)\right| = \frac{1}{2}\left|\frac{v'}{v}(x) - \frac{u'}{u}(x) - \frac{v'}{v}(-x) + \frac{u'}{u}(-x)\right| \tag{19}$$

$$= o(x^{A-1}) + O(x^{-1}) = o(x^{A-1}) \qquad (x \to \infty).$$

Hence and from (17)

$$|\alpha| = o(a^{A-B}) \qquad (a \to \infty). \tag{20}$$

Similarly, using also (12)

$$\left|\left(\frac{V'}{V}\right)'(x)\right| = o(x^{A-2}) + O(x^{-2}) = o(x^{A-2}) \qquad (x \to \infty). \tag{21}$$

Thus from (17)

$$|\beta| = o(a^{A-2B}) \qquad (a \to \infty). \tag{22}$$

Now, in order to show condition (9) for Q_1 and for some $A_1,\ B_1$ instead of Q, A, B, respectively, we write

(23)
$$\frac{(x\bar{Q}'(x))'}{\bar{Q}'(x)} = 1 + \frac{\frac{xQ''(x)}{Q'(x)} - \frac{x}{Q'(x)}\left(\frac{V'}{V}\right)'(x)}{1 - \frac{1}{Q'(x)}\frac{V'}{V}(x)}.$$

Here, by the definition of $V(x)$, (13), (19), (20)-(22) and (14)

$$\frac{1}{Q'(x)}\frac{V'}{V}(x) = \begin{cases} \frac{Bx^{B-1}}{Q'(x)}[\alpha + 2\beta(x^B - a^B)] = O(|\alpha|a^{B-A} + |\beta|a^{2B-A} = o(1) & \text{if } |x| \le a, \\ \frac{o(|x|^{A-1})}{Q'(1)|x|^{A-1}} = o(1) & \text{if } |x| > a, \end{cases}$$

i.e.

(24)
$$M_1 := \left\| \frac{1}{Q'(x)}\frac{V'}{V}(x) \right\| < 1$$

provided a is large enough.

Similarly, using (14) and (20)-(22)

$$\frac{x}{Q'(x)}\left(\frac{V'}{V}\right)'(x) = \begin{cases} \frac{B(B-1)x^{B-1}(\alpha - 2\beta a^B) + 2B(2B-1)\beta x^{2B-1}}{Q'(x)} & \\ = O((|\alpha| + |\beta|a^B)a^{B-A} + |\beta|a^{2B-A}) = o(1) & \text{if } |x| \le a, \\ \frac{o(|x|^{A-1})}{Q'(1)|x|^{A-1}} = o(1) & \text{if } |x| > a, \end{cases}$$

i.e.

(25)
$$M_2 := \left\| \frac{x}{Q'(x)}\left(\frac{V'}{V}\right)'(x) \right\| < A$$

provided a is large enough.[3] Now (9) and (23)-(25) yield

$$1 < 1 + \frac{A - M_2}{2} < \frac{(xQ_1'(x))'}{Q_1'(x)} < 1 + \frac{B}{1 - M_1} < \infty \qquad (x \in \mathbf{R})$$

which shows that $w_1(x)$ is indeed a Freud weight.

We now construct the point system realizing the optimal order of magnitude of the Lebesgue constant. Let

(26)
$$r := \sum_{i=1}^{s} m_i,$$

and consider the roots of the polynomial $p_{n+r-2}(x)$ of degree $n + r - 2$ orthogonal with respect to the Freud weight $w_1(x)^2$. Let n be sufficiently large, and for each $1 \le i \le s$, let $y_{i,1}, y_{i,2}, \ldots, y_{i,m_i+2}$ be the $m_i + 2$ roots of this polynomial nearest to t_i, in such an order that

(27)
$$|t_i - y_{i,1}| \le |t_i - y_{i,2}| \le \ldots \le |t_i - y_{i,m_i+2}| \qquad (i = 1, \ldots, s).$$

We drop the first $m_i + 1$ of these roots from, and add

[3] The "O" and "o" signs in the previous formulas refer to either $x \to \infty$ or $a \to \infty$.

$$
z_i := \begin{cases} \frac{\lambda y_{i,1}+y_{i,2}}{\lambda+1} & \text{if } \operatorname{sgn}(t_i - y_{i,1}) = \operatorname{sgn}(t_i - y_{i,2}), \\ \frac{\lambda t_i + y_{i,2}}{\lambda+1} & \text{otherwise} \end{cases} \qquad (i = 1, \dots, s) \tag{28}
$$

to the set of roots of p_{n+r-2}, where $\lambda > 0$ is a constant to be chosen later. In this way we get $n-2$ roots. Further let $z_0 > 0$ be a point where the norm $\|w_1 p_{n+r-2}\|$ is attained; we add $\pm z_0$ to the previous system of nodes. These n nodes $x_1, \dots, x_n$ will be our system. In other words, these are the roots of the polynomial

$$
\omega_n(x) := p_{n+r-2}(x)(x^2 - z_0^2) \prod_{i=1}^{s} \frac{x - z_i}{\prod_{j=1}^{m_i+1}(x - y_{i,j})} \tag{29}
$$

of degree n.

We shall prove a series of lemmas estimating quantities related to this polynomial. Let a_n be the Mhaskar–Rahmanov–Saff number belonging to the Freud weight $w_1 = e^{-Q_1}$ (i.e. Q_1 satisfies (a) of Definition 2), that is for any polynomial p of degree at most n we have

$$
\|w_1 p\| = \max_{|x| \le a_n} w_1(x)|p(x)|
$$

(cf. e.g. Mhaskar and Saff [5]).

Lemma 1. *Given a Freud weight $w_1 = e^{-Q_1}$, there exist constants $0 < c_4 < 1 < c_5$ such that for any polynomial of degree at most n we have*

$$
w_1(x)|p(x)| \le \|w_1 p\| c_4^n \qquad (|x| \ge c_5 a_n).
$$

Proof. By the above definition of a_n and the monotonicity of Q_1 we get

$$
|p(x)| \le \frac{\|w_1 p\|}{w_1(a_n)} \qquad (|x| \le a_n).
$$

Then, as it is well-known, outside this interval p can be estimated as

$$
|p(x)| \le \frac{\|w_1 p\|}{w_1(a_n)} \left(\frac{2|x|}{a_n} \right)^n \qquad (|x| \ge a_n).
$$

Hence, using mean value theorem, the relation

$$
a_n Q_1'(a_n) \ge c_6 n,
$$

as well as the monotonicity of Q_1' (cf. Levin–Lubinsky [6, Lemma 3.1] and [7, Lemma 5.1(d)]) we obtain for $c_5 \ge 1/c_6$

$$
w_1(x)|p(x)| \le \|w_1 p\| \left(\frac{2}{a_n} \right)^n |x|^n e^{Q_1(a_n) - Q_1(x)} \le \|w_1 p\| \left(\frac{2}{a_n} \right) |x|^n e^{-(x-a_n)Q_1'(a_n)}
$$

$$
\le \|w_1 p\| \left(\frac{2}{a_n} \right)^n |x|^n e^{-c_6 (x-a_n) n / a_n} \le \|w_1 p\| (2c_5)^n e^{-c_6 (c_5 - 1) n} \le \|w_1 p\| c_4^n \qquad (|x| \ge c_5 a_n)
$$

provided c_5 is large enough.

Lemma 2. *Let $x \in \mathbf{R}$ and*

$$
|x - x_j| := \min_{1 \le k \le n} |x - x_k|. \tag{30}
$$

Then we have

$$w(x)\left|\frac{\omega_n(x)}{x-x_j}\right| = O\left(\frac{v(x)na_n^{1/2}\psi_n^{5/4}(x)}{u(x)V(x)}\right) \qquad (x\in\mathbf{R}), \tag{31}$$

where

$$\psi_n(x) := \max\{n^{-2/3}, 1-|x|/a_n\}. \tag{32}$$

Here in case

$$|x-x_j| \le \frac{\eta a_n}{n\psi_n(x)^{1/2}} \tag{33}$$

with a small enough $\eta > 0$ the estimate is sharp in the sense of the order of magnitude.

Proof. We distinguish three cases.

Case 1: $x_j = z_0$ (the case $x_j = -z_0$ can be handled similarly). Then we get from (29)

$$w(x)\left|\frac{\omega_n(x)}{x-x_j}\right| = \frac{v(x)}{V(x)}w_1(x)|p_{n+r-2}(x)|(x+z_0)\prod_{i=1}^{s}\frac{|x-z_i|}{\prod_{j=1}^{m_i+1}|x-y_{i,j}|}.$$

Here by the definition of $y_{i,j}$ we have

$$|x-y_{i,j}| \sim |x-t_i| \qquad (j=1,\dots,m_i+1,\ i=1,\dots,s) \tag{34}$$

$$|x-z_i| \sim |x-t_i| \qquad (i=1,\dots,s), \tag{35}$$

and[4]

$$w_1(x)|p_{n+r-2}(x)(x+z_0)| \tag{36}$$

$$= \begin{cases} O(a_n^{-1/2}\psi_n(x)^{-1/4}a_n) = O(na_n^{1/2}\psi_n(x)^{5/4}) & \text{if } a_n \le |x| \le c_5a_n, \\ O(n^{1/6}a_n^{1/2}c_4^n) & \text{if } |x| \ge c_5a_n \end{cases}$$

(cf. [3, Lemma 4.2(b)] and [4, Lemma 2]). Note that in case (33), with a sufficiently small $\eta > 0$, the last estimate is sharp (cf. [3, Lemma 4.2(d)]). Using these relations, as well as (16), we get

$$w(x)\left|\frac{\omega_n(x)}{x-x_j}\right| = O\left(\frac{v(x)}{u(x)V(x)}a_n^{-1/2}\psi_n(x)^{-1/4}a_n\right) = O\left(\frac{v(x)}{u(x)V(x)}a_n^{1/2}\psi_n(x)^{-1/4}\right),$$

which is equivalent to (31), since in this case $\psi_n(x) \sim \psi_n(a_n) = n^{-2/3}$ (cf. [4, Lemma 2]). Also, the estimate is sharp if (33) holds.

Case 2: x_j is a z_μ. Then

$$w(x)\left|\frac{\omega_n(x)}{x-x_j}\right| = \frac{v(x)}{V(x)}\frac{w_1(x)|p_{n+r-2}(x)|(z_0^2-x^2)}{\prod_{j=1}^{m_\mu+1}|x-y_{\mu,j}|}\prod_{\substack{i=1\\ i\ne\mu}}^{s}\frac{|x-z_i|}{\prod_{j=1}^{m_i+1}|x-y_{i,j}|}.$$

[4] Here we use the relation $a_{n+r-1} \sim a_n$ (cf. [3, Lemma 4.5(c)]).

Here $z_0^2 - x^2 \sim a_n^2$, further (3) holds again. Moreover, (35) also holds except for $i = \mu$. Using again (36), as well as (32), $\psi_n(x) \sim 1$ and $|x - y_{\mu,j}| \sim a_n/n$ we get

$$w(x)\left|\frac{\omega_n(x)}{x - x_j}\right| = O\left(\frac{v(x)}{V(x)u(x)}\frac{a_n^{-1/2}a_n^2}{\frac{a_n}{n}}\right) = O\left(\frac{v(x)}{V(x)u(x)}na_n^{1/2}\right),$$

which proves (31) in this case. Again, the estimate is sharp when (33) holds.

Case 3: x_j is not $\pm z_0, z_1, \ldots, z_s$. Then

$$w(x)\left|\frac{\omega_n(x)}{x - x_j}\right| = \frac{v(x)}{V(x)}\frac{w_1(x)|p_{n+r-2}(x)|}{|x - x_j|}|x^2 - z_0^2|\prod_{i=1}^{s}\frac{|x - z_i|}{\prod_{j=1}^{m_i+1}|x - y_{i,j}|}.$$

Now (34)–(35) still holds, and we also have

$$\frac{w_1(x)|p_{n+r-2}(x)|}{|x - x_j|}|x^2 - z_0^2| = O(na_n^{-3/2}\psi_n(x)^{1/4}a_n^2\psi_n(x)) = O(na_n^{1/2}\psi_n(x)^{5/4})$$

$$(|x| \le a_n(1 + cn^{-2/3}))$$

(cf. [3, Lemma 4.2(d)]; and this is sharp in case (33) holds);

$$\frac{w_1(x)|p_{n+r-2}(x)|}{|x - x_j|}|x^2 - z_0^2| = w_1(x)|p_{n+r-2}(x)| \cdot \left|\frac{x - z_0}{x - x_j}\right| \cdot |x + z_0|$$

$$= O(a_n^{-1/2}\psi_n(x)^{-1/4}a_n) = O(na_n^{1/2}\psi_n(x)^{5/4}) \qquad (a_n(1 + cn^{-2/3}) \le |x| \le c_5 a_n)$$

(cf. [3, Lemma 4.2(b)]), since $|z_0|$, $|x_j| \le a_n(1 + (c/2)n^{-2/3})$ if c is large enough (cf. [3, Lemma 4.4]); finally by our Lemma 1

$$\frac{w_1(x)|p_{n+r-2}(x)|}{|x - x_j|}|x^2 - z_0^2| = O(na_n^{1/2}c_4^n) \qquad (|x| \ge c_5 a_n)$$

which is more than stated.

Collecting these estimates we obtain

$$w(x)\left|\frac{\omega_n(x)}{x - x_j}\right| = O\left(\frac{v(x)}{V(x)}na_n^{-3/2}\psi_n(x)^{1/4}a_n^2\psi_n(x)\prod_{i=1}^{s}\frac{1}{|x - t_i|^{m_i}}\right)$$

$$= O\left(\frac{v(x)}{V(x)u(x)}na_n^{1/2}\psi_n(x)^{5/4}\right),$$

and the lemma is completely proved.

Corollary 1. *We have*

$$w(x)|\omega_n(x)| = O\left(\frac{v(x)a_n^{3/2}\psi_n^{3/4}(x)}{u(x)V(x)}\right) \qquad (x \in \mathbf{R}).$$

This follows from (31) by taking into account that

$$|x - x_j| = O\left(\frac{a_n}{n\psi_n(x)^{1/2}}\right),$$

which is a consequence of the root distance relation

$$\Delta x_i := x_i - x_{i+1} \sim \frac{a_n}{n\psi_n(x_i)^{1/2}} \qquad (i = 1, \ldots, n-1) \tag{37}$$

(cf. [3, Lemma 4.4]). The applicability of the last relations for x_i's instead of the original roots of the polynomial p_{n+r-2} follows from the construction of these nodes.

Corollary 2. *We have*

$$w(x_j)|\omega_n'(x_j)| \sim \frac{v(x_j)\psi_n(x_j)^{5/4}}{u(x_j)V(x_j)} n a_n^{1/2} \qquad (j=1,\dots,n).$$

This follows again from (31) by letting $x \to x_j$ and using the sharpness of the estimate.

We now return to the proof of Theorem 2. Using the notation (30), Lemma 2 and Corollaries 1 and 2 we get

$$w(x)\sum_{k=1}^{n}\frac{|l_k(x)|}{w(x_k)} = O\left(\frac{v(x)}{u(x)}\frac{u(x_j)}{v(x_j)}\frac{V(x_j)}{V(x)}\left(\frac{\psi_n(x)}{\psi_n(x_j)}\right)^{5/4}\right. \tag{38}$$

$$\left. + \frac{v(x)\psi_n(x)^{3/4}a_n}{u(x)V(x)n}\sum_{\substack{k=1\\k\neq j}}^{n}\frac{u(x_k)V(x_k)}{v(x_k)\psi_n(x_k)^{5/4}|x-x_k|}\right).$$

Here in the first term $\psi_n(x) = O(\psi_n(x_j))$, and in case $|x| \geq a$ we have $\frac{v(x)}{u(x)V(x)} = O(1)$ by (15). If $|x| < a$ then again by (10)

$$\frac{v(x)}{v(x_j)} = O\left(\frac{u(x)}{u(x_j)}\left(1+\left|\frac{x-t_i}{x_j-t_i}\right|\right)\right) = O\left(\frac{u(x)}{u(x_j)}\right),$$

where t_i is the nearest to x, and by (15) $V(x) \sim 1$, $V(x_j) \sim 1$. This shows that the first term in (38) is $O(1)$.

For the rest of the right-hand side in (38), applying (37) we get

$$O\left(\sum_{k\neq j}\frac{v(x)}{u(x)V(x)}\frac{u(x_k)}{v(x_k)V(x_k)}\left(\frac{\psi_n(x)}{\psi_n(x_k)}\right)^{3/4}\frac{\Delta x_k}{|x-x_k|}\right)$$

$$\leq \sum_{i=1}^{s}\sum_{\substack{|x-t_i|\leq|x_k-t_i|\leq c_3\\k\neq j}} + \sum_{i=1}^{s}\sum_{\substack{|x_k-t_i|<|x-t_i|\leq c_3\\k\neq j}} + \sum_{\substack{|x_k-t_i|\geq c_3,\frac{1}{2}(t_{i-1}+t_i)\leq x_k\leq\frac{1}{2}(t_i+t_{i+1})\\k\neq j}}$$

$$+ \sum_{\substack{x_k<\frac{1}{2}(t_0+t_1)\text{or}\,x_k>\frac{1}{2}(t_s+t_{s+1})\\k\neq j}} = A_1 + A_2 + A_3 + A_4,$$

where now $t_0 = -a = t_{s+1}$.

For A_1, we get from (10), (15) and (32)

$$\frac{v(x)}{v(x_k)} = O\left(\frac{u(x)}{u(x_k)}\right),$$

$$V(x) \sim 1, \quad V(x_k) \sim 1 \tag{39}$$

and

$$\frac{\psi_n(x)}{\psi_n(x_k)} = O(1), \tag{40}$$

whence

$$A_1 = O\left(\sum_{k\neq j} \frac{\Delta x_k}{|x - x_k|}\right) = O(\log n) \tag{41}$$

(cf. [4, Lemma 6]). For A_2, (39) and (40) still hold, and again by (10)

$$\frac{v(x)}{v(x_k)} = O\left(\frac{u(x)}{u(x_k)}\left|\frac{x - t_i}{x_k - t_i}\right|\right) = O\left(\frac{u(x)}{u(x_k)}\left(1 + \left|\frac{x - t_i}{x_k - t_i}\right|\right)\right),$$

whence

$$A_2 = O(A_1) + O\left(\sum_{k\neq j} \frac{\Delta x_k}{|t_i - x_k|}\right) = O(\log n),$$

again by Lemma 6 in [3].

For A_3, by (10) and (15)

$$v(x) \sim u(x)V(x), \quad u(x_k) \sim v(x_k)V(x_k),$$

and (40) still holds. Therefore A_3 has the same estimate as A_1 in (41).

Finally, A_4 is estimated the same way as A_3. Theorem 2 is completely proved.

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