NIKOL'SKII INEQUALITY BETWEEN THE UNIFORM NORM AND INTEGRAL NORM WITH BESSEL WEIGHT FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE ON THE HALF-LINE

V. ARESTOV[a,b,1,2], A. BABENKO[b,a,1,2], M. DEIKALOVA[a,b,1,2], and Á. HORVÁTH[c,1]

[a] *Institute of Natural Sciences and Mathematics, Ural Federal University, Yekaterinburg, Russia*
e-mails: vitalii.arestov@urfu.ru, marina.deikalova@urfu.ru
[b] *Krasovskii Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, Yekaterinburg, Russia*
e-mail: babenko@imm.uran.ru
[c] *Institute of Mathematics, Budapest University of Technology and Economics, Budapest, Hungary*
e-mail: ahorvath@math.bme.hu

(Received October 09, 2017)

Abstract

We study the Nikol'skii type inequality for even entire functions of given exponential type between the uniform norm on the half-line $[0,\infty)$ and the norm $\left(\int_0^\infty |f(x)|^q\, x^{2\alpha+1} dx\right)^{1/q}$ of the space $L^q((0,\infty), x^{2\alpha+1})$ with the Bessel weight for $1 \le q < \infty$ and $\alpha > -1/2$. An extremal function is characterized. In particular, we prove that the uniform norm of an extremal function is attained only at the end point $x = 0$ of the half-line. To prove these results, we use the Bessel generalized translation.

Key words and phrases: entire functions of exponential type, Nikol'skii type inequality, Bessel generalized translation.

1 INTRODUCTION

1.1 Notation. Problem statement

For $1 \le q < \infty$ and $\alpha > -1$, denote by $L^q_\alpha = L^q((0,\infty), x^{2\alpha+1})$ the space of complex-valued Lebesgue measurable functions f on the half-line $\mathbb{R}_+ = [0,\infty)$ such that the function $|f(x)|^q\, x^{2\alpha+1}$ is integrable over $(0,\infty)$. The space L^q_α is equipped with the norm

$$\|f\|_{q,\alpha} = \|f\|_{L^q_\alpha} = \left(\int_0^\infty |f(x)|^q\, x^{2\alpha+1} dx\right)^{1/q}, \quad f \in L^q_\alpha.$$

In the case $q = \infty$ ($\alpha > -1$), we assume that L^∞_α is the space $L^\infty = L^\infty(0,\infty)$ of functions f measurable and essentially bounded on $\mathbb{R}_+$ with the norm

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(x)| \colon x \in (0,\infty)\}, \quad f \in L^\infty.$$

Along with L^∞, we consider the space $C = C[0,\infty)$ of functions continuous and bounded on the half-line $[0,\infty)$ with the uniform norm

$$\|f\|_{C[0,\infty)} = \sup\{|f(x)| \colon x \in [0,\infty)\}.$$

Denote by $\mathscr{E}(\sigma, q, \alpha)$ the set of even entire functions of exponential type (at most) $\sigma > 0$ whose restrictions to the half-line $[0,\infty)$ belong to the space L^q_α. Platonov [29] studied the approximative and extremal properties of the class $\mathscr{E}(\sigma, q, \alpha)$ in the space L^q_α in details. In particular, he proved that, for $1 \le q < p \le \infty$ and $\alpha > -1/2$, the Nikol'skii type inequality

$$\|f\|_{p,\alpha} \le K\, \sigma^{(2\alpha+2)(1/q-1/p)}\, \|f\|_{q,\alpha}, \quad f \in \mathscr{E}(\sigma, q, \alpha), \tag{1.1}$$

[1]This work was supported by the Russian Academic Excellence Project (agreement no. 02.A03.21.0006 of August 27, 2013, between the Ministry of Education and Science of the Russian Federation and Ural Federal University).

[2]V.Arestov, A.Babenko, and M.Deikalova were supported by the Russian Foundation for Basic Research (project no. 15-01-02705) and by the Program for State Support of Leading Scientific Schools of the Russian Federation (project no. NSh-9356.2016.1).

holds with some constant $K = K(q, p, \alpha)$ (see [29, Theorem 3.5]; this result was announced earlier in [28, Theorem 2]).

In the present paper, we will discuss inequality (1.1) for $p = \infty$, i.e., the inequality

$$\|f\|_C \le M\,\|f\|_{q,\alpha}, \quad f \in \mathscr{E}(\sigma, q, \alpha), \tag{1.2}$$

with the best (i.e., the smallest possible) constant $M = M(\sigma, \alpha, q)$. The aim of the present paper is to study extremal functions in inequality (1.2), i.e., functions $\rho_\sigma \in \mathscr{E}(\sigma, q, \alpha)$, $\rho_\sigma \not\equiv 0$, for which this inequality becomes an equality. In particular, we will study the uniqueness of an extremal function. It is clear that, if a function ρ_σ is extremal, then the function $c\rho_\sigma$ for any constant $c \neq 0$ is also extremal. If ρ_σ is an extremal function in inequality (1.2) and any other extremal function has the form $c\,\rho_\sigma$, $c \in \mathbb{C}$, then we will say that ρ_σ is the *unique* extremal function in inequality (1.2).

If $\alpha = \frac{n}{2} - 1$, where n is a nonnegative integer, then the space L^q_α is isometric to the subspace of spherically symmetrical functions from the space $L^q(\mathbb{R}^n)$. Similarly, the space $\mathscr{E}(\sigma, q, \alpha)$ is related to the space of entire functions of n (complex) variables of exponential spherical type σ. Thus, for $\alpha = \frac{n}{2} - 1$, $n \in \mathbb{N}$, inequality (1.1) and, in particular, (1.2), is contained in Theorem 3.3.5 of Nikol'skii's monograph [27].

Extremal (and especially approximative) properties of entire functions of exponential type of one and several complex variables is a large part of function theory. Such problems were studied by S.N.Bernstein, S.M.Nikol'skii, B.M.Levitan, B.Ya. Levin, N.I. Akhiezer, R.P.Jr. Boas, S.S. Platonov, Q.I. Rahman, G. Schmeisser, V.I. Ivanov, D.V. Gorbachev, O.L. Vinogradov, A.V. Gladkaya, M.I. Ganzburg, S.Yu. Tikhonov, and others; see [2, 13, 18–20, 23, 26, 27, 29, 31, 35] and the references therein. The related topic of extremal properties of algebraic polynomials on an interval, domains of the complex plane, Euclidean sphere, and other manifolds and trigonometric polynomials in one and several variables is even greater; see monographs [9, 13, 15, 17, 25, 30, 32, 34, 38], papers [5–7, 14], and the references therein. In what follows, we will refer only to results directly relevant to the subject of the present paper.

All functional spaces considered in this paper are complex. In addition, let us agree to say exponential type σ instead of exponential type at most σ.

1.2 Nikol'skii inequality for the end point of the half-line

The related to (1.2) inequality

$$|f(0)| \le D\|f\|_{q,\alpha}, \quad f \in \mathscr{E}(\sigma, q, \alpha), \tag{1.3}$$

with the best constant $D = D(\sigma, q, \alpha)$ plays an important role in what follows. Obviously, $D \le M$. We will show that, in fact, $D = M$ at least for $\alpha > -1/2$.

Consider the set

$$\mathscr{E}[1](\sigma, q, \alpha) = \{f \in \mathscr{E}(\sigma, q, \alpha)\colon f(0) = 1\} \tag{1.4}$$

of entire functions from $\mathscr{E}(\sigma, q, \alpha)$ equal to 1 at the point 0. Define

$$\Delta = \inf\{\|f\|_{q,\alpha}\colon f \in \mathscr{E}[1](\sigma, q, \alpha)\}. \tag{1.5}$$

It is clear that $D = 1/\Delta$. Thus, the problem on the exact inequality (1.3) coincides with problem (1.5) about the smallest deviation from zero of class (1.4) of entire functions.

Problems on entire functions that deviate least from zero were studied by Bernstein [11, 12], Akhiezer [1], Vinogradov, Gladkaya [35], and others. However, in comparison with similar problems for algebraic and trigonometric polynomials, problems for entire functions are much less studied.

Value (1.5) can be interpreted as the best approximation in the space L^q_α of an arbitrary function from set (1.4) by the subspace

$$\mathscr{E}[0](\sigma, q, \alpha) = \{f \in \mathscr{E}(\sigma, q, \alpha)\colon f(0) = 0\} \tag{1.6}$$

of functions from $\mathscr{E}(\sigma, q, \alpha)$ vanishing at the point 0. Therefore, it is reasonable to expect that the following statement is valid.

Theorem 1 *For $1 \le q < \infty$, $\alpha > -1$, and $\sigma > 0$, an extremal function $\varrho_\sigma = \varrho_{\sigma,q,\alpha} \in \mathscr{E}(\sigma, q, \alpha)$, $\varrho_\sigma \not\equiv 0$, in inequality* (1.3) *exists and is characterized by the property of "orthogonality" to set* (1.6):

$$\int_0^\infty f(x)\, x^{2\alpha+1} |\varrho_\sigma(x)|^{q-1} \operatorname{sign} \overline{\varrho}_\sigma(x) dx = 0, \quad f \in \mathscr{E}[0](\sigma, q, \alpha). \tag{1.7}$$

For $1 < q < \infty$, an extremal function in inequality (1.3) *is unique.*

1.3 Main result

The following statement is the main result of the present paper.

Theorem 2 *For $\alpha > -1/2$, $1 \le q < \infty$, and $\sigma > 0$, the following statements are valid.*

(1) *The best constants in inequalities* (1.2) *and* (1.3) *coincide*:

$$M(\sigma, q, \alpha) = D(\sigma, q, \alpha). \tag{1.8}$$

(2) *Inequalities* (1.2) *and* (1.3) *have the same set of extremal functions. An extremal function $\varrho_{\sigma,q,\alpha}$ of inequalities* (1.2) *and* (1.3) *is characterized by property* (1.7). *For $1 < q < \infty$, this function is unique.*

(3) *For $1 \le q < \infty$, the uniform norm on the half-line $[0, \infty)$ of any function extremal in inequality* (1.2) *is attained only at the point $x = 0$.*

By now, the authors do not known whether an extremal function in inequalities (1.2) and (1.3) is unique for $q = 1$.

2 Entire functions that deviate least from zero

The main aim of this section is to prove Theorem 1.

2.1 Auxiliary statement

Lemmas 1 and 2 are either known or can be proved by using known arguments. Nevertheless, to make the presentation complete and convenient, we give their proofs. The statement of Lemma 1 is a kind of the compactness property on the set of entire functions. This statement is well known and was used by a number of authors; see, for example, [11], [2, Ch. IV, Sect. 83], [29, the proof of Theorem 3.6], and [27, Ch. 3, Sects. 3.3, 3.5]. We prove Lemma 1 mostly following [2, Ch. IV, Sect. 83].

Lemma 1 *Any sequence of entire functions of exponential type σ collectively bounded on the real line contains a subsequence uniformly convergent to an entire function of exponential type σ on every compact subset of the complex plane.*

Proof. Assume that $\{f_n\}_{n=1}^\infty$ is a sequence of entire functions of exponential type σ collectively bounded on the real line, more exactly, such that

$$M = \sup_{n \ge 1} \|f_n\|_{C(\mathbb{R})} < \infty.$$

In the set $\mathscr{E}(\sigma)$ of entire functions of exponential type σ bounded on the real line, the following inequalities hold for points $z = x + iy \in \mathbb{C}$ (see, for example, [2, Ch. IV, Sect. 83, (3), (4)]):

$$|f(z)| \le M e^{\sigma|y|}, \quad |f'(z)| \le M\sigma e^{\sigma|y|}, \quad f \in \mathscr{E}(\sigma); \tag{2.1}$$

here, $M = \|f\|_{C(\mathbb{R})}$. Note that the latter inequality follows from the former and known Bernstein's inequality for entire functions of exponential type (see, for example, [2, Ch. IV, Sect. 83]).

Inequalities (2.1) imply that, on any compact set $Q \subset \mathbb{C}$, the sequence $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded and equicontinuous family of functions. By the Arzelá–Ascoli theorem (see, for example, [16, Ch. IV, Sect. 6]), the sequence $\{f_n\}_{n=1}^{\infty}$ contains a subsequence $\{f_{n_\nu}\}_{\nu=1}^{\infty}$ uniformly convergent on Q. Besides, this fact can be deduced from Montel's theorem about the compactness of a family of analytic functions with respect to the uniform convergence in the interior of a domain (see, for example, [24, Ch. 4, Sect. 1]).

Restricting ourselves by considering only closed circles $Q_N = \{|z| \le N,\ z \in \mathbb{C}\}$ centered at the point 0 with radii that are nonnegative integers, and applying Cantor's diagonal process, we conclude that the subsequence $\{f_{n_\nu}\}_{\nu=1}^{\infty}$ can be chosen independent from the compact set Q. Denote by f the limiting function; this is an entire function. Using the first inequality from (2.1), we conclude that the function f has exponential type σ. The lemma is proved.

Lemma 2 *For $\alpha > -1$ and $1 \le q < \infty$, the following statements are valid with respect to extremal functions of inequality* (1.3) *and of the equivalent problem* (1.5).

(1) *An extremal function ϱ_σ in inequality* (1.3) *and in the equivalent problem* (1.5) *exists.*

(2) *An extremal function ϱ_σ is characterized by the "orthogonality" property* (1.7).

(3) *An extremal function ϱ_σ can have only real zeros and at least one zero exists.*

(4) *An extremal function ϱ_σ is real-valued on the real line.*

Proof. (1) Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions from $\mathscr{E}[1](\sigma, q, \alpha)$ minimizing value (1.5), more exactly, possessing the property $\|f_k\|_{q,\alpha} \to \Delta,\ k \to \infty$. According to inequality (1.2), this sequence is bounded on the half-line $[0, \infty)$. By Lemma 1, the sequence $\{f_k\}$ contains a subsequence uniformly convergent on any compact set of the complex plane to an entire function $\varrho = \varrho_\sigma$ of exponential type σ; it is convenient to assume that the sequence $\{f_k\}$ itself has this property. Obviously, the function ϱ is even and $\varrho(0) = 1$.

For any $R > 0$, we have

$$\left(\int_0^R |\varrho(x)|^q x^{2\alpha+1} dx \right)^{1/q} = \lim_{k \to \infty} \left(\int_0^R |f_k(x)|^q x^{2\alpha+1} dx \right)^{1/q} \le \Delta.$$

Hence, $\varrho \in L^q_\alpha$ and $\|\varrho\|_{q,\alpha} \le \Delta$. Consequently, $\varrho \in \mathscr{E}[1](\sigma, q, \alpha)$ and, for the function ϱ, the minimum is attained in (1.5); i.e., the function ϱ is extremal in (1.5) and (1.3).

(2) The second statement of the lemma is validated by known duality arguments (see, for example, [22, Ch. 2], [4]). The finiteness of the constant in inequality (1.3) means that $f(0)$ is a bounded linear functional on the subspace $\mathscr{E}(\sigma, q, \alpha)$ (with the norm of the space L^q_α) and its norm is D. According to the Hahn–Banach theorem, the functional $f(0)$ can be extended to the whole space L^q_α as a bounded linear functional Ξf with the same norm: $\|\Xi\|_{(L^q_\alpha)^*} = D$. For $1 \le q < \infty$, the conjugate space for the space L^q_α is $L^{q'}_\alpha,\ 1/q' + 1/q = 1$; in particular,

$$\Xi f = \int_0^\infty f(x) \overline{\xi}(x) dx, \quad f \in L^q_\alpha, \tag{2.2}$$

where $\xi \in L^{q'}_\alpha$; moreover, $\|\xi\|_{L^{q'}_\alpha} = \|\Xi\|_{(L^q_\alpha)^*} = D$.

Since the functional Ξ is a norm-preserving extension of the functional $f(0)$ from $\mathscr{E}(\sigma, q, \alpha)$ to L^q_α, the norm of the functional Ξ in the space L^q_α is attained at a function $\varrho = \varrho_\sigma$ extremal in inequality (1.3). Applying Hölder's inequality in (2.2) and taking into account the conditions under which this inequality becomes an equality, we conclude that the formula $\xi(x) = c\,|\varrho_\sigma(x)|^{q-1} \operatorname{sign} \overline{\varrho}_\sigma(x)$, where c is a constant, hods almost everywhere on the half-line $(0, \infty)$. For $q = 1$, this holds because ϱ, being an entire function, cannot vanish on a set of positive measure from the half-line $(0, \infty)$.

It follows that

$$f(0) = c \int_0^\infty f(x) x^{2\alpha+1} |\varrho_\sigma(x)|^{q-1} \operatorname{sign} \overline{\varrho}_\sigma(x) dx, \quad f \in \mathscr{E}(\sigma, q, \alpha),$$

where $c \in \mathbb{C}$ is a constant. This representation implies property (1.7) of the function ϱ_σ extremal in inequality (1.3).

It remains to prove that condition (1.7) is sufficient for a function $\varrho_\sigma \in \mathscr{E}[1](\sigma,q,\alpha)$ to be extremal in problem (1.5). Using the function ϱ_σ, we define the functional

$$\Xi^0 f = c\int_0^\infty f(x)x^{2\alpha+1}|\varrho_\sigma(x)|^{q-1}\operatorname{sign}\overline{\varrho}_\sigma(x)dx, \quad f\in L^q_\alpha, \tag{2.3}$$

where the constant c is chosen from the condition $\|\Xi^0\|_{(L^q_\alpha)^*}=1$. Obviously, the norm of functional (2.3) is attained at the function ϱ_σ and, hence, $|\Xi^0\varrho_\sigma| = \|\varrho_\sigma\|_{L^q_\alpha}$. In view of property (1.7), for an arbitrary function $f\in\mathscr{E}[1](\sigma,q,\alpha)$, we have

$$\|f\|_{L^q_\alpha} \ge |\Xi^0 f| = |\Xi^0(\varrho_\sigma + (f-\varrho_\sigma))| = |\Xi^0\varrho_\sigma| = \|\varrho_\sigma\|_{L^q_\alpha}.$$

Thus, the function ϱ_σ is extremal in problem (1.5).

(3) Let us prove the third statement of the lemma. Assume that a function $f\in\mathscr{E}[1](\sigma,q,\alpha)$ has a zero ζ that is not real. Since the function f is even, the point $-\zeta$ is also its zero. Consider the function

$$g(z) = \epsilon^2\frac{f(z)}{z^2-\zeta^2}(z^2-|\zeta|^2), \quad \epsilon = \operatorname{sign}\zeta = \frac{\zeta}{|\zeta|}.$$

This is an entire function of exponential type σ and $g(0)=1$. For real $x\neq 0$, we have $|x^2-|\zeta|^2| < |x^2-\zeta^2|$. Consequently, $g\in L^q_\alpha$ and $\|g\|_{L^q_\alpha} < \|f\|_{L^q_\alpha}$. Thus, the function $g\in\mathscr{E}[1](\sigma,q,\alpha)$ has a smaller norm in L^q_α in comparison with the function f. Thus, a function from $\mathscr{E}[1](\sigma,q,\alpha)$ having zeros outside the real line cannot be extremal.

For integer $k > (\alpha+1)/q$, the function

$$f_0(z) = \left(\frac{\sin^2\lambda z}{z}\right)^{2k}, \quad \lambda = \frac{\sigma}{4k},$$

is an entire function of exponential type σ, even, belongs to the space L^q_α on the half-line $(0,\infty)$, and $f_0(0)=0$. Thus, $f_0\in\mathscr{E}[0](\sigma,q,\alpha)$. Consequently, property (1.7) must hold for this function. Since the function f_0 is nonnegative on the half-line $(0,\infty)$, we conclude that a function ϱ_σ extremal in inequality (1.3) and problem (1.5) cannot be of constant sign on the half-line $(0,\infty)$.

(4) Let us prove that an extremal function $\varrho=\varrho_\sigma$ is real-valued on the half-line $[0,\infty)$. Together with the function ϱ, the function $\overline{\varrho}$ defined by the relation

$$\overline{\varrho}(z) = \overline{\varrho(\overline{z})}, \quad z\in\mathbb{C},$$

is entire, has type σ, and $\overline{\varrho}(0)=1$. The absolute values of the functions ϱ and $\overline{\varrho}$ coincide on the real line: $|\varrho(x)| = |\overline{\varrho}(x)|$, $x\in\mathbb{R}$. Consequently, the function $\overline{\varrho}$ belongs to the space L^q_α and the norms of the functions ϱ and $\overline{\varrho}$ in L^q_α coincide. Thus, $\overline{\varrho}\in\mathscr{E}[1](\sigma,q,\alpha)$ and, along with the function ϱ, the function $\overline{\varrho}$ is also extremal in problem (1.5). By the inequality

$$\|\varrho+\overline{\varrho}\|_{q,\alpha} \le \|\varrho\|_{q,\alpha} + \|\overline{\varrho}\|_{q,\alpha}, \tag{2.4}$$

the function $g=(\varrho+\overline{\varrho})/2$ is also extremal, and inequality (2.4) turns into an equality. For $1\le q<\infty$, this fact implies that the functions ϱ and $\overline{\varrho}$ have the same sign on the half-line $[0,\infty)$, which, in this case, means that they are real-valued on $[0,\infty)$. Lemma 2 is proved completely.

2.2 The proof of Theorem 1

All statements of Theorem 1 except for the property of uniqueness of an extremal function are contained in Lemma 2. The space L^q_α for $1<q<\infty$ is strictly normed; hence, a function extremal in problem (1.5) (and, hence, in inequality (1.3)) is unique. Thus, Theorem 1 is proved.

3 Fourier–Bessel transform and its application

In this section, we present some information about Bessel functions and the Fourier–Bessel transform, sometimes called the Fourier–Hankel transform, which will be need in what follows. Using the Fourier–Bessel transform, we will give a solution of the above problems for $q = 2$.

3.1 Bessel functions and some of their properties

Bessel function J_α (of the first kind) of order α plays an important role in mathematics and its applications; a widespread bibliography is devoted to the properties of this function (see monographs [10, 37] and textbook [21]). This function is considered for complex values of the parameter α and the independent variable. In the present paper, we assume that α is real and, moreover, $\alpha > -1$. The Bessel function is defined by the formula (see, for example, [37, Ch. III, Sect. 3.1 (8)], [10, Ch. 7, Sect. 7.2, (2)], [21, Ch. 2])

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\alpha+1)} \left(\frac{z}{2}\right)^{2k}. \tag{3.1}$$

By the D'Alembert test, the series on the right-hand side of (3.1) (absolutely) converges everywhere in the complex plane $\mathbb{C}$; hence, its sum is an entire function with nonzero value at the point 0. Consequently, if α is a (nonnegative) integer, then J_α is a single-valued analytic function. For noninteger values of α, the function J_α is multivalued; this function is defined everywhere in the complex plane in the case $\alpha \geq 0$, and everywhere except the point 0 in the case $\alpha < 0$. Let us list some specific cases (see [37, Ch. III, Sect. 3.4, (3)+(6)]):

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z, \quad J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \quad J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z\right).$$

For $\alpha > -1/2$, the Bessel function can be represented in the form of Poisson's integral (see, for example, [37, Ch. III, Sect. 3.3 (1)]):

$$J_\alpha(z) = \frac{(z/2)^\alpha}{h_\alpha} \int_0^\pi \cos(z\cos\theta) \sin^{2\alpha}\theta d\theta, \quad h_\alpha = \Gamma\left(\alpha + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right);$$

as a consequence [37, Ch. III, Sect. 3.3 (3)],

$$J_\alpha(z) = \frac{(z/2)^\alpha}{h_\alpha} \int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(zt) d\theta. \tag{3.2}$$

The normed Bessel function

$$j_\alpha(z) = \Gamma(\alpha+1)\left(\frac{2}{z}\right)^\alpha J_\alpha(z)$$

is of prime importance. According to (3.1), the function j_α is the sum of the series

$$j_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(k+\alpha+1)} \left(\frac{z}{2}\right)^{2k}. \tag{3.3}$$

Series (3.3) converges in the whole complex plane $\mathbb{C}$; hence, function (3.3) is entire. In particular, we have

$$j_{-\frac{1}{2}}(z) = \cos z, \quad j_{\frac{1}{2}}(z) = \frac{\sin z}{z}, \quad j_{\frac{3}{2}}(z) = \frac{3}{z^2}\left(\frac{\sin z}{z} - \cos z\right) \tag{3.4}$$

As a consequence of (3.2),

$$j_\alpha(z) = \frac{\Gamma(\alpha+1)}{h_\alpha} \int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(zt) dt. \tag{3.5}$$

For $\alpha \ge -\frac{1}{2}$, we have the inequality [37, Ch. III, Sect. 3.31 (1)]

$$|j_\alpha(z)| \le e^{|\mathrm{Im}\, z|}, \quad z \in \mathbb{C}. \tag{3.6}$$

Inequality (3.6) for $\alpha > -1/2$ follows from (3.5); for $\alpha = -1/2$, (3.6) follows from the explicit form of the function $j_{-1/2}$ and (3.4). Moreover, estimate (3.6) implies that the (entire) function j_α has exponential type 1.

The function j_α has the following properties:

$$|j_\alpha(t)| \le j_\alpha(0) = 1, \quad \alpha \ge -\frac{1}{2}, \quad t \in \mathbb{R}; \tag{3.7}$$

$$\lim_{u\to\infty} j_\alpha(u) = 0, \quad \alpha > -\frac{1}{2}. \tag{3.8}$$

Property (3.7) can be found in [37, Ch. III, Sects. 3.3, 3.31] and [10, Ch. 7, Sect. 7.3, (4)]. Property (3.8) follows from known asymptotic expansions of $J_\alpha(z)$ as $z \to \infty$ [37, Ch. VII, Sect. 7.21], [10, Ch. 7, Sect. 7.13]. This property can also be easily proved with the use of representation (3.5).

3.2 Fourier–Bessel (Fourier–Hankel) transform

It is the most natural to consider main notions and constructions of this section in the space $L_2^\alpha = L_2(\mathbb{R}_+, x^{2\alpha+1})$; this is a Hilbert space with the inner product

$$(f, g) = (f, g)_{L_2^\alpha} = \int_0^\infty f(x)\overline{g(x)} x^{2\alpha+1}\, dx, \quad f, g \in L_2^\alpha.$$

An important tool for studying problems in the space L_2^α is the Fourier–Bessel (Fourier–Hankel) transform

$$f(x) = \widehat{g}(x) = \frac{1}{2^\alpha \Gamma(\alpha+1)} \int_0^\infty g(y) j_\alpha(xy) y^{2\alpha+1} dy; \tag{3.9}$$

the inverse transform is defined by the same formula. The integral in (3.9) for functions $f \in L_2^\alpha$ is understood as the limit in L_2^α as $R \to \infty$ of the family of functions

$$f_R(x) = \frac{1}{2^\alpha \Gamma(\alpha+1)} \int_0^R g(y) j_\alpha(xy) y^{2\alpha+1} dy.$$

For $\alpha = -1/2$ (by the first formula in (3.4)), the Fourier–Bessel transform (3.9) is the Fourier cosine transform.

The Fourier–Bessel transform (for all $\alpha \ge -1/2$) is a unitary operator in the space L_2^α:

$$\|\widehat{f}\|_{L_2^\alpha} = \|f\|_{L_2^\alpha}, \quad f \in L_2^\alpha.$$

Moreover, for the Fourier–Bessel transform, Parceval's identity holds:

$$(\widehat{f}, \widehat{g})_{L_2^\alpha} = (f, g)_{L_2^\alpha}, \quad f, g \in L_2^\alpha.$$

These facts can be found in [23, Sect. 2], see also [29] and the references therein.

3.3 Pointwise Nikol'skii inequality for $q = 2$

Along with inequality (1.3), the more general inequality

$$|f(z)| \le D(z)\, \|f\|_{q,\alpha}, \quad f \in \mathscr{E}(\sigma, q, \alpha), \tag{3.10}$$

for points $z \in \mathbb{C}$ with the best constant $D(z) = D(z; \sigma, q, \alpha)$ is of interest. For $q = 2$, using the Fourier–Bessel transform and well known arguments, we can find the value $D(z)$ explicitly for all $z \in \mathbb{C}$.

Let $q = 2$ and $\alpha \geq -1/2$. The Fourier–Bessel transform of a function $f \in \mathscr{E}(\sigma, 2, \alpha)$ is supported on the interval $[0, \sigma]$; see, for example, [29]. Consequently,

$$f(z) = \widehat{g}(z) = \frac{1}{2^\alpha \Gamma(\alpha+1)} \int_0^\sigma g(y) j_\alpha(zy) y^{2\alpha+1} dy, \quad x \in \mathbb{C}.$$

Applying the Cauchy–Bunyakovskii inequality, we obtain

$$|f(z)| = \frac{1}{2^\alpha \Gamma(\alpha+1)} \left| \int_0^\sigma g(y) j_\alpha(zy) y^{2\alpha+1} dy \right| \leq$$

$$\leq \frac{1}{2^\alpha \Gamma(\alpha+1)} \left(\int_0^\sigma |j_\alpha(zy)|^2 y^{2\alpha+1} dy \right)^{1/2} \|g\|_{L^2((0,\infty), x^{2\alpha+1})}.$$

Hence,

$$D(z; \sigma, 2, \alpha) = \frac{1}{2^\alpha \Gamma(\alpha+1)} \left(\int_0^\sigma |j_\alpha(zy)|^2 y^{2\alpha+1} dy \right)^{1/2} \tag{3.11}$$

and the function f such that

$$g(y) = \widehat{f}(y) = \begin{cases} \overline{j_\alpha(zy)}, & y \in [0, \sigma], \\ 0, & y > \sigma, \end{cases}$$

is the unique extremal function in (3.10).

By property (3.7), statement (3.11) allows us to find an exact value of the best constant in the corresponding inequality (1.2). Indeed, we have

$$M(\sigma, 2, \alpha) = \max\{D(z; \sigma, 2, \alpha) \colon z \in [0, \infty)\} = D(0) = \frac{\sigma^{\alpha+1}}{2^\alpha \Gamma(\alpha+1) \sqrt{2(\alpha+1)}}.$$

4 Generalized Bessel translation: The proof of Theorem 2

In the proof of Theorem 2, we will use the generalized translation generated by the Bessel function and some of its properties. This section is devoted to the necessary facts about the Bessel translation.

4.1 Bessel generalized translation operator

The Bessel generalized translation operator with step $t \in [0, \infty)$ for $\alpha > -1/2$ is said to be the operator

$$T_t f(x) = T_t^\alpha f(x) = \gamma(\alpha) \int_0^\pi f\left(\sqrt{t^2 + x^2 - 2xt\cos\varphi}\right) \sin^{2\alpha}\varphi \, d\varphi; \tag{4.1}$$

here,

$$\gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\alpha + \frac{1}{2}\right)} = \frac{1}{\displaystyle\int_0^\pi \sin^{2\alpha}\varphi d\varphi}.$$

Making the change $\eta = \cos\varphi$, we obtain the representation

$$T_t f(x) = \gamma(\alpha) \int_{-1}^1 f\left(\sqrt{t^2 + x^2 - 2xt\eta}\right) \left(\sqrt{1 - \eta^2}\right)^{2\alpha - 1} d\eta. \tag{4.2}$$

The translation operator (4.1) is generated by the identity

$$T_t\eta_y(x) = \eta_y(t)\eta_y(x), \quad t, x \geq 0, \tag{4.3}$$

for functions $\eta_y(x) = j_\alpha(yx)$ depending on the parameter $y \geq 0$; identity (4.3) is called a product formula (for Bessel functions (3.1)). The product formula (4.3) was probably first obtained in 1875 by L.Gegenbuer [37, Sect. 11.41, (16)].

Note that the generalized translation operator for $\alpha = -1/2$ is defined by the formula

$$T_t f(x) = \frac{1}{2}\Big\{f(x+t) + f(|x-t|)\Big\}.$$

Properties of the generalized translation operator were studied in details by Levitan [23, Sect. 7]. In particular, he proved [23, Sect. 7, (7.5)] that the operator T_t is self-adjoint for all $\alpha \geq -1/2$; more exactly, if a function $f \in L^1_\alpha$ is continuous and $g \in C[0,\infty)$, then

$$\int_0^\infty (T_t f)(x) g(x) x^{2\alpha+1} dx = \int_0^\infty f(x)(T_t an)(x) x^{2\alpha+1} dx. \tag{4.4}$$

The generalized translation operator finds important applications in mathematics, in particular, in approximation theory where by means of the operator T_t the smoothness of functions is defined; see, for example, [8, 29] and the references therein.

There are several ways (with equivalent results) of constructing and studying the Fourier–Bessel transform and the generalized translation operator T_t in the spaces L^q_α. The most natural way is based on considerations from the theory of generalized functions. Namely this method was used in [29] where further references can be found. Let $\mathscr{S}$ be the space of test functions on the real line, i.e., the space of infinitely differentiable functions on the line vanishing at infinity together with their derivatives of any order faster than absolute values of their arguments; let this space be equipped with the standard topology; see, for example, [33, 36]. Let $\mathscr{S}'$ be the corresponding space of generalized functions, i.e., the set of continuous linear functionals on $\mathscr{S}$. In the space $\mathscr{S}$, consider the subspace $\mathscr{S}_+$ of even functions with the topology induced from $\mathscr{S}$. Denote by $\mathscr{S}_+{}'$ the corresponding space of generalized functions, i.e., the set of continuous linear functionals on $\mathscr{S}_+$. For a value of a functional $f \in \mathscr{S}_+{}'$ at a test function $\phi \in \mathscr{S}_+$, the standard notation $\langle f, \phi\rangle$ is used. The space L^q_α is embedded to the space $\mathscr{S}_+{}'$ if we define the value of a functional $f \in L^q_\alpha$ at a test function $\phi \in \mathscr{S}_+$ by the formula

$$\langle f, \phi\rangle = \int_0^\infty f(x)\phi(x) x^{2\alpha+1} dx.$$

Another method is to define the operator and obtain its desired properties on a sufficiently narrow class of smooth functions dense in the space L^q_α and extend the operator by continuity to the whole space L^q_α. This was done in [29] for the operator T_t with the use of the space $\mathscr{S}_+$. On this way, in [29, see (2.24) and (2.21)], it is proved that, for all $\alpha > -1/2$, $1 \leq q \leq \infty$, and $t \geq 0$, the operator T_t is a bounded linear operator in L^q_α and the following inequality holds for its norm:

$$\|T_t\|_{q,\alpha} = \|T_t\|_{L^q_\alpha \to L^q_\alpha} \leq 1.$$

It follows from (4.3) and (3.7) that, in fact, the equality holds:

$$\|T_t\|_{q,\alpha} = 1. \tag{4.5}$$

The boundedness of the operator T_t in the space L^q_α gives us the possibility, by using a well known argument, to extend formula (4.4) to pairs of functions $f \in L^q_\alpha$ and $g \in (L^q_\alpha)^*$. More exactly, the following statement is valid, which we present here without a proof.

Lemma 3 *For $\alpha > -1/2$, $t > 0$, and $1 \leq q < \infty$, formula* (4.4) *holds for the following pairs of functions*:
(1) $f \in L^q_\alpha$ *and* $g \in L^{q'}_\alpha$ *in the case* $1 < q < \infty$;
(2) $f \in L^1_\alpha$ *and* $g \in C[0,\infty)$ *in the case* $q = 1$.

Lemma 4 *For $\alpha > -1/2$, $t > 0$, and $1 \le q < \infty$, the Bessel generalized translation operator T_t maps the set $\mathscr{E}(\sigma, q, \alpha)$ to itself:*

$$T_t\mathscr{E}(\sigma, q, \alpha) \subset \mathscr{E}(\sigma, q, \alpha).$$

Proof. The value of the generalized translation operator $T_t f(z) = (T_t f)(z)$ for $z \in \mathbb{C}$ at a function $f \in \mathscr{E}(\sigma, q, \alpha)$ can be defined at least in two ways: (i) to define $T_t f(z) = (T_t f)(z)$ by formula (4.1) or, which is the same, (4.2) not only on $[0, \infty)$ but in the whole complex plane; (ii) to verify that, for $f \in \mathscr{E}(\sigma, q, \alpha)$, function (4.1) can be continued from the half-line $[0, \infty)$ to the whole complex plane. By the (interior) uniqueness theorem for analytic functions (see, for example, [24, Ch. 3, Sect. 6]), these two approaches coincide. We will use the former.

A function $f \in \mathscr{E}(\sigma, q, \alpha)$ is entire and even. Consequently, its power series expansion has the form

$$f(z) = \sum_{k=0}^{\infty} c_k z^{2k}, \quad z \in \mathbb{C}.$$

The function

$$h(w) = \sum_{k=0}^{\infty} c_k w^{k}, \quad w \in \mathbb{C},$$

is also entire and $f(z) = h(z^2)$, $z \in \mathbb{C}$. The function $f \in \mathscr{E}(\sigma, q, \alpha)$ has exponential type at most σ; this means that

$$\varlimsup_{|z|\to\infty} \frac{\ln |f(z)|}{|z|} \le \sigma.$$

In terms of the function h, this relation takes the form

$$\varlimsup_{|w|\to\infty} \frac{\ln |h(w)|}{\sqrt{|w|}} \le \sigma. \tag{4.6}$$

Using formula (4.2), we define in the complex plane the function

$$g(z) = T_t f(z) = \gamma(\alpha) \int_{-1}^{1} h(t^2 + z^2 - 2zt\eta) \left(\sqrt{1-\eta^2}\right)^{2\alpha-1} d\eta, \quad z \in \mathbb{C}, \tag{4.7}$$

which is, obviously, entire and even. Property (4.6) means that, for any $\varepsilon > 0$, there exists a number $R = R(\varepsilon) > 0$ such that the following inequality holds for $|w| > R$: $\ln |h(w)| \le (\sigma + \varepsilon)\sqrt{|w|}$ or, which is the same, the inequality

$$|h(w)| < \exp\left((\sigma + \varepsilon)\sqrt{|w|}\right), \quad |w| > R. \tag{4.8}$$

For the argument of the function h in (4.7), we have

$$|t^2 + z^2 - 2zt\eta| \le t^2 + |z|^2 + 2|z|t = (|z| + t)^2, \tag{4.9}$$

$$|t^2 + z^2 - 2zt\eta| \ge |z|^2 - (t^2 + 2|z|t) = (|z| - t)^2 - 2t^2. \tag{4.10}$$

Assume that

$$|z| > t + \sqrt{R + 2t^2}.$$

Then, using (4.10), it is easy to verify that $|t^2 + z^2 - 2zt\eta| > R$. Therefore, for $w = t^2 + z^2 - 2zt\eta$, estimate (4.8) holds. Using inequality (4.9), we obtain

$$|h(t^2 + z^2 - 2zt\eta)| < \exp\left((\sigma + \varepsilon)(|z| + t)\right), \quad |z| > t + \sqrt{R + 2t^2}.$$

This implies the following estimate for function (4.2):

$$|g(z)| < \exp\left(t(\sigma+\varepsilon)\right)\exp\left((\sigma+\varepsilon)|z|\right), \quad |z| > t+\sqrt{R+2t^2}.$$

Hence, the function g has the property

$$\overline{\lim_{|z|\to\infty}}\frac{\ln|g(z)|}{|z|} \le \sigma+\varepsilon.$$

In view of the arbitrariness of $\varepsilon > 0$, this inequality also holds for $\varepsilon = 0$. This means that g is an entire function of exponential type at most σ.

Property (4.5) implies that the restriction of the function g to the real line belongs to the space L^q_α; moreover, $\|g\|_{L^q_\alpha} \le \|f\|_{L^q_\alpha}$. Lemma 4 is proved completely.

4.2 Bessel generalized translation operator in $L^q((0,\infty), x^{2\alpha+1})$, $1 \le q < \infty$, $\alpha > -1/2$

In addition to statement (4.5), we need to know whether the norm of the operator T_t for $t > 0$ in L^q_α is attained (or not). To obtain this information, we first transform the expression for the operator T_t in the space $C[0,\infty)$. The function $u = \sqrt{t^2+x^2-2xt\eta}$ in representation (4.2) decreases in $\eta \in [-1,1]$ from $x+t$ to $|x-t|$. In the latter integral, we pass from the variable η to the varivable $u = \sqrt{t^2+x^2-2xt\eta}$. We have $u^2 = t^2+x^2-2xt\eta$ and, consequently,

$$\eta = \eta(u) = \frac{t^2+x^2-u^2}{2xt}, \quad u\,du = -xt\,d\eta.$$

As a result, we obtain the following representation for $(T_tf)(x)$ for $xt > 0$:

$$T_tf(x) = \int_{|x-t|}^{x+t} f(u)F(t,x,u)\,du, \tag{4.11}$$

where

$$F(t,x,u) = \gamma(\alpha)\left(\sqrt{1-\eta^2}\right)^{2\alpha-1}\Big|_{\eta=\eta(u)}\frac{u}{xt} =$$

$$= \gamma(\alpha)\left(\sqrt{(u^2-(x-t)^2)\left((x+t)^2-u^2\right)}\right)^{2\alpha-1}\frac{2u}{(2xt)^{2\alpha}}.$$

For fixed $xt > 0$, the function $F(t,x,u)$ is positive in the variable $u \in (|x-t|, x+t)$ and

$$\int_{|x-t|}^{x+t} F(t,x,u)\,du = 1.$$

In certain situations, it will be convenient to consider the function $F(t,x,u)$ for $u \in (0,\infty)$, setting $F(t,x,u) = 0$ for $u \in (0,\infty)\setminus(|x-t|, x+t)$.

4.2.1 Bessel translation in $L^1((0,\infty), x^{2\alpha+1})$, $\alpha > -1/2$

In what follows, we sometimes assume that the point x in (4.11) belongs to the set

$$X(t) = \{x \in (0,\infty):\ x \ne t\} = (0,\infty)\setminus\{t\}. \tag{4.12}$$

This assumption provides the fact that the integral in (4.11) is taken over the (finite) interval $[|x-t|, x+t]$ from the half-line $(0,\infty)$.

Lemma 5 *For $\alpha > -1/2,\ t > 0,$ and $q = 1$, the following statements hold for any function $f \in L^1_\alpha$:*

(1) *for $x \in X(t)$, the integral*

$$\int_0^\infty f(u)F(t,x,u)du = \int_{|x-t|}^{x+t} f(u)F(t,x,u)du \tag{4.13}$$

exists and is a continuous function on the set $X(t)$;

(2) *function (4.13) is integrable with the weight $x^{2\alpha+1}$ over $(0,\infty)$ and*

$$(T_t f)(x) = \int_0^\infty f(u)F(t,x,u)du. \tag{4.14}$$

Proof. A function $f \in L^1_\alpha$ is integrable over any half-line $[a,\infty),\ a > 0$. Therefore, for any function $f \in L^1_\alpha$, integral (4.13) over the set (4.12) exists and is a continuous function.

Let $\{\varphi_n\}_{n=1}^\infty$ be a sequence of functions from $\mathscr{S}_+$ convergent to a function f in L^1_α. Consider the corresponding sequence of values (4.1) or, which is the same, (4.11) of the translation operator:

$$\psi_n(x) = (T_t\varphi_n)(x) = \int_0^\infty \varphi_n(u)F(t,x,u)du. \tag{4.15}$$

The sequence $\{\psi_n\}$ converges in L^1_α to $T_t f$. Consequently, we can extract from $\{\psi_n\}$ a subsequence convergent to $T_t f$ almost everywhere on $(0,\infty)$. We assume that the sequence $\{\psi_n\}$ itself has this property. On the other hand, as it is easy to understand, the sequence of functions (4.15) converges at any point $x \in X(t)$ to function (4.13). Consequently, equality (4.14) holds (almost everywhere on $(0,\infty)$). The lemma is proved.

Lemma 6 *For $\alpha > -1/2,\ t > 0,$ and $q = 1$, the norm of the operator T_t^α in the space L^1_α is attained at a function $f \in L^1_\alpha$ almost everywhere nonzero on $(0,\infty)$ if and only if the function f is of constant sign (almost everywhere) on $(0,\infty)$.*

Proof. Taking the specific function $g \equiv 1$ in the second statement of Lemma 3, we obtain the relation

$$\int_0^\infty (T_t f)(x)x^{2\alpha+1}dx = \int_0^\infty f(x)x^{2\alpha+1}dx, \quad f \in L^1_\alpha. \tag{4.16}$$

The operator T_t is positive. Therefore, if the function f is of constant sign (almost everywhere) on $(0,\infty)$, then $T_t f$ has the same sign. Formula (4.16) makes it possible to conclude that the norm of the operator T_t in L^1_α is attained at functions from L^1_α that are of constant sign on $(0,\infty)$.

By (4.14), for any function $f \in L^1_\alpha$, the following inequality holds for almost all $x \in (0,\infty)$:

$$|(T_t f)(x)| = \left|\int_{|x-t|}^{x+t} f(u)F(t,x,u)du\right| \le \int_{|x-t|}^{x+t} |f(u)|F(t,x,u)du = (T_t|f|)(x). \tag{4.17}$$

This and equality (4.16) imply the relations

$$\|T_t f\|_{L^1_\alpha} \le \|T_t|f|\,\|_{L^1_\alpha} = \|f\|_{L^1_\alpha}, \quad f \in L^1_\alpha. \tag{4.18}$$

In order to inequality (4.18) become an equality at a function $f \in L^1_\alpha$, it is necessary and sufficient that inequality (4.17) become an equality at this function for almost all $x \in (0,\infty)$. Inequality (4.17) turns into an equality if and only if the function f is of constant sign almost everywhere on the interval $I(t,x) = (|x-t|, x+t)$. The functions $|x-t|$ and $x+t$ are continuous in $x \in (0,\infty)$. Therefore, inequality (4.18) turns into an equality at a function $f \in L^1_\alpha$ if and only if the function f is of constant sign (almost everywhere) on $I(t,x)$ for any $x \in (0,\infty)$.

The family of intervals $\{I(x,t) = (|x-t|, x+t),\ x \in (t,\infty)\}$ covers the half-line $(0,\infty)$. Therefore, for any closed interval $[a,b] \subset (0,\infty)$, there exists a finite number of intervals $I(t,x),\ x \in (0,\infty)$ covering this closed interval. If the norm of the operator T_t is attained at a function f and two interval $I' = I(t,x')$ and $I'' = I(t,x'')$ have a nonempty intersection, then the function f is of constant sign on the union $I' \bigcup I''$ of the intervals. Hence, it is easy conclude that f is of constant sign on the interval $[a,b]$ and, consequently, on the half-line $(0,\infty)$. Lemma 6 is proved.

4.2.2 Bessel Translation in $L^q((0,\infty), x^{2\alpha+1})$, $1 < q < \infty$, $\alpha > -1/2$

Lemma 7 *For $\alpha > -1/2$, $t > 0$, and $1 < q < \infty$, the following statements hold.*

(1) *For any function $f \in L^q_\alpha$, the integral*

$$\int_0^\infty f(u)F(t,x,u)du = \int_{|x-t|}^{x+t} f(u)F(t,x,u)du$$

for all $x \in (0,\infty)$, $x \neq t$, exists and belongs to the space L^q_α and

$$(T_t f)(x) = \int_0^\infty f(u)F(t,x,u)du.$$

(2) *The norm of the operator T_t in the space L^q_α is 1 and not attained.*

Proof. For $1 < q < \infty$ and any $0 < A < \infty$, we have the inclusion

$$L^q((0,A), x^{2\alpha+1}) \subset L^1((0,A), x^{2\alpha+1}).$$

By Lemma 5, the right-hand side of formula (4.14) is defined and, obviously, is a linear operator

$$(\widetilde{T}_t f)(x) = \int_{|x-t|}^{x+t} f(u)F(t,x,u)du$$

on the space L^q_α. Let us check that $\widetilde{T}_t f \in L^q_\alpha$. Indeed, let $f \in L^q_\alpha$. For $x \in (0,\infty)$ and $t > 0$, by Hölder's inequality, we have

$$\left|(\widetilde{T}_t f)(x)\right| = \left|\int_{|x-t|}^{x+t} f(u)F(t,x,u)du\right| \le \left(\int_{|x-t|}^{x+t} |f(u)|^q \, F(t,x,u)du\right)^{1/q}. \tag{4.19}$$

Applying this estimate and equality (4.16) to the function $|f|^q$, we obtain

$$\|\widetilde{T}_t f\|^q_{L^q_\alpha} \le \|\widetilde{T}_t(|f|^q)\|_{L^1_\alpha} = \||f|^q\|_{L^1_\alpha} = \|f\|^q_{L^q_\alpha}. \tag{4.20}$$

Thus, $\widetilde{T}_t$ is a bounded linear operator in the space L^q_α and its norm in L^q_α is at most 1.

Let $f \in L^q_\alpha$ and $\{f_n\}$ be a sequence of functions from $\mathscr{S}_+$ convergent to f in L^q_α. The sequence of functions $\psi_n = T_t f_n = \widetilde{T}_t f_n$ converges in L^q_α both to $T_t f$ and to $\widetilde{T}_t f$. Thus, the first statement of Lemma 7 is proved.

The fact that the norm of the operator T_t in the space L^q_α is attained at a function $f \not\equiv 0$ means that the first inequality in (4.20) turns into an equality at this function. For this, it is necessary and sufficient that inequality (4.19) become an equality for almost all $x \in (0,\infty)$. Inequality (4.19) turns into an equality if and only if the function f is constant almost everywhere on the interval $I(t,x) = (|x-t|, x+t)$. Similar to the proof of Lemma 6, we conclude that this holds if and only if f is constant almost everywhere on $(0,\infty)$. However, such a function does not belong to the space L^q_α, $1 < q < \infty$. Lemma 7 is proved.

4.3 Proof of the main theorem (Theorem 2)

Obviously, the best constants in inequalities (1.2) and (1.3) are related by the inequality $D \le M$. Let us show that, in fact, they coincide: $D = M$, i.e., (1.8) holds. Let us use the generalized translation operator (4.1).

A function $f \in \mathscr{E}(\sigma, q, \alpha)$ has the property

$$f(x) \to 0, \quad x \to \infty \tag{4.21}$$

on the half-line $(0,\infty)$. Indeed, by inequality (1.2), a function $f \in \mathscr{E}(\sigma, q, \alpha)$ is bounded on the half-line $[0,\infty)$; moreover, this function is even. According to Bernstein's inequality, its derivative is also bounded;

more exactly, $\|f'\|_{C[0,\infty)} \le \sigma\|f\|_{C[0,\infty)}$. In addition, as noted above, a function $f \in L^q((0,\infty), x^{2\alpha+1})$ belongs to the space $L^q(a,\infty)$ (with the unit weight) for any $a > 0$. In this situation, on any half-line $[a,\infty)$, $a > 0$, we have the inequality

$$\|f\|_{C[a,\infty)} \le K\|f\|^{\alpha}_{L_q(a,\infty)}\|f'\|^{\beta}_{C[a,\infty)}, \quad \alpha = \frac{q}{1+q}, \quad \beta = \frac{1}{1+q}, \tag{4.22}$$

with some finite constant $K = L(q)$ independent of the function f and the parameter $a > 0$; see the review paper [3, Sect. 4] and the references therein. By the assumption $1 \le q < \infty$, the right-hand side of (4.22) tends to zero as $a \to \infty$. Thus, property (4.21) is proved.

Let $f \in \mathscr{E}(\sigma, q, \alpha)$. By property (4.21), there exists a point $t = t(f) \in [0,\infty)$ at which the uniform norm of the function f on the half-line $[0,\infty)$ is attained. According to Lemma 4, the function

$$g(x) = (T_t f)(x), \quad x \in [0,\infty),$$

also belongs to the class $\mathscr{E}(\sigma, q, \alpha)$ and, in accordance with definition (4.1) of the translation operator, has the property $g(0) = f(t)$.

Using inequality (1.3) and the property $\|T_t\|_{q,\alpha} = 1$, we obtain

$$\|f\|_{C[0,\infty)} = |f(t)| = |g(0)| \le D\|g\|_{L^q_\alpha} \le D\|f\|_{L^q_\alpha}. \tag{4.23}$$

So that $\|f\|_C \le D\|f\|_{L^q_\alpha}$. In view of the arbitrariness of $f \in \mathscr{E}(\sigma, q, \alpha)$, this implies the inequality $M \le D$. Equality (1.8) is proved.

Recall that we denote by ϱ_σ the function extremal in inequality (1.3). We have

$$D\,\|\varrho_\sigma\|_{L^q_\alpha} = |\varrho_\sigma(0)| \le \|\varrho_\sigma\|_C \le M\,\|\varrho_\sigma\|_{L^q_\alpha}.$$

Hence, in view of (1.8), it follows that

$$\|\varrho_\sigma\|_C = |\varrho_\sigma(0)|$$

and the function ϱ_σ is extremal in inequality (1.2).

It remains to verify that ϱ_σ is the unique extremal function in inequality (1.2). If the uniform norm of a function f_σ extremal in inequality (1.2) is attained at the end point $x = 0$ of the half-line, then this function is also extremal in inequality (1.3). By Theorem 1, for $1 < q < \infty$ such function, up to a constant factor, coincides with ϱ_σ. For $q = 1$, we can only assert that f_σ is an extremal function in inequality (1.3).

Let us check that the uniform norm of any extremal function in the inequality (1.2) cannot be attained at points of the half-line $(0,\infty)$. We will argue by contradiction. Assume that the uniform norm of an extremal function $f_\sigma \in \mathscr{E}(\sigma, q, \alpha)$ in inequality (1.2) is attained at a point $t \in (0,\infty)$. Both inequalities in (4.23) and, in particular, the second inequality must turn into equalities at this function f_σ. This means that the norm of the operator T_t is attained at the function f_σ.

For $1 < q < \infty$, this is impossible in view of Lemma 7.

Let us discuss the case $q = 1$. The norm of the operator T_t in L^1_α is attained at a function $f_\sigma \not\equiv 0$. Being an entire function, $f_\sigma \not\equiv 0$ cannot vanish on a set of positive measure from the half-line. By Lemma 1, the function f_σ is of constant sign on $(0,\infty)$. By formulas (4.1), the function $g_\sigma = T_t f_\sigma$ is also of constant sign on $(0,\infty)$.

The first inequality in (4.23) must turn into an equality at the function f_σ. Consequently, the function $g_\sigma = T_t f_\sigma$ is extremal in inequality (1.3). According to Lemma 2, the function g_σ cannot be of constant sign. Theorem 2 is proved completely.

5 Acknowledgements

The authors are grateful to R. Akopyan for careful reading the manuscript and a number of useful comments.

References

[1] N. I. Akhiezer, On entire functions of finite degree deviating least from zero, *Mat. Sbornik*, **31**:73 (1952), 415–438 (in Russian).

[2] N. I. Akhiezer, *Lectures on Approximation Theory*, Nauka (Moscow, 1965) (in Russian).

[3] V. V. Arestov, Approximation of unbounded operators by bounded operators and related extremal problems, *Uspehi Mat. Nauk*, **51**:6 (1996), 89–124 (in Russian); translated in *Russ. Math. Surv.*, **51**:6 (1996), 1093–1126.

[4] V. V. Arestov, Characterization of extremal elements in certain linear problems, *Ural Math. J.*, **3**:2 (2017).

[5] V. Arestov and M. Deikalova, Nikol'skii inequality between the uniform norm and L_q-norm with ultraspherical weight of algebraic polynomials on an interval, *Comput. Methods Funct. Theory*, **15**:4 (2015), 689–708.

[6] V. Arestov and M. Deikalova, Nikol'skii inequality between the uniform norm and L_q-norm with Jacobi weight of algebraic polynomials on an interval, *Analysis Math.*, **42**:2 (2016), 91–120.

[7] V. Arestov, M. Deikalova, and A. Horváth, On Nikolskii type inequality between the uniform norm and the integral q-norm with Laguerre weight of algebraic polynomials on the half-line, *J. Approx. Theory*, **222** (2017), 40–54.

[8] A. G. Babenko, Exact Jackson–Stechkin inequality in the space $L^2(\mathbb{R}^m)$, *Trudy Inst. Mat. Mekh. UrO RAN*, **5**, 183–198 (1998) (in Russian).

[9] V. F. Babenko, N. P. Korneichuk, and A. A. Ligun, *Extremal Properties of Polynomials and Splines*, Naukova Dumka (Kiev, 1992) (in Russian); translated Nova Science (New York, 1996).

[10] H. Bateman and A. Erdélyi, *Higher Transcendental Functions*, McGraw-Hill Book Company (New York, 1953), Vol. 2.

[11] S. N. Bernstein, On the best approximation of continuous functions on the whole real axis by entire functions of given degree, III, *Dokl. Akad. Nauk SSSR*, **52**:7 (1946), 565–568 (in Russian).

[12] S. N. Bernstein, Functions of finite degree and functions of finite semidegree, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **13** (1949), 111–124 (in Russian).

[13] S. N. Bernstein, *Collected Works* (Izd. Akad. Nauk SSSR, Moscow, 1952), Vol. 1 (in Russian).

[14] B. Bojanov, Polynomial inequalities, *Proc. Int. Conf. "Open Problems in Approximation Theory" (Voneshta voda, Bulgaria, June 18–24, 1993),* Science Culture Technology Publishing (Singapore, 1994), 25–42.

[15] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer (New York 1995).

[16] N. Dunford and J. Schwartz, *Linear Operators*: *General Theory*, Interscience (New York, 1958); translated Inostrannaya Lit. (Moscow, 1962) (in Russian).

[17] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Springer (Berlin 1993).

[18] M. I. Ganzburg, Sharp constants in V.A. Markov–Bernstein type inequalities of different metrics, *J. Approx. Theory*, **215** (2017), 92–105.

[19] M. I. Ganzburg and S. Yu. Tikhonov, On sharp constants in Bernstein–Nikol'skii inequalities, *Constructive Approximation*, **45** (2017), 449–466.

[20] D. V. Gorbachev, *Selected Problems in Functional Analysis and Approximation Theory and Their Applications*, 2nd ed. Grif i K (Tula, 2005) (in Russian).

[21] A. Gray and G. B. Mathews, *A treatise on Bessel Fuctions and Their Applications to Physics*, Mackmillan and Co. (London, 1895).

[22] N. P. Korneichuk, *Extremal Problems of Approximation Theory*, Nauka (Moscow, 1976) (in Russian).

[23] B. M. Levitan, Expansion in Fourier series and integrals with Bessel functions, *Uspekhi Mat. Nauk*, **6**:2 (1951), 102–143 (in Russian).

[24] A. I. Markushevich, *Theory of Analytic Functions*, Nauka (Moscow, 1968) (in Russian).

[25] G. V. Milovanović, D. S. Mitrinovic, and Th. M. Rassias, *Topics in Polynomials*: *Extremal Problems, Inequalities, Zeros*, World Scientific (Singapore, 1994).

[26] S. M. Nikol'skii, Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of several variables, *Trudy Mat. Inst. Steklov*, **38** (1951), 244–278 (in Russian).

[27] S. M. Nikol'skii, *Approximation of Functions of Several Variables and Imbedding Theorems*, Nauka (Moscow, 1969) (in Russain); translated Springer (New York, 1975).

[28] S. S. Platonov, Analogs of Bernstein and Nikol'skii inequalities for one class of exponential-type entire functions *Dokl. Akad. Nauk*, **398**:2 (2004), 168–171 (in Russian); translated in *Dokl. Math.*, **70**:2 (2004), 711–713.

[29] S. S. Platonov, Bessel harmonic analysis and approximation of functions on the half-line, *Izv. RAN, Ser. Mat.*, **71** (2007), 149–196 (in Russian); translated in *Izv. Math.*, **71**:5 (2007), 1001–1048.

[30] G. Pólya, G. Szegő, *Problems and Theorems in Analysis*, Springer (Berlin, 1972); translated Nauka (Moscow, 1978) (in Russian), Vol. 2.

[31] Q. I. Rahman and G. Schmeisser, L^p inequalities for entire functions of exponential type, *Trans. Amer. Math. Soc.*, **320**:1 (1990), 91–103.

[32] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, Oxford Univ. Press (Oxford, 2002).

[33] G. E. Shilov, *Mathematical Analysis*: *The Second Special Course*, Nauka (Moscow, 1965) (in Russian).

[34] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Fizmatgiz (Moscow, 1960); translated Pergamon (New York, 1963).

[35] A. V. Gladkaya and O. L. Vinogradov, Entire functions that deviate least from zero in the uniform and the integral metrics with a weight, *Algebra i Analiz* **26**:2 (2014), 10–28 (in Russian); translated in *St. Petersburg Math. J.*, **26** (2015), 867–879.

[36] V. S. Vladimirov, *Generalized Functions in Mathematical Physics*, Mir (Moscow, 1979) (in Russian).

[37] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press (Cambridge, 1995). translated Inostr. Lit. (Moscow, 1949) (in Russian).

[38] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press (New York, 1959); translated Mir (Moscow, 1965) (in Russian), Vols. 1, 2.