On Nikol'skii type inequality between the uniform norm and the integral q-norm with Laguerre weight of algebraic polynomials on the half-line[☆]

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Abstract

We study the Nikol'skii type inequality for algebraic polynomials on the half-line $[0,\infty)$ between the "uniform" norm $\sup\{|f(x)|e^{-x/2}\colon x\in[0,\infty)\}$ and the norm $\left(\int_0^\infty \left|f(x)e^{-x/2}\right|^q x^\alpha dx\right)^{1/q}$ of the space $\mathcal{L}^q_\alpha$ with the Laguerre weight for $1\le q<\infty$ and $\alpha\ge 0$. It is proved that the polynomial with a fixed leading coefficient that deviates least from zero in the space $\mathcal{L}^q_{\alpha+1}$ is the unique extremal polynomial in the Nikol'skii inequality. To prove this result, we use the Laguerre translation. The properties of the norm of the Laguerre translation in the spaces $\mathcal{L}^q_\alpha$ are studied.

Keywords: algebraic polynomial, Nikol'skii inequality, Laguerre weight, Laguerre translation

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1. Nikol'skii inequality

1.1. Notation. Statement and discussion of the problems

For $1 \le q < \infty$ and $\alpha > -1$, denote by $\mathcal{L}^q_\alpha = \mathcal{L}^q_{x^\alpha}(\mathbb{R}_+)$ the space of complex-valued Lebesgue measurable functions f on the half-line $\mathbb{R}_+ = [0, \infty)$ and such that the function $\left|f(x)e^{-x/2}\right|^q x^\alpha$ is integrable over $\mathbb{R}_+$. The space $\mathcal{L}^q_\alpha$ is equipped with the norm

$$\|f\|^*_{q,\alpha} = \|f\|_{\mathcal{L}^q_\alpha} = \left(\int_0^\infty \left|f(x)e^{-x/2}\right|^q x^\alpha dx\right)^{\frac{1}{q}}.$$

For $f \in \mathcal{L}^q_\alpha$, define

$$\tilde{f}(x) := f(x)e^{-x/2}. \tag{1.1}$$

With this notation, we have

$$\|f\|^*_{q,\alpha} = \left(\int_0^\infty \left|\tilde{f}(x)\right|^q x^\alpha dx\right)^{\frac{1}{q}} < \infty.$$

The space $\mathcal{L}^2_\alpha$ is a Hilbert space with the inner product

$$\langle f, g\rangle^* := \int_0^\infty f(x)\overline{g}(x)e^{-x}x^\alpha dx = \int_0^\infty \tilde{f}(x)\tilde{\overline{g}}(x)x^\alpha dx, \qquad f, g \in \mathcal{L}^2_\alpha. \tag{1.2}$$

In the case $q = \infty$ ($\alpha > -1$), we assume that $\mathcal{L}^\infty = \mathcal{L}^\infty(\mathbb{R}_+)$ is the space of measurable functions f such that product (1.1) is essentially bounded on $\mathbb{R}_+$. This space is equipped with the norm

$$\|f\|^*_\infty = \operatorname{ess\,sup}\{|e^{-x/2}f(x)| \colon x \in (0, \infty)\}.$$

Let $\mathscr{P}_n = \mathscr{P}_n(\mathbb{C})$, $n \ge 0$, be the set of univariate algebraic polynomials of degree (at most) n with complex coefficients. Denote by $M_n = M(n, q, \alpha)$ the best (i.e. the smallest possible) constant in the inequality

$$\|p_n\|^*_\infty \le M_n \|p_n\|^*_{q,\alpha}, \quad p_n \in \mathscr{P}_n. \tag{1.3}$$

The aim of the present paper is to study the extremal polynomials in inequality (1.3), i.e., polynomials $\rho_n \in \mathscr{P}_n$, $\rho_n \not\equiv 0$, for which this inequality becomes an equality. In particular, we will study the uniqueness of extremal polynomials. It is clear that if a polynomial ρ_n is extremal, then the polynomial $c\rho_n$ for any constant $c \ne 0$ is also extremal. If ρ_n is an extremal polynomial in inequality (1.3) and any other extremal polynomial has the form $c\rho_n$, $c \neq 0$, then we will say that ρ_n is the unique extremal polynomial in inequality (1.3).

Inequality (1.3) is a specific case of inequalities between different metrics or Nikol'skii inequalities. Such inequalities appeared for the first time in Nikol'skii's paper [13] and, a short time later, in a paper of Szegő and Zygmund [17]. Similar inequalities and, more generally, inequalities between the uniform norm and integral norms with weights of derivatives of algebraic polynomials and the polynomials themselves were studied over a period of more than 150 years. Much information and further references on this subject can be found in monographs [9, 11] and paper [12].

Along with inequality (1.3), we consider an auxiliary inequality

$$|p_n(0)| \leq D_n \, \|p_n\|^*_{q,\alpha}, \quad p_n \in \mathscr{P}_n, \tag{1.4}$$

with the best constant $D_n = D(n, q, \alpha)$. This inequality is also of independent interest. It is clear that $D_n \leq M_n$. We will show below that, in fact, $D_n = M_n$ at least for $\alpha \geq 0$.

1.2. Polynomials that deviate least from zero

Denote by $\varrho_n = \varrho_{n,q,\alpha+1}$ the polynomial of degree n with "unit" leading coefficient that deviates least from zero in the space $\mathcal{L}^q_{\alpha+1}$. More exactly, the polynomial ϱ_n is a solution of the problem

$$\min\{\|p_n\|_{\mathcal{L}^q_{\alpha+1}} : p_n \in \mathscr{P}^1_n\} = \|\varrho_n\|_{\mathcal{L}^q_{\alpha+1}}, \tag{1.5}$$

where $\mathscr{P}^1_n$ is the set of polynomials $p_n(x) = (-1)^n x^n + \sum_{k=0}^{n-1} a_k x^k$ of degree n with leading coefficient $(-1)^n$.

Polynomials that deviate least from zero appeared for the first time in studies of Chebyshev. He found [4] the polynomial with fixed leading coefficient that deviates least from zero in the space $C[-1, 1]$. At present, this polynomial is called the Chebyshev polynomial of the first kind. Korkin and Zolotarev [7] solved a similar problem in $L(-1, 1)$, where the extremal polynomial is the Chebyshev polynomial of the second kind. By now, there are many studies devoted to this subject area, see monographs [9, 11] and the references therein and in [1, 2]. Problem (1.5) (in itself) was studied by Mhaskar and Saff [10]; under some conditions on the parameters of the problem, they found the asymptotic behavior of quantity (1.5) and the limiting distribution of zeros of the extremal polynomials.

1.3. Main result

The following statement is the main result of the present paper.

Theorem 1. *For $\alpha \geq 0,\ 1 \leq q < \infty,$ and $n \geq 1,$ the following statements are valid.*

(1) *The best constants in inequalities* (1.3) *and* (1.4) *coincide*:

$$M(n, q, \alpha) = D(n, q, \alpha). \tag{1.6}$$

(2) *The polynomial $\varrho_{n,q,\alpha+1}$ that deviates least from zero with respect to the norm of the space $\mathcal{L}^q_{\alpha+1}$ is the unique* (*up to a constant factor*) *extremal polynomial in both inequalities* (1.3) *and* (1.4).

(3) *The polynomial $\varrho_{n,q,\alpha+1}$ and hence any polynomial p_n that is extremal in inequality* (1.3) *have the property that the uniform norm of the function $\widetilde{p}_n(x) = p_n(x)e^{-x/2}$ on the half-line $[0, \infty)$ is attained only at the point $x = 0$.*

An essential step in the proof of Theorem 1 is to prove the fact that, for a polynomial p_n extremal in inequality (1.3), the product $\widetilde{p}_n(x) = p_n(x)e^{-x/2}$ attains its uniform norm only at the endpoint $x = 0$ of the half-line $[0, \infty)$. To prove this fact, we use the generalized translation operator associated with the so-called Laguerre weight. It is important to know the properties of the norm of this operator in the space $\mathcal{L}^q_\alpha$; the second section of the paper is devoted to these issues.

Inequalities similar to (1.3) with ultraspherical weight and Jacobi weight were studied in [1, 2], respectively. Analogs of theorem 1 were obtained there. To prove them, we applied the generalized translation associated with a Jacobi weight. The study of inequality(1.3) in the present paper has essential peculiarities. The reason is, in particular, in the fact that, in contrast to the Jacobi translation, the Laguerre translation is not a positive operator.

Theorem 1 reduces the problem of studying inequality (1.3) to studying problem (1.5), which, in our opinion, is considerably simpler. For example, a solution of problem (1.5) and hence of problems (1.3) and (1.4) can be found explicitly; this will be discussed in Subsection 1.4.

1.4. Laguerre polynomials

Let $\{L_\nu^{(\alpha)}\}_{\nu=0}^\infty$ for $\alpha > -1$ be a system of Laguerre polynomials [15, Ch. V, Sect. 5.1, (5.1.5), (5.1.7)]

$$\begin{cases} L_\nu^{(\alpha)}(x) = \dfrac{1}{\nu!} e^x x^{-\alpha} \dfrac{d^\nu}{dx^\nu}(e^{-x} x^{\alpha+\nu}), & x > 0; \\ L_\nu^{(\alpha)}(0) = A_\nu^\alpha = \dbinom{\nu+\alpha}{\nu} = \dfrac{\Gamma(\nu+\alpha+1)}{\nu!\,\Gamma(\alpha+1)}, & x = 0, \end{cases}$$

orthogonal on $\mathbb{R}_+$ with respect to inner product (1.2). In addition [15, Ch. V, Sect. 5.1, (5.1.1)],

$$\langle L_\nu^{(\alpha)}, L_\nu^{(\alpha)} \rangle^* = \|L_\nu^{(\alpha)}\|_{\mathcal{L}_\alpha^2}^2 = \Gamma(\alpha+1) \binom{\nu+\alpha}{\nu} = \frac{\Gamma(\nu+\alpha+1)}{\nu!}.$$

The system of Laguerre polynomials $\{L_\nu^{(\alpha)}\}_{\nu=0}^\infty$ forms an orthogonal basis in the space $\mathcal{L}_\alpha^2$. Thus, an arbitrary function $f \in \mathcal{L}_\alpha^2$ is expanded into Fourier–Laguerre series

$$f(x) = \sum_{\nu=0}^\infty f_\nu L_\nu^{(\alpha)}(x), \quad f_\nu = \frac{\langle f, L_\nu^{(\alpha)} \rangle^*}{\langle L_\nu^{(\alpha)}, L_\nu^{(\alpha)} \rangle^*}. \tag{1.7}$$

For two functions $f, g \in \mathcal{L}_\alpha^2$, the generalized version of Parseval's identity holds:

$$\langle f, g \rangle^* = \sum_{\nu=0}^\infty \delta_\nu f_\nu \overline{g}_\nu, \quad \delta_\nu = \langle L_\nu^{(\alpha)}, L_\nu^{(\alpha)} \rangle^* = \|L_\nu^{(\alpha)}\|_{\mathcal{L}_\alpha^2}^2.$$

In particular, the norm of a function $f \in \mathcal{L}_\alpha^2$ can be expressed in terms of its Fourier–Laguerre coefficients $\{f_\nu\}$ by Parseval's identity:

$$\|f\|_{\mathcal{L}_\alpha^2}^2 = \sum_{\nu=0}^\infty \delta_\nu \, |f_\nu|^2. \tag{1.8}$$

Using Parseval's identity, it is not hard to solve problems (1.4) and (1.5) for $q = 2$ explicitly. Consider the pointwise inequality

$$|p_n(z)| \le D_n(z) \, \|p_n\|_{\mathcal{L}_\alpha^2}, \quad p_n \in \mathscr{P}_n,$$

for a fixed $z \in \mathbb{C}$ with the best constant $D_n(z) = D(n, q, \alpha; z)$, which is more general than inequality (1.4). There are many studies devoted to such inequalities, see monographs [9, Ch. 4], [11, Sect. 6.1], [15, Sect. 7.71].

A polynomial $p_n \in \mathscr{P}_n$ can be represented in the form of linear combination of Laguerre polynomials:

$$p_n(x) = \sum_{\nu=0}^{n} c_\nu L_\nu^{(\alpha)}(x).$$

Using the Cauchy–Bunyakovskii inequality, we obtain

$$\begin{aligned} |p_n(z)| &= \sum_{\nu=0}^{n} |c_\nu||L_\nu^{(\alpha)}(z)| = \sum_{\nu=0}^{n} \left(\sqrt{\delta_\nu}|c_\nu|\right)\left(\frac{1}{\sqrt{\delta_\nu}}|L_\nu^{(\alpha)}(z)|\right) \\ &\le \left(\sum_{\nu=0}^{n} \delta_\nu |c_\nu|^2\right)^{1/2} \left(\sum_{\nu=0}^{n} \delta_\nu^{-1} |L_\nu^{(\alpha)}(z)|^2\right)^{1/2}. \end{aligned}$$

Hence, for any $z \in \mathbb{C}$, the formula

$$|D_n(z)|^2 = \frac{1}{\Gamma(\alpha+1)} \sum_{\nu=0}^{n} \binom{\nu+\alpha}{\nu}^{-1} |L_\nu^{(\alpha)}(z)|^2$$

holds and the unique (up to a constant factor) extremal polynomial is

$$\rho_n(x) = \rho_n^{(\alpha)}(x;z) = \sum_{\nu=0}^{n} \binom{\nu+\alpha}{\nu}^{-1} L_\nu^{(\alpha)}(z) L_\nu^{(\alpha)}(x), \tag{1.9}$$

which, by the formula $\rho_n^{(\alpha)}(x;z) = \Gamma(\alpha+1)\mathcal{K}_n^{(\alpha)}(x;z)$, is expressed in terms of the Christoffel–Darboux kernel $\mathcal{K}_n^{(\alpha)}$ for Laguerre polynomials [15, Sect. 5.1, (5.1.11)].

Let $z = 0$. In this case, polynomial (1.9) takes the form

$$\rho_n(x) = \rho_n^{(\alpha)}(x;0) = \sum_{\nu=0}^{n} L_\nu^{(\alpha)}(x);$$

for this polynomial, the following formula holds [15, Sect. 5.1, (5.1.13)]:

$$\rho_n(x) = \rho_n^{(\alpha)}(x;0) = L_n^{(\alpha+1)}(x). \tag{1.10}$$

The solution of problem (1.5) for $q = 2$ is the polynomial $\varrho_{n,2,\alpha+1} = n!\, L_n^{(\alpha+1)}$, which differs from (1.10) only by the normalizing factor. This fact is the point of Theorem 1 on the connection between problems (1.4) and (1.5) for $q = 2$. This connection for all values of the parameter $1 \le q < \infty$ will be discussed in Theorem 3.

2. Laguerre translation

2.1. Product formula for Laguerre polynomials

Watson [18, p. 21] obtained the following formula for Laguerre polynomials with $\alpha > -1/2$:

$$\frac{n! L_n^{(\alpha)}(x) L_n^{(\alpha)}(t)}{\Gamma(n+\alpha+1)} = \frac{1}{\sqrt{\pi}} \int_0^\pi L_n^{(\alpha)}(x+t+2\sqrt{xt}\cos\theta)\Phi^{(\alpha)}(x,t,\theta)\,d\theta, \tag{2.1}$$

$$\Phi^{(\alpha)}(x,t,\theta) = \begin{cases} \exp(-\sqrt{xt}\cos\theta) \times \dfrac{J_{\alpha-1/2}(\sqrt{xt}\sin\theta)}{\left(\frac{1}{2}\sqrt{xt}\sin\theta\right)^{\alpha-1/2}} \sin^{2\alpha}\theta, & xt > 0, \\ \dfrac{1}{\Gamma\left(\alpha + \dfrac{1}{2}\right)} \sin^{2\alpha}\theta, & xt = 0, \end{cases}$$

where $J_{\alpha-1/2}$ is the Bessel function of order $\alpha - 1/2$. Since (see, for example [19, Ch. III, Sect. 3.4, (6)])

$$J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos z, \tag{2.2}$$

formula (2.1) for $\alpha = 0$ takes the form

$$L_n^{(0)}(x) L_n^{(0)}(t) = \frac{1}{\pi} \int_0^\pi L_n^{(0)}(x+t+2\sqrt{xt}\cos\theta)\Phi^{(0)}(x,t,\theta)\,d\theta, \tag{2.3}$$

$$\Phi^{(0)}(x,t,\theta) = \exp(-\sqrt{xt}\cos\theta)\cos(\sqrt{xt}\sin\theta).$$

As mentioned in [18, p. 19] and [3, Sect. 1], formula (2.3) was obtained earlier by G.Hardy (approximately, in 1934). For $\alpha > 0$, using the representation of the Bessel function as the Poisson integral [19, Ch. III, Sect. 3.3, (1)]

$$J_{\alpha-\frac{1}{2}}(\xi) = \frac{\left(\frac{\xi}{2}\right)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\,\Gamma\left(\frac{1}{2}\right)} \int_0^\pi \cos(\xi\cos\psi)\sin^{2\alpha-1}\psi\,d\psi, \tag{2.4}$$

we can write formula (2.1) in the following form [18, p. 21]:

$$\frac{n! L_n^{(\alpha)}(x) L_n^{(\alpha)}(t)}{\Gamma(n+\alpha+1)} = \frac{1}{\pi\Gamma(\alpha)} \int_0^\pi \int_0^\pi L_n^{(\alpha)}(x+t+2\sqrt{xt}\cos\theta)\Psi^{(\alpha)}(x,t,\theta,\psi)\,d\psi\,d\theta, \tag{2.5}$$

$$\Psi^{(\alpha)}(x,t,\theta,\psi) = \exp(-\sqrt{xt}\cos\theta)\cos(\sqrt{xt}\sin\theta\cos\psi) \times \sin^{2\alpha}\theta \times \sin^{2\alpha-1}\psi.$$

Each of formulas (2.1), (2.3), and (2.5) may be called a product formula for Laguerre polynomials. Product formulas have several applications. In particular, Watson [18, p. 21], using a product formula, obtained the following estimate for Laguerre polynomials for $\alpha \geq 0$:

$$e^{-x/2}\left|L_n^{(\alpha)}(x)\right| \leq \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}, \quad x \geq 0. \tag{2.6}$$

This estimate for $\alpha = 0$ was obtained earlier by Szegő by another method [16, Sect. 2, p. 343]; moreover, he proved that this estimate is strict for $x > 0$.

Using Watson's ideas [18, p. 21], we can rectify inequality (2.6). We will do this in terms of the polynomials

$$R_n^{(\alpha)}(x) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} = \frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x). \tag{2.7}$$

Inequality (2.6) for polynomials (2.7) takes the form

$$e^{-x/2}\left|R_n^{(\alpha)}(x)\right| \leq 1, \quad x \geq 0. \tag{2.8}$$

For $\alpha \geq 0$, define a function $u(x) = u^{(\alpha)}(x)$, $x \geq 0$, as follows. For $\alpha = 0$, we set

$$u(x) = u^{(0)}(x) = \frac{1}{\pi}\int_0^{\pi} |\cos(x\sin\theta)|\,d\theta; \tag{2.9}$$

for $\alpha > 0$, we set

$$u(x) = u^{(\alpha)}(x) = \frac{\alpha}{\pi}\int_0^{\pi}\int_0^{\pi} |\cos(x\sin\theta\cos\psi)|\sin^{2\alpha}\theta\sin^{2\alpha-1}\psi\,d\psi\,d\theta. \tag{2.10}$$

The function $u^{(\alpha)}$ has the property

$$u^{(\alpha)}(0) = 1; \quad 0 < u^{(\alpha)}(x) < 1, \quad x > 0. \tag{2.11}$$

Indeed, for $\alpha > 0$ and $x > 0$, based on definition (2.10), we have

$$0 < u^{(\alpha)}(x) < u^{(\alpha)}(0) = \frac{\alpha}{\pi}\int_0^{\pi}\int_0^{\pi} \sin^{2\alpha}\theta\sin^{2\alpha-1}\psi\,d\psi\,d\theta = 1.$$

In the case $\alpha = 0$, property (2.11) easily follows from definition (2.9). Statement (2.11) is verified.

Lemma 1. *For $\alpha \geq 0$ and all $n \geq 0$, the following pointwise estimate holds*:

$$e^{-x/2}\left|R_n^{(\alpha)}(x)\right| \leq \sqrt{u^{(\alpha)}(x)}, \quad x \geq 0; \tag{2.12}$$

Proof is implemented by means of arguments used by Watson [18, p. 21] for the proof of estimate (2.6). Let $\alpha > 0$. We put $t = x$ in (2.5), multiply the obtained relation by e^{-x}, and pass from $L_n^{(\alpha)}$ to $R_n^{(\alpha)}$ by formula (2.7). As a result, we obtain

$$\begin{gathered}\left(e^{-x/2}R_n^{(\alpha)}(x)\right)^2 \\ = \frac{\alpha}{\pi}\int_0^\pi\int_0^\pi R_n^{(\alpha)}\big(2x(1+\cos\theta)\big)\exp\big(-x(1+\cos\theta)\big)V^{(\alpha)}(x,\theta,\psi)\,d\psi\,d\theta, \\ V^{(\alpha)}(x,\theta,\psi) = \cos(x\sin\theta\cos\psi)\times\sin^{2\alpha}\theta\times\sin^{2\alpha-1}\psi.\end{gathered} \tag{2.13}$$

By (2.8), the estimate $|R_n^{(\alpha)}(2x(1+\cos\theta))\exp(-x(1+\cos\theta))| \leq 1$ is valid. Therefore, (2.13) implies the estimate

$$\left(e^{-x/2}R_n^{(\alpha)}(x)\right)^2 \leq \frac{\alpha}{\pi}\int_0^\pi\int_0^\pi |V^{(\alpha)}(x,\theta,\psi)|\,d\psi\,d\theta = u^{(\alpha)}(x), \quad x \geq 0,$$

which yields (2.12).

Property (2.12) for $\alpha = 0$ can be verified by the same scheme based on product formula (2.3) and definition (2.9). The lemma is proved. □

2.2. Laguerre translation in the spaces $\mathcal{L}_\alpha^q$, $1 \leq q \leq \infty$, $\alpha \geq 0$.

Product formula (2.1) serves as a basis for the definition of the generalized translation operator associated with the Laguerre weight or shortly the Laguerre translation for $\alpha > -1/2$. Let us write it in the following equivalent form:

$$L_n^{(\alpha)}(x)R_n^{(\alpha)}(t) = \frac{1}{\sqrt{\pi}}\int_0^\pi L_n^{(\alpha)}(x+t+2\sqrt{xt}\cos\theta)W^{(\alpha)}(x,t,\theta)\,d\theta, \tag{2.14}$$

$$W^{(\alpha)}(x,t,\theta) = \Gamma(\alpha+1)\Phi^{(\alpha)}(x,t,\theta); \quad x,t \geq 0.$$

Based on representation (2.14), we call the operator T_t^α, defined for $\alpha > -1/2$ by the formula

$$T_t^\alpha(f;x) = \frac{1}{\sqrt{\pi}}\int_0^\pi f(x+t+2\sqrt{xt}\cos\theta)W^{(\alpha)}(x,t,\theta)\,d\theta \tag{2.15}$$

the *Laguerre translation with step* $t \in [0, \infty)$. For the properties of this operator, see [6] and the references therein. The Laguerre translation operator for $t = 0$ is the identity operator.

An important tool for studying the operator T_t^α is the following integral representation obtained in [6, (2.2)] for $\alpha > -1/2$:

$$T_t^\alpha(f; x) = \int_0^\infty f(z)K(x, t, z)e^{-z}z^\alpha dz \tag{2.16}$$

with kernel

$$K(x, t, z) = \begin{cases} \dfrac{C_\alpha}{(xtz)^\alpha} e^{(x+t+z)/2} J_{\alpha-\frac{1}{2}}(r(x, t, z))r^{\alpha-\frac{1}{2}}(x, t, z), & z \in I(x, t), \\ 0, & z \notin I(x, t), \end{cases} \tag{2.17}$$

where

$$I(x, t) = \Big((\sqrt{x} - \sqrt{t})^2, (\sqrt{x} + \sqrt{t})^2\Big),$$
$$r(x, t, z) = \frac{1}{2}\sqrt{2(xt + xz + tz) - x^2 - t^2 - z^2},$$
$$C_\alpha = \frac{\Gamma(\alpha + 1)2^{\alpha-1}}{\sqrt{2\pi}}. \tag{2.18}$$

The kernel $K(x, t, z)$ is symmetric in each of its variables [6, p. 164].

By means of (2.16), it is proved in [6] that, for $\alpha \geq 0$, $1 \leq q \leq \infty$, and $t \geq 0$, the operator T_t^α is a bounded linear operator in the space $\mathcal{L}_\alpha^q$ and the following estimate is valid for the norm $\|T_t^\alpha\|_{q,\alpha}^* = \|T_t^\alpha\|_{\mathcal{L}_\alpha^q \to \mathcal{L}_\alpha^q}$ of this operator [6, Theorem 1]:

$$\|T_t^\alpha\|_{q,\alpha}^* \leq e^{t/2}. \tag{2.19}$$

Moreover, the following relation holds for $q = 1$ [6, Corollary 3]:

$$\|T_t^\alpha\|_{1,\alpha}^* = e^{t/2}. \tag{2.20}$$

Let us rectify statements (2.19) and (2.20) with the aim of their further application in Lemmas 2 and 3 below. By definition (2.15) and formula (2.14), we have

$$T_t^\alpha(L_n^{(\alpha)}; x) = L_n^{(\alpha)}(x)R_n^{(\alpha)}(t). \tag{2.21}$$

Hence, we obtain the following lower estimate for the norm of the operator T_t^α (for $\alpha > -1$ and $1 \leq q \leq \infty$):

$$\|T_t^\alpha\|_{q,\alpha}^* \geq \sup\{|R_n^{(\alpha)}(t)| \colon n \geq 0\}. \tag{2.22}$$

For $q = 2$ $(\alpha \geq 0)$, there is an equality in (2.22):

$$\|T_t^\alpha\|_{2,\alpha}^* = \sup\{|R_n^{(\alpha)}(t)|\colon n \geq 0\}. \tag{2.23}$$

Indeed, based on relation (2.21), the linearity and boundedness of the operator T_t^α, using the Fourier–Laguerre expansion of functions $f \in \mathcal{L}_\alpha^2$, we can write the operator T_t^α as the series

$$T_t^\alpha(f;x) = \sum_{\nu=0}^{\infty} f_\nu L_\nu^{(\alpha)}(x) R_\nu^{(\alpha)}(t). \tag{2.24}$$

Note that, sometimes, it is this relation taken as a definition of the Laguerre translation. Now, (twice) applying Parseval's identity (1.8), we obtain

$$\|T_t^\alpha f\|_{\mathcal{L}_\alpha^2}^2 = \sum_{\nu=0}^{\infty} \delta_\nu\,|f_\nu|^2 |R_\nu^{(\alpha)}(t)|^2 \leq \sup\{|R_n^{(\alpha)}(t)|^2\colon n \geq 0\} \times \sum_{\nu=0}^{\infty} \delta_\nu\,|f_\nu|^2$$

$$= \sup\{|R_n^{(\alpha)}(t)|^2\colon n \geq 0\} \times \|f\|_{\mathcal{L}_\alpha^2}^2.$$

This and (2.22) imply (2.23).

Lemma 2. *For* $1 < q < \infty,\ \alpha \geq 0,$ *and* $t > 0$, *the following strict inequality holds*:

$$\|T_t^\alpha\|_{q,\alpha}^* < e^{t/2}. \tag{2.25}$$

Proof. Inequality (2.25) for $q = 2$ follows from (2.23), Lemma 1, and (2.11). Now, to prove the statements of the lemma for $1 < q < 2$ and $2 < q < \infty$, we have to use M. Riesz' theorem on the convexity of linear operators (see, for example, [14, Ch. V, Sect. 1, Theorem 1.3] or [5, Ch. VI, Sect. 10, Theorem 11]) and estimates (2.19) for $q = 1$ and $q = \infty$, respectively. $\square$

For $q = 1$, in addition to (2.20), the following statement holds.

Lemma 3. *For* $\alpha \geq 0$ *and all* $t > 0$, *the norm of the operator* T_t^α *in the space* $\mathcal{L}_\alpha^1$ *is not attained.*

Proof. To prove the lemma, we will partially repeat the proof of the estimate $\|T_t^\alpha\|_{1,\alpha}^* \le e^{t/2}$ from [6, p. 165], implementing some steps more informatively. For any function $f \in \mathcal{L}_\alpha^1$, by (2.16), we have

$$\|T_t^\alpha f\|_{1,\alpha}^* = \int_0^\infty |T_t^\alpha(f;x)| e^{-x/2} x^\alpha dx$$
$$\le \int_0^\infty \left(\int_0^\infty |f(z)||K(x,t,z)| e^{-z} z^\alpha\, dz \right) e^{-x/2} x^\alpha dx$$
$$= \int_0^\infty |f(z)| e^{-z/2} z^\alpha \left(e^{-z/2} \int_0^\infty |K(x,t,z)|\, e^{-x/2} x^\alpha dx \right) dz.$$

Thus, the inequality

$$\|T_t^\alpha f\|_{1,\alpha}^* \le \int_0^\infty |f(z)| e^{-z/2} z^\alpha\, \Omega(t,z)\, dz \tag{2.26}$$

is valid, where

$$\Omega(t,z) = e^{-z/2} \int_0^\infty |K(x,t,z)|\, e^{-x/2} x^\alpha dx. \tag{2.27}$$

Let us transform and estimate function (2.27). By the invariance property of kernel (2.17), we have

$$\Omega(t,z) = e^{-z/2} \int_{\left(\sqrt{z}-\sqrt{t}\right)^2}^{\left(\sqrt{z}+\sqrt{t}\right)^2} |K(z,t,x)|\; e^{-x/2} x^\alpha dx$$

$$= C_\alpha \int_{\left(\sqrt{z}-\sqrt{t}\right)^2}^{\left(\sqrt{z}+\sqrt{t}\right)^2} \frac{e^{-z/2}}{(xtz)^\alpha} e^{(x+t+z)/2} \left| J_{\alpha-\frac{1}{2}}(r(z,t,x)) \right| r^{\alpha-\frac{1}{2}}(z,t,x)\, e^{-x/2} x^\alpha dx$$

$$= C_\alpha e^{t/2} \int_{\left(\sqrt{z}-\sqrt{t}\right)^2}^{\left(\sqrt{z}+\sqrt{t}\right)^2} \frac{1}{(tz)^\alpha} \left| J_{\alpha-\frac{1}{2}}(r(z,t,x)) \right| r^{\alpha-\frac{1}{2}(z,t,x)}\, dx. \tag{2.28}$$

Following [6, Sect. 2], we pass from the variable x to the variable θ in the latter integral by the formula

$$x = x(\theta) = z + t + 2\sqrt{zt}\, \cos\theta, \quad \theta \in [0,\pi]. \tag{2.29}$$

We have

$$\sqrt{zt}\,\sin\theta = \sqrt{zt(1-\cos^2\theta)} = \sqrt{zt - \left(\frac{x-z-t}{2}\right)^2} = r(z,t,x).$$

In addition, (2.29) implies $dx = -2\sqrt{zt}\,\sin\theta\,d\theta$. Substituting these relations into (2.28), we obtain

$$\Omega(t,z) = 2C_\alpha e^{t/2}\int_0^\pi \frac{1}{(tz)^\alpha}\left|J_{\alpha-\frac{1}{2}}(\sqrt{zt}\,\sin\theta)\right|(\sqrt{zt}\,\sin\theta)^{\alpha+\frac{1}{2}}\,d\theta. \qquad (2.30)$$

For $\alpha > 0$, using (2.4) and (2.18) in (2.30), we obtain the representation

$$\Omega(t,z) = \frac{\alpha}{\pi}\,e^{t/2}\int_0^\pi \left|\int_0^\pi \cos(\sqrt{zt}\,\sin\theta\cos\psi)\sin^{2\alpha-1}\psi\,d\psi\right|\sin^{2\alpha}\theta\,d\theta. \quad (2.31)$$

Representations (2.31) and (2.10) imply the estimate

$$\Omega(t,z) \le e^{t/2}\,u^{(\alpha)}(\sqrt{zt}).$$

Substituting this estimate into (2.26), we obtain

$$\|T_t^\alpha f\|_{1,\alpha}^* \le e^{t/2}\int_0^\infty |f(z)|e^{-z/2}z^\alpha\,u^{(\alpha)}(\sqrt{zt})\,dz,$$

By property (2.11), the following inequality is valid:

$$\int_0^\infty |f(z)|e^{-z/2}z^\alpha\,u^{(\alpha)}(\sqrt{zt})\,dz \le \int_0^\infty |f(z)|e^{-z/2}z^\alpha\,dz = \|f\|_{1,\alpha}^*. \qquad (2.32)$$

Moreover, if $t > 0$ and the function f is nonzero on a set of positive measure, the last inequality is strict; hence, the strict inequality $\|T_t^\alpha f\|_{1,\alpha}^* < \|f\|_{1,\alpha}^*$ holds. The lemma for $\alpha > 0$ is proved.

By (2.2) and (2.18), representation (2.30) for $\alpha = 0$ takes the form

$$\Omega(t,z) = \frac{e^{t/2}}{\pi}\int_0^\pi \left|\cos(\sqrt{zt}\,\sin\theta)\right|\,d\theta = e^{t/2}\,u^{(0)}(\sqrt{zt}).$$

Hence, as in the case $\alpha > 0$, we conclude that, for $\alpha = 0$ and $t > 0$, there is no function at which the norm of the Laguerre translation operator in the space $\mathcal{L}_0^1$ is attained. The lemma is proved completely. $\square$

2.3. A modified Laguerre translation

For $1 \le q < \infty$ and $\alpha > -1$, denote by $L^q_\alpha = L^q_{x^\alpha}(\mathbb{R}_+)$ the set of complex-valued Lebesgue measurable functions f on the half-line $\mathbb{R}_+$ such that the integral in the relation

$$\|f\|_{q,\alpha} = \|f\|_{L^q_\alpha} := \left(\int_0^\infty |f(x)|^q\, x^\alpha dx\right)^{\frac{1}{q}}, \qquad (2.33)$$

which defines the norm of the space L^q_α, converges. In the case $q = \infty$, we assume that $L^\infty_\alpha = L^\infty(0,\infty)$ is the space of measurable functions f essentially bounded on $\mathbb{R}_+$. This space is equipped with the norm

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(x)|\colon x \in (0,\infty)\}. \qquad (2.34)$$

The mapping

$$f \in \mathcal{L}^q_\alpha \to \tilde{f}(x) := f(x)e^{-x/2} \qquad (2.35)$$

is the bijection between the spaces $\mathcal{L}^q_\alpha$ and L^q_α; in addition, this mapping preserves the norm:

$$\|f\|^*_{q,\alpha} = \|\tilde{f}\|_{q,\alpha}.$$

An analog of translation operator (2.15) in the spaces L^q_α was introduced and studied in [6]. In [6], this operator was denoted by τ^α_t. Here, we denote this operator by Θ^α_t and also call it the Laguerre translation operator. It is defined by the formula [6, (2.5)]

$$\Theta^\alpha_t(f;x) := T^\alpha_t(\check{f};x)e^{-(t+x)/2}, \quad \check{f}(z) = f(z)e^{z/2}. \qquad (2.36)$$

The operator Θ^α_t is a bounded linear operator in the spaces L^q_α for all $1 \le q \le \infty$ and $\alpha \ge 0$. Moreover, the norm $\|\Theta^\alpha_t\|_{q,\alpha} = \|\Theta^\alpha_t\|_{L^q_\alpha \to L^q_\alpha}$ of this operator in the space L^q_α is connected with the norm of the operator T^α_t in $\mathcal{L}^q_\alpha$ by the relation

$$\|\Theta^\alpha_t\|_{q,\alpha} = e^{-t/2}\|T^\alpha_t\|^*_{q,\alpha}. \qquad (2.37)$$

Indeed,

$$\begin{aligned}\|\Theta^\alpha_t\|_{q,\alpha} &= \sup\{\|\Theta^\alpha_t f\|_{q,\alpha}\colon \|f\|_{L^q_\alpha} \le 1\} = e^{-t/2}\sup\{\|T^\alpha_t(\check{f})\|^*_{q,\alpha}\colon \|f\|_{L^q_\alpha} \le 1\}\\ &= e^{-t/2}\sup\{\|T^\alpha_t(\check{f})\|^*_{q,\alpha}\colon \|\check{f}\|_{\mathcal{L}^q_\alpha} \le 1\} = e^{-t/2}\|T^\alpha_t\|^*_{q,\alpha}.\end{aligned}$$

In what follows, we will use version (2.36) of the Laguerre translation. Statements (2.19) and (2.20) proved in [6] for $\alpha \ge 0$ can be written in the forms

$$\|\Theta^\alpha_t\|_{q,\alpha} \le 1 \qquad \text{for} \quad 1 \le q \le \infty; \qquad (2.38)$$

$$\|\Theta_t^\alpha\|_{1,\alpha} = 1 \qquad \text{for} \quad q = 1. \tag{2.39}$$

According to Lemma 2, the following strict inequality is valid for $1 < q < \infty$ and $t > 0$:

$$\|\Theta_t^\alpha\|_{q,\alpha} < 1.$$

By Lemma 3, in addition to (2.39), we can assert that, for $t > 0$, the norm of the operator Θ_t^α in the space L_α^1 is not attained.

3. Modified versions of the problems. Proof of the main results

3.1. Reformulation of the problems

For a polynomial $p_n \in \mathscr{P}_n$, we will call the function $\widetilde{p}_n(x) = e^{-x/2}p_n(x)$ an *e-polynomial* (of degree n). Denote by $\widetilde{\mathscr{P}_n}$ the set of all e-polynomials of degree n. Let us write the main problems in terms of norms (2.33) and (2.34) for e-polynomials. Inequalities (1.3) and (1.4) are equivalent to the inequalities

$$\|g_n\|_\infty \le M_n\, \|g_n\|_{q,\alpha}, \quad g_n \in \widetilde{\mathscr{P}_n}, \tag{3.1}$$

$$|g_n(0)| \le D_n\, \|g_n\|_{q,\alpha}, \quad g_n \in \widetilde{\mathscr{P}_n}. \tag{3.2}$$

Consider the set $\widetilde{\mathscr{P}_n^1} = \{g_n = \widetilde{p}_n,\ p_n \in \mathscr{P}_n^1\}$ of e-polynomials of degree n with the "unit" leading coefficient. Problem (1.5) is equivalent to the problem on the e-polynomial $g_n^* = g_{n,q,\alpha+1}^* \in \widetilde{\mathscr{P}_n^1}$ that deviates least from zero with respect to the norm of the space $L_{\alpha+1}^q$, i.e., has the property

$$\min\left\{\|g_n\|_{L_{\alpha+1}^q} :\ g_n \in \widetilde{\mathscr{P}_n^1}\right\} = \|g_n^*\|_{q,\alpha+1}. \tag{3.3}$$

An e-polynomial $g_n^* = \widetilde{p}_n^*$ is a solution of (3.3) if and only if the polynomial $p_n^* \in \mathscr{P}_n^1$ is a solution of (1.5).

For us, it is convenient to study inequalities (3.1) and (3.2) and problem (3.3) instead of inequalities (1.3) and (1.4) and problem (1.5). In these new terms, Theorem 1 takes the following form.

Theorem 2. *For $\alpha \ge 0$, $1 \le q < \infty$, and $n \ge 1$, the following statements are valid.*

(1) *The best constants in inequalities* (3.1) *and* (3.2) *coincide*:

$$M(n,q,\alpha) = D(n,q,\alpha). \tag{3.4}$$

(2) *The e-polynomial* $g^*_{n,q,\alpha+1}$ *that deviates least from zero with respect to the norm of the space* $L^q_{\alpha+1}$ *is the unique extremal polynomial in both inequalities* (3.1) *and* (3.2)*.*

(3) *The e-polynomial* $g^*_{n,q,\alpha+1}$ *and hence any e-polynomial* g_n *extremal in inequality* (3.1) *attain their uniform norm on the half-line* $[0,\infty)$ *only at the point* $x = 0$*.*

3.2. The connection between inequality (3.2) *and problem* (3.3)

According to the next theorem, problems (3.2) and (3.3) have the same solution. Similar statement for an arbitrary weight on a finite interval was proved in [1]. Theorem 3 is proved by the same scheme; however, in this situation, the proof has peculiarities; therefore, we present it here.

Theorem 3. *For* $1 \le q < \infty$, $\alpha > -1$, *and* $n \ge 1$, *the e-polynomial* $g^*_n = g^*_{n,q,\alpha}$ *that is the solution of problem* (3.3) *is the unique extremal polynomial in inequality* (3.2)*.*

Proof. The characteristic property of the e-polynomial $g^*_n = g^*_{n,q,\alpha+1}$ extremal in problem (3.3) is that

$$\int_0^\infty |g^*_n(x)|^{q-1}\operatorname{sgn} g^*_n(x) g_{n-1}(x) x^{\alpha+1} dx = 0 \quad \text{for all} \quad g_{n-1} \in \widetilde{\mathscr{P}}_{n-1}; \tag{3.5}$$

see, for example, [8, Ch. 3, Sect. 3.3, Theorems 3.3.1, 3.3.2]. This property, in particular, implies that all n zeros of the e-polynomial g^*_n are simple and lie on the half-line $(0,\infty)$. This, in turn, implies that $g^*_n(0) > 0$.

An arbitrary e-polynomial $g_n \in \widetilde{\mathscr{P}}_n$ has the form $g_n(x) = e^{-x/2}p_n(x)$, where $p_n \in \mathscr{P}_n$. Let us represent a polynomial p_n in the form

$$p_n(x) = x r_{n-1}(x) + p_n(0), \quad r_{n-1}(x) = \frac{p_n(x) - p_n(0)}{x} \in \mathscr{P}_{n-1}.$$

Now, we have

$$\begin{aligned} &\int_0^\infty g_n(x)|g^*_n(x)|^{q-1} \left(\operatorname{sgn} g^*_n(x)\right)\, x^\alpha\, dx \\ &= \int_0^\infty r_{n-1}(x) e^{-x/2} |g^*_n(x)|^{q-1} \left(\operatorname{sgn} g^*_n(x)\right) x^{\alpha+1}\, dx \\ &\quad + p_n(0) \int_0^\infty e^{-x/2}|g^*_n(x)|^{q-1} \left(\operatorname{sgn} g^*_n(x)\right) x^\alpha dx. \end{aligned}$$

By (3.5), the next-to-last integral is zero. Consequently, for any e-polynomial $g_n \in \widetilde{\mathscr{P}}_n$, the following relation holds:

$$
\begin{aligned}
&\int_0^\infty g_n(x)|g_n^*(x)|^{q-1}\,(\operatorname{sgn} g_n^*(x))\ x^\alpha\,dx \\
&= g_n(0)\int_0^\infty e^{-x/2}|g_n^*(x)|^{q-1}\,(\operatorname{sgn} g_n^*(x))\,x^\alpha dx.
\end{aligned}
\tag{3.6}
$$

Let us determine the sign of the integral

$$
I(n,q,\alpha) = \int_0^\infty e^{-x/2}|g_n^*(x)|^{q-1}\,(\operatorname{sgn} g_n^*(x))\,x^\alpha dx.
$$

Substituting the e-polynomial $g_n = g_n^*$ into (3.6), we obtain the equality

$$
\int_0^\infty x^\alpha |g_n^*(x)|^q dx = g_n^*(0) I(n,q,\alpha). \tag{3.7}
$$

Since $g_n^*(0) > 0$, it follows from (3.7) that $I(n,q,\alpha) > 0$.

Relation (3.6) can be now written in the form

$$
g_n(0) = \frac{1}{I(n,q,\alpha)}\int_0^\infty g_n(x)|g_n^*(x)|^{q-1}\,(\operatorname{sgn} g_n^*(x))\,x^\alpha\,dx, \quad g_n \in \widetilde{\mathscr{P}}_n. \tag{3.8}
$$

From (3.8), using Hölder's inequality, we obtain for $g_n \in \widetilde{\mathscr{P}}_n$ the estimate

$$
|g_n(0)| \le \frac{1}{I(n,q,\alpha)}\left(\int_0^\infty x^\alpha |g_n(x)|^q dx\right)^{\frac{1}{q}}\left(\int_0^\infty x^\alpha |g_n^*(x)|^q dx\right)^{\frac{q-1}{q}}. \tag{3.9}
$$

At the e-polynomial g_n^*, inequality (3.9) turns into an equality; this can be easily verified, for example, with the use of identity (3.8). Consequently, inequality (3.9) is inequality (3.2); moreover,

$$
D(n,q,\alpha) = \frac{(\|g_n^*\|_{q,\alpha})^{q-1}}{I(n,q,\alpha)}.
$$

Based on the conditions under which Hölder's inequality turns into an equality, it is easy to conclude that, for $1 \le q < \infty$, inequality (3.9) turns into an equality only for the e-polynomials cg_n^*, where $c \in \mathbb{R}$. Thus, the e-polynomial g_n^* is the unique extremal polynomial in inequality (3.2). Theorem 3 is proved. □

3.3. Proof of Theorems 2 and 1

For the constants M_n and D_n in inequalities (3.1) and (3.2), the inequality $D_n \le M_n$ holds. Let us show that, in fact, the constants coincide, i.e., (3.4) holds. Let $f \in \widetilde{\mathscr{P}}_n$, and let the uniform norm of f be attained at some point $t \in [0, \infty)$. Consider the function $g(x) = \Theta_t^\alpha(f; x)$. The function $\check{f}(z) = f(z)e^{z/2}$ is a polynomial of degree n:

$$\check{f}(z) = \sum_{\nu=0}^{n} c_\nu L_\nu^{(\alpha)}(z).$$

By (2.24), the function

$$T_t^\alpha(\check{f}; x) = \sum_{\nu=0}^{n} c_\nu L_\nu^{(\alpha)}(x) R_\nu^{(\alpha)}(t) \tag{3.10}$$

is also a polynomial of degree n. Finally, by (2.36), we have

$$g(x) = \Theta_t^\alpha(f; x) = T_t^\alpha(\check{f}; x)e^{-(t+x)/2} = e^{-t/2}e^{-x/2} \sum_{\nu=0}^{n} c_\nu L_\nu^{(\alpha)}(x) R_\nu^{(\alpha)}(t)$$

and, consequently, $g \in \widetilde{\mathscr{P}}_n$. By (3.10) and (2.7), the relation $T_t^\alpha(\check{f}; 0) = \check{f}(t)$ holds. Therefore, $g(0) = \check{f}(t)e^{-t/2} = f(t)$. Applying inequality (3.2), we obtain

$$\|f\|_\infty = |f(t)| = |g(0)| \le D_n \|g\|_{L_\alpha^q} \le D_n \, \|\Theta_t^\alpha\|_{q,\alpha} \, \|f\|_{L_\alpha^q}. \tag{3.11}$$

By (3.11) and (2.38), it follows that $\|f\|_\infty \le D_n \, \|f\|_{L_\alpha^q}, \ f \in \widetilde{\mathscr{P}}_n$. Therefore, the inequality $M_n \le D_n$ and hence equality (3.4) hold.

Recall that $g_n^* = g_{n,q,\alpha+1}^*$ stands for the e-polynomial that solves problem (3.3). By Theorem 3, this is the unique extremal polynomial in inequality (3.2). We have

$$D_n \, \|g_n^*\|_{L_\alpha^q} = |g_n^*(0)| \le \|g_n^*\|_\infty \le M_n \, \|g_n^*\|_{L_\alpha^q}.$$

Hence, in view of the equality $D_n = M_n$, we have

$$\|g_n^*\|_\infty = |g_n^*(0)|$$

and the polynomial g_n^* is extremal in inequality (3.1).

It remains to verify that g_n^* is the unique extremal e-polynomial in inequality (3.1). Let $f \in \widetilde{\mathscr{P}}_n$ be an extremal e-polynomial in inequality (3.1). Its uniform norm is necessarily attained at some point $t \in [0, \infty)$. For the e-polynomial f with this value of the parameter t, both inequalities (3.11) must turn into equalities; in particular, the latter inequality. By (2.37) and Lemmas 2 and 3, this is impossible for $t > 0$. Consequently, $t = 0$ and, hence, f is extremal in inequality (3.2). According to Theorem 3, this means that $f = cg_n^*$, $c \in \mathbb{C}$. Thus, indeed, g_n^* is the unique extremal e-polynomial in inequality (3.1). This completes the proof of Theorem 2, and hence the proof of its equivalent–Theorem 1. $\square$

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