The Dirichlet problem in weighted norm

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ABSTRACT. We study the classical Dirichlet problem in the disc with the weighted uniform norm for the weight function $w(x) = v(x)\prod_{j=1}^{s}\left|\sin\left(\frac{x-x_j}{2}\right)\right|^{\lambda_j}$, $\{\lambda_j\}_{j=1}^{s}$ are positive numbers and v is a strictly positive continuous function on the circle. Remarkably the problem has solution if and only if none of the numbers $\{\lambda_j\}_{j=1}^{s}$ is natural.

1. Introduction

Dirichlet problem is a widely studied topic and it turned out to be fundamental in many areas of mathematics and physics. Several different methods were developed to solve the problem in several levels of generalization. We are interested in Poisson integral method. According to Brelot's theorem, for a domain in the complex plane such that its boundary is compact and of positive capacity, and for a finite Borel measurable function defined on the boundary, the Dirichlet problem is solvable if and only if the function is integrable with respect to harmonic measures, and the solution is given by the generalized Poisson integral, cf. [**14**]. More details about this general setting can be found in e.g. [**5**]. Our purpose is finding some balance between generality and speciality which can be useful for applications.

One direction to generalization as a weighted problem on the unit disc is the theory of weighted Laplacian. Although it is also a rich and developing area (cf. e.g. [**12**] and the references therein), we turn our attention to weights on the boundary. The development of weighted approximation in the last century led to application of potential theoretic methods to investigate the fundamental properties of asymptotics of orthogonal polynomials, Christoffel functions, discrepancy of measures, etc. Recently the focus is intended to weighted problems with measures (or the Radon-Nikodym derivatives the weights) behaving like $|x-x_0|^{\alpha}dx$ around some x_0 in their support, cf. e.g. [**3**] or [**10**]. As via Green function method and Poisson integral method the Dirichlet problem establishes connections between potential theory, approximation theory (cf. e.g. [**11**]) and Fourier analysis (and several other areas of mathematics what we are not discussing here), the problem to examine Dirichlet problem with weights behaving like $|x-x_0|^{\alpha}$ arises naturally.

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The solution our problem led to introducing closed and minimal systems in some weighted spaces, which derived from the original (exponential) unweighted system by deleting finite many elements. This process reminds of the structure of exceptional orthogonal polynomials (cf. e.g. [**4**]), so it would be nice to find some relationship between the two ideas.

The Dirichlet problem in the $L^p(\psi), 1 \leq p < \infty$ metric, where the weight function $\psi > 0$ has singularities was studied in [**7**]. Similar problem in the weighted L^1 space was investigated in [**1**]. The Dirichlet problem in some classes of functions which have singularities on the boundary of the region was considered earlier in [**16**]. A preliminary version of the present paper appeared in [**6**]. In this note we examine the problem in a weighted space of continuous functions.

2. Definitions, results

Set $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and let

$$w(x) = v(x) \prod_{j=1}^{s} \left| \sin\left(\frac{x - x_j}{2}\right) \right|^{\lambda_j}, \tag{1}$$

where $v(x)$ is a positive continuous function on $\mathbb{T}$ such that for some $C_0 > 0$

$$\max_{(x,t)\in\mathbb{T}^2} \frac{v(x)}{v(t)} \leq C_0; \tag{2}$$

$X = \{x_1, x_2, \dots, x_s\} \subset \mathbb{T}$ is a set of distinct points, and $\Lambda = \{\lambda_j\}_{j=1}^s$ is a collection of positive real numbers.

By $C(w)$ is denoted the linear space of all complex valued functions f defined on $\mathbb{T}$ such that fw is continuous on $\mathbb{T}$ and

$$\lim_{x\to x_j} f(x)w(x) = 0, \quad j = 1, \dots, s. \tag{3}$$

It is easy to check that $C(w)$ is a Banach space with the norm

$$\|f\|_{C(w)} = \max_{x\in\mathbb{T}} |f(x)|w(x). \tag{4}$$

for any $f \in C(w)$. The space of continuous complex valued functions defined on $\mathbb{T}$ with the standard norm will be denoted by $C_{\mathbb{T}}$.

We study the following classical

Dirichlet problem. For any $f \in C(w)$ find a harmonic function u_f on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ such that

$$\lim_{r\to 1-} \|u_f(r,\theta) - f(\theta)\|_{C(w)} = 0, \tag{5}$$

where $z = re^{i\theta}$.

The solution of the classical Dirichlet problem when the weight function has no singularities is represented by the convolution of f with the Poisson kernel

$$P_r(x) := 1 + 2\sum_{n=1}^{\infty} r^n \cos nx = \frac{1 - r^2}{1 - 2r\cos x + r^2}, \quad 0 < r < 1. \tag{6}$$

In our case, when the weight function has essential singularities, the solution can not be represented as a convolution. In this case modified Poisson kernels (see [**8**], [**9**]) replace the Poisson kernel.

We set $k_j := [\lambda_j]$, where $[\lambda]$ is the integer part of the number λ, $\lambda - 1 < [\lambda] \leq \lambda$. For $x \in \mathbb{R}$ let

$$\omega(x) = \omega_{X,\Lambda}(x) := \prod_{j=1}^{s} \sin^{k_j}\left(\frac{x - x_j}{2}\right) \tag{7}$$

if $|\Lambda|_* := \sum_{j=1}^{s} k_j \geq 1$ and

$$\omega(x) = \omega_{X,\Lambda}(x) :\equiv 1 \qquad \text{if} \quad |\Lambda|_* = 0. \tag{8}$$

If $|\Lambda|_* > 0$ and $|\Lambda|_* = 2n - 1$, where $n \in \mathbb{N}$ we denote by $T_{j,l}(x)$ the trigonometric polynomials of degree n such that

$$T_{j,l}^{(m)}(x_i) = \delta_{l,m}\delta_{i,j} \quad 1 \leq i, j \leq s;\ 0 \leq m \leq k_i - 1;\ 0 \leq l \leq k_j - 1. \tag{9}$$

If in the above formula $k_j = 0$ then no polynomials $T_{j,l}$ are defined.

For the uniqueness of the solution when $|\Lambda|_* = 2n$, $n \in \mathbb{N}$ we put an additional condition on the trigonometric polynomials $T_{j,l}(x)$. That condition is formulated in terms of the leading coefficients of the trigonometric polynomial $\omega_{X,\Lambda}(x)$ defined by (7)

$$\omega_{X,\Lambda}(x) = a_n \cos nx + b_n \sin nx + \cdots . \tag{10}$$

We set that

$$\frac{b_n^{(j,l)}}{a_n^{(j,l)}} = -\frac{a_n}{b_n} \quad \text{and} \quad a_n^{(j,l)} = 0 \quad \text{if} \quad b_n = 0, \tag{11}$$

where

$$T_{j,l}(x) = a_n^{(j,l)} \cos nx + b_n^{(j,l)} \sin nx + \cdots \quad (1 \leq j \leq s, 0 \leq l \leq k_j - 1).$$

The modified Poisson kernels ([**7**], (1.9)) are defined as follows:

$$P_{X,\Lambda,r}(x,t) := P_r(t - x) - \sum_{j=1}^{s} \sum_{l=0}^{k_j - 1} P_r^{(l)}(x_j - x) T_{j,l}(t) \tag{12}$$

if $|\Lambda|_* > 0$ and

$$P_{X,\Lambda,r}(x,t) := P_r(t - x) \quad \text{if} \quad |\Lambda|_* = 0, \tag{13}$$

where $P_r^{(l)}(x) := \frac{d^l}{dx^l} P_r(x)$. Note that if $k_j = 0$ then the term with the index j is absent in the formula (12). Set

$$O_j(\rho) = \{t \in \mathbb{T} : |t - x_j| < \rho\}, \qquad \text{where} \quad 1 \leq j \leq s \quad \text{and} \quad \rho > 0.$$

Further in the text constants will be denoted by C, C_j, C_j' and they may be different in different inequalities.

We prove the following main theorem.

THEOREM 1. *Let $\Lambda \bigcap \mathbb{N} = \emptyset$ and let w be a weight function, where w satisfies the conditions (1) and (2). Then for any $f \in C(w)$ there exists a unique harmonic function u_f on the unit disk D such that (5) holds. Moreover,*

$$u_f(r,\theta) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) P_{X,\Lambda,r}(\theta, t)\, dt, \tag{14}$$

where the kernel $P_{X,\Lambda,r}$ is defined by (12).

The proof of the above theorem is based on the following result.

Theorem 2. *For any weight function w, where w satisfies the conditions (1), (2) and $\Lambda \bigcap \mathbb{N} = \emptyset$ there exists $C > 0$ such that*

$$\sup_{0<r<1} \sup_{x\in\mathbb{T}} w(x) \int_{\mathbb{T}} \frac{1}{w(t)} |P_{X,\Lambda,r}(x,t)| dt \leq C. \tag{15}$$

The following theorem shows that if $\Lambda \bigcap \mathbb{N} \neq \emptyset$ then Theorem 2 is not true.

Theorem 3. *Let w be a weight function , where w satisfies the conditions (1), (2) and $\Lambda \bigcap \mathbb{N} \neq \emptyset$. Then there exists $f \in C(w)$ such that*

$$\limsup_{r\to 1-} \sup_{x\in\mathbb{T}} w(x) \int_{\mathbb{T}} f(t) P_{X,\Lambda,r}(x,t) dt = +\infty. \tag{16}$$

Further we will use the following terminology. A system of elements $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ in a Banach space B will be called closed system if any element of B can be arbitrarily approximated by a finite linear combination of elements of Φ. We will say that Φ is complete with respect to the dual space B^* if the condition

$$\phi^*(\varphi_n) = 0, \quad \text{for all} \quad n \in \mathbb{N},$$

where $\phi^* \in B^*$ yields that ϕ^* is the trivial element of the space B^*. The system $\Phi = \{\varphi_n\}_{n=1}^{\infty} \subset B$ is called a minimal system if there exists $\Phi^* = \{\phi_n^*\}_{n=1}^{\infty} \subset B^*$ such that

$$\phi_n^*(\varphi_k) = \delta_{nk} \qquad\qquad n, k \in \mathbb{N}, \tag{17}$$

where δ_{nk} is the Kronecker symbol. We will say that a system of elements $\Phi = \{\varphi_n\}_{n=1}^{\infty} \subset B$ is an $A-$basis of the Banach space B if Φ is closed and minimal in B and for any $x \in B$

$$\lim_{r\longrightarrow 1-} \|x - \sum_{n=1}^{\infty} r^n \phi_n^*(x)\varphi_n\|_B = 0,$$

where $\Phi^* = \{\phi_n^*\}_{n=1}^{\infty} \subset B^*$ is the uniquely defined system in the dual space for which the condition (17) holds. We will say that the system Φ^* is the conjugate system of Φ. For the convenience of the reader we will formulate the analogue of Banach's theorem for the summation bases for the Abel-Poisson method. We will not bring the proof because it is similar to the proof of Banach's original proof [**2**] with some technical modifications. Some references about summation bases can be found in [**15**] and [**7**].

Lemma 1. *Let $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ is a closed and minimal system in a separable Banach space B. Then Φ is an $A-$basis of B if and only if there exists a constant $C > 0$ such that for any $x \in B$*

$$\sup_{0<r<1} \|\sum_{n=1}^{\infty} r^n \phi_n^*(x)\varphi_n\|_B \leq C\|x\|_B. \tag{18}$$

3. Auxiliary results

In the proof of Theorem 2 the kernels $P_{X,\Lambda,r}(x,t)$ are decomposed into sums of kernels $B_{r,j}(x,t)$ $(1 \le j \le s)$. For that purpose we use the identity

$$\sum_{j=1}^{s} T_{j,0}(t) \equiv 1, \tag{19}$$

where it is supposed that $T_{j,0} \equiv 0$, if $k_j = 0$.

By (12) and (19) we have

$$P_{X,\Lambda,r}(x,t) = \sum_{j=1}^{s} B_{r,j}(x,t), \tag{20}$$

where $B_{r,j}(x,t) \equiv 0$ if $k_j = 0$, and

$$B_{r,j}(x,t) = P_r(t-x)T_{j,0}(t) - \sum_{l=0}^{k_j-1} P_r^{(l)}(x_j - x)T_{j,l}(t) \tag{21}$$

if $k_j > 0$. The kernel (21) is decomposed applying Lemmas 2 and 3(see [**8**]). We set

$$\xi_r(t) = 1 - 2r\cos t + r^2. \tag{22}$$

Next Lemma is proved in [**8**].

LEMMA 2. *Let $|\Lambda|_* = 2N+1 (N = 0, 1, \dots)$. Then for every $j (1 \le j \le s)$ there are functions $G_r^* := G_{r,j}^*$ and $G_r^{**} := G_{r,j}^{**}$ such that*

$$B_{r,j}(x,t) = P_r(t-x)\omega(t)\big[G_r^*(x)\sin\frac{t-x_j}{2} + G_r^{**}(x)\cos\frac{t-x_j}{2}\big]$$

and

$$|G_r^*(x)| \le C[\xi_r(x-x_j)]^{-\frac{k_j+1}{2}},$$
$$|G_r^{**}(x)| \le C[\xi_r(x-x_j)]^{-\frac{k_j}{2}},$$

where $C > 0$ is independent of r, j and x.

The proof of the following lemma is similar to the proof of Lemma 2.

LEMMA 3. *Let $|\Lambda|_* = 2N (N = 1, 2 \dots)$. Then for every $j (1 \le j \le s)$ there are functions $G_r^* := G_{r,j}^*$, $G_r^{**} := G_{r,j}^{**}$ and $G_r^{***} := G_{r,j}^{***}$ such that*

$$B_{r,j}(x,t) = P_r(t-x)\omega(t)\big[G_r^*(x)\sin(t-x_j) + G_r^{**}(x)\cos(t-x_j) + G_r^{***}(x)\big]$$

and

$$|G_r^*(x)| \le C[\xi_r(x-x_j)]^{-\frac{k_j+1}{2}},$$
$$|G_r^{**}(x)| \le C[\xi_r(x-x_j)]^{-\frac{k_j}{2}}$$

and

$$|G_r^{***}(x)| \le C[\xi_r(x-x_j)]^{-\frac{k_j}{2}},$$

where $C > 0$ is independent of r, j and x.

Let δ_x be the Dirac measure concentrated at a given point $x \in \mathbb{T}$. We consider the finite dimensional subspace of the dual space $C_{\mathbb{T}}^*$ generated by the Dirac measures $\delta_{x_j}, 1 \le j \le s$ which will be denoted by $\mathcal{M}_X$. From the Hahn-Banach theorem we obtain the following description of the dual space $C^*(w)$ of $C(w)$.

LEMMA 4. *Let w be a weight function, where w satisfies the conditions (1), (2). Then $\tau \in C^*(w)$ if and only if there exists a unique class of equivalences $E_\tau \in C^*_{\mathbb{T}}/\mathcal{M}_X$ of complex Borel measures such that*

$$\tau(f) = \int_{\mathbb{T}} f(t)w(t)d\mu(t) \qquad \forall f \in C(w) \quad \textit{and} \quad \forall \mu \in E_\tau,$$

and

$$\|\tau\|_{C^*(w)} = \|E_\tau\|_{C^*/\mathcal{M}_X}.$$

We take a system of functions $\mathcal{T}_\Lambda$ which in the space $C(w)$ will replace the trigonometric system. Let $\mathbb{Z}^*_\Lambda = \{k \in \mathbb{Z} : \, k = -n, n, -n-1, n+1, \dots\}$ if $|\Lambda|_* = 2n-1$, and $\mathbb{Z}^*_\Lambda = \{k \in \mathbb{Z} : \, k = -n-1, n+1, \dots\}$ if $|\Lambda|_* = 2n$.

Set

$$\mathcal{T}_\Lambda = \{e^{ikx} : \, k \in \mathbb{Z}^*_\Lambda\} \qquad \text{if} \quad |\Lambda|_* = 2n-1,$$

where $n = 1, \dots$; and if $|\Lambda|_* = 2n$ we put

$$\mathcal{T}_\Lambda = \{a_n \cos nx + b_n \sin nx, e^{ikx} : \, k \in \mathbb{Z}^*_\Lambda\},$$

where the numbers a_n, b_n are the leading coefficients of the polynomial (10).

LEMMA 5. *Let w be a weight function, where w satisfies the conditions (1) and (2). Then the system $\mathcal{T}_\Lambda$ is closed and minimal in $C(w)$ with the conjugate system $\{E_k\}_{k\in\mathbb{Z}^*_\Lambda}$ if $|\Lambda|_* = 2n-1$ and with the conjugate system $\{E_n, E_k\}_{k\in\mathbb{Z}^*_\Lambda}$ if $|\Lambda|_* = 2n$, where $E_k \in C^*/\mathcal{M}_X$. Moreover, absolutely continuous complex Borel measures $dg_k \in E_k$ for $k \in \mathbb{Z}^*_\Lambda$ are defined by the equations*

$$dg_k(x) = \frac{1}{2\pi w(x)} \left(e^{ikx} - \sum_{j=1}^{s} \sum_{l=0}^{k_{j-1}} \frac{d^{\,l}}{dt^l} e^{ikt} \Big|_{t=x_j} T_{j,l}(x) \right) dx,$$

when $|\Lambda|_* = 2n-1$ and if $|\Lambda|_* = 2n$

$$dg_n(x) = \frac{1}{\pi(a_n^2 + b_n^2)w(x)} \Big(a_n \cos nx + b_n \sin nx \tag{23}$$

$$- \sum_{j=1}^{s} \sum_{l=0}^{k_j-1} \frac{d^{\,l}}{dt^l} (a_n \cos nt + b_n \sin nt) \Big|_{t=x_j} T_{j,l}(x) \Big) dx,$$

and for $k \in \mathbb{Z}^*_\Lambda$

$$dg_k(x) = \frac{1}{w(x)2\pi} \left(e^{ikx} - \sum_{j=1}^{s} \sum_{l=0}^{k_j-1} \frac{d^l}{dt^l} e^{ikt} \Big|_{t=x_j} T_{j,l}(x) \right) dx. \tag{24}$$

PROOF. We will bring the proof for the case $|\Lambda|_* = 2n$. When $|\Lambda|_* = 2n-1$ the proof is similar. Suppose that for some $\phi^* \in C^*(w)$

$$\phi^*(a_n \cos nx + b_n \sin nx) = 0$$

and

$$\phi^*(e^{ikx}) = 0 \quad \text{for all} \quad k \in \mathbb{Z}^*_\Lambda.$$

Then by Lemma 4 there exists a unique class of equivalences of Borel measures $E_{\phi^*} \in C^*/\mathcal{M}_X$ such that for all $\mu \in E_{\phi^*}$

$$\phi^*(a_n \cos nx + b_n \sin nx) = \int_{\mathbb{T}} (a_n \cos nt + b_n \sin nt) w(t) d\mu(t) = 0$$

and

$$\phi^*(e^{ikx}) = \int_{\mathbb{T}} e^{-ikt} w(t) d\mu(t) = 0 \qquad \forall \mu \in E_{\phi^*} \quad \text{and} \quad \forall k \in \mathbb{Z}^*_\Lambda.$$

We put

$$\alpha_n(\mu) = \frac{1}{\pi(a_n^2 + b_n^2)} \int_{\mathbb{T}} (b_n \cos nt - a_n \sin nt) w(t) d\mu(t)$$

and

$$\alpha_m(\mu) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-imt} w(t) d\mu(t) \quad \text{for} \quad |m| \leq n-1.$$

Hence, by the closedness of the trigonometrical system in $C_{\mathbb{T}}$ we obtain that for any $\mu \in E_{\phi^*}$

$$w(t) d\mu(t) = \big[\alpha_n(\mu)\big(b_n \cos nt - a_n \sin nt\big) + \sum_{|m| \leq n-1} \alpha_{-m}(\mu) e^{imt}\big] dt.$$

Hence, if $\mu_0(t) \in E_\phi$ is such that $\mu_0(\{x_j\}) = 0, 1 \leq j \leq s$ then by (1), (2) and (7) we obtain that

$$\alpha_n(\mu_0)\big(b_n \cos nt - a_n \sin nt\big) + \sum_{|m| \leq n-1} \alpha_m(\mu_0) e^{imt} = C \cdot \omega_{X,\Lambda}(t),$$

where $C \in \mathbb{C}$. From the last equality and (10) immediately follows that $\alpha_n(\mu_0) = 0$. Which yields $C = 0$ and consequently $\alpha_m(\mu_0) = 0$ for all $|m| \leq n-1$. Thus $E_{\phi^*} = \mathcal{M}_X$ which proves that the system $\mathcal{T}_\Lambda$ is closed in $C(w)$. One can easily check that the absolutely continuous Borel measures (23), (24) are finite Borel measures which satisfy the conditions

$$\int_{\mathbb{T}} (a_n \cos nt + b_n \sin nt) w(t) dg_k(t) = \delta_{nk}, \text{ for } k = n \text{ and all } k \in \mathbb{Z}^*_\Lambda;$$

$$\int_{\mathbb{T}} e^{-ijt} w(t) dg_k(t) = \delta_{jk}, \text{ for } k = n \text{ and for all } j, k \in \mathbb{Z}^*_\Lambda.$$

Hence, the system is also minimal in $C(w)$. □

For any $0 < a < 1$ we define $\Delta_a \in C_{\mathbb{T}}$ as follows

$$\Delta_a(x) = \begin{cases} 1 & \text{if} \quad x \in [-\frac{a}{2}, \frac{a}{2}]; \\ 2a^{-1}(x+a) & \text{if} \quad x \in [-a, -\frac{a}{2}); \\ -2a^{-1}(x-a) & \text{if} \quad x \in (\frac{a}{2}, a]; \\ 0 & \text{elsewhere.} \end{cases}$$

4. Proof of Main theorems

Proof of Theorem 2. We set $\delta = \min_{i \neq j}\{\frac{1}{2}, \frac{1}{4}|x_i - x_j|\}$.

For the convenience of the reader at first let us consider the case $|\Lambda|_* = 0$. By (13) the inequality (15) can be written in the following form:

$$\mathcal{I}(r,x) =: w(x) \int_{\mathbb{T}} \frac{1}{w(t)} P_r(x-t) dt \leq C \qquad \text{for any} \quad x \in \mathbb{T}. \tag{25}$$

To prove (25) we write

$$\mathcal{I}(r,x) = \sum_{j=1}^{s} w(x) \int_{\mathbb{T}} \frac{1}{w(t)} \Delta_\delta(t - x_j) P_r(x-t) dt$$

$$+w(x)\int_{\mathbb{T}}\frac{1}{w(t)}\big[1-\sum_{j=1}^{s}\Delta_\delta(t-x_j)\big]P_r(x-t)dt=:\sum_{j=1}^{s}\mathcal{I}_j(r,x)+\mathcal{I}_0(r,x).$$

Fix any $j(1\leq j\leq s)$ and consider three cases:
1) $x\in\mathbb{T}\setminus O_j(2\delta)$;
2) $x\in O_j(2\delta)\setminus O_j(\frac{\delta}{2})$;
3) $x\in O_j(\frac{\delta}{2})$.

In the case 1) the well known estimates for the Poisson kernel (e.g., [**17**], ch. 3) yield

$$\mathcal{I}_j(r,x)\leq w(x)\int_{O_j(\delta)}\frac{1}{w(t)}dt\ \min\left\{\frac{2}{1-r},\frac{1-r}{2r\sin^2\frac{\delta}{2}}\right\}. \tag{26}$$

Recall that $0<\lambda_j<1$. Hence, by (1) and (2) we obtain that for some $C_j>0$

$$\mathcal{I}_j(r,x)\leq C_j \tag{27}$$

for any $0<r<1$.

In the case 2) the estimate (27) is trivial if $1-r\geq\frac{\delta}{4}$. If $1-r<\frac{\delta}{4}$ then we write

$$\mathcal{I}_j(r,x)=w(x)\left\{\int_{O_j(\frac{\delta}{4})}+\int_{O_j(2\delta)\setminus O_j(\frac{\delta}{4})}\right\}\frac{1}{w(t)}\Delta_\delta(t-x_j)P_r(x-t)dt.$$

Afterwards conditions (1), (2) yield that the function $\frac{w(x)}{w(t)}$ is bounded uniformly on the set

$$\Pi_j(\delta)=\left\{(x,t)\in\mathbb{T}^2:\frac{\delta}{2}<|x-x_j|<2\delta\quad\&\quad\frac{\delta}{4}\leq|t-x_j|\leq2\delta\right\}.$$

Thus the second integral on the right hand of the above equality is uniformly bounded for any $0<r<1$. Writing

$$\begin{aligned}w(x)\int_{O_j(\frac{\delta}{4})}&\frac{\Delta_\delta(t-x_j)}{w(t)}P_r(x-t)dt\\ &\leq\ w(x)\int_{O_j(\frac{\delta}{4})}\frac{1}{w(t)}dt\ \min\left\{\frac{2}{1-r},\frac{1-r}{2r\sin^2\frac{\delta}{8}}\right\}\leq C_j'\end{aligned}$$

for some $C_j'>0$, we finish the proof for the case 2).

In the case 3) the estimate (27) is trivial if $1-r\geq\frac{\delta}{4}$. If $x\in O_j(2-2r)$ and $1-r<\frac{\delta}{4}$ then we have

$$\mathcal{I}_j(r,x)=w(x)\left\{\int_{O_j(1-r)}+\int_{O_j(2\delta)\setminus O_j(1-r)}\right\}\frac{1}{w(t)}\Delta_\delta(t-x_j)P_r(x-t)dt.$$

By (1) and (2) we have that the function $\frac{w(x)}{w(t)}$ is bounded by a $C>0$ independent of any t from the set $1-r\leq|t-x_j|\leq2\delta$. Thus the second integral on the right hand of the above equality is bounded. Afterwards we deduce

$$w(x)\int_{O_j(1-r)}\frac{\Delta_\delta(t-x_j)}{w(t)}P_r(x-t)dt\leq\frac{w(x)}{1-r}\int_{O_j(1-r)}\frac{1}{w(t)}dt\ \leq C'$$

for any $0<r<1$, where $C'>0$.

If $x \in O_j(\frac{\delta}{2}) \setminus O_j(2-2r)$ then we write

$$\begin{aligned}\mathcal{I}_j(r,x) &= w(x)\left\{\int_{O_j(1-r)} + \int_{1-r\le|t-x_j|<2|x-x_j|} + \int_{2|x-x_j|\le|t-x_j|\le 2\delta}\right\}\cdots dt \\ &=: \mathcal{I}_j^{(1)}(r,x) + \mathcal{I}_j^{(2)}(r,x) + \mathcal{I}_j^{(3)}(r,x).\end{aligned}$$

The boundedness of $\mathcal{I}_j^{(3)}(r,x)$ follows immediately. One should only observe that by (1), (2) $\frac{w(x)}{w(t)}$ is bounded by an absolute constant for any t from the set $2|x-x_j| \le |t-x_j| \le 2\delta$.

Afterwards we write

$$\begin{aligned}\mathcal{I}_j^{(1)}(r,x) &= w(x)\int_{O_j(1-r)} \frac{\Delta_\delta(t-x_j)}{w(t)} P_r(x-t)dt \\ &\le \frac{w(x)}{2\sin^2\frac{|x-x_j|}{4}}(1-r)\int_{O_j(1-r)}\frac{1}{w(t)}dt \\ &\le C\frac{w(x)}{2\sin^2\frac{|x-x_j|}{4}}(1-r)^{2-\lambda_j} \le C'', \quad \forall\, x \in O_j(\frac{\delta}{2}) \setminus O_j(2-2r),\end{aligned}$$

where $C'' > 0$.

We study $\mathcal{I}_j^{(2)}(r,x)$ at first in the case $x > x_j$ and $x \in O_j(\frac{\delta}{2}) \setminus O_j(2-2r)$. In this case we set

$$\{t \in \mathbb{T} : 1-r \le |t-x_j| < 2|x-x_j|\} = \bigcup_{i=1}^{4} \Omega_{x,r}^{(i)},$$

where

$$\begin{aligned}\Omega_{x,r}^{(1)} &= \{t \in \mathbb{T} : x_j + 1 - r \le t < x - (1-r)\}\,; \\ \Omega_{x,r}^{(2)} &= \{t \in \mathbb{T} : x - (1-r) \le t < x + 1 - r\}\,; \\ \Omega_{x,r}^{(3)} &= \{t \in \mathbb{T} : x + 1 - r \le t < x_j + 2(x-x_j)\}\,; \\ \Omega_{x,r}^{(4)} &= \{t \in \mathbb{T} : x_j - 2(x-x_j) < t \le x_j - (1-r)\}\,.\end{aligned}$$

Afterwards we write

$$\mathcal{I}_j^{(2)}(r,x) = \sum_{i=1}^{4} \mathcal{I}_j^{(2,i)}(r,x),$$

where

$$\mathcal{I}_j^{(2,i)}(r,x) = w(x)\int_{\Omega_{x,r}^{(i)}} \frac{\Delta_\delta(t-x_j)}{w(t)} P_r(x-t)dt \qquad \text{for } 1 \le i \le 4.$$

Having in mind that $x > x_j$ and $x \in O_j(\frac{\delta}{2}) \setminus O_j(2-2r)$ we begin with the estimate of the first integral

$$\begin{aligned}
\mathcal{I}_j^{(2,1)}(r,x) &= w(x)\int_{\Omega_{x,r}^{(1)}} \frac{\Delta_\delta(t-x_j)}{w(t)} P_r(x-t)dt \\
\leq\ & w(x)\int_{[x_j+(1-r),\frac{x_j+x}{2}]} \frac{1-r}{w(t)}|\sin\frac{x-t}{2}|^{-2}dt \\
+\ & w(x)\int_{[\frac{x_j+x}{2},x-(1-r)]} \frac{1-r}{w(t)}|\sin\frac{x-t}{2}|^{-2}dt \\
\leq\ & C(x-x_j)^{\lambda_j}|\sin\frac{x-x_j}{4}|^{-2}(1-r)^{2-\lambda_j} \\
+\ & C'(1-r)\frac{|\sin\frac{x-x_j}{2}|^{\lambda_j}}{|\sin\frac{x-x_j}{2}|^{\lambda_j}}\int_{1-r}^{\frac{x-x_j}{2}}\frac{1}{y^2}dy \leq C''.
\end{aligned}$$

Then we estimate the next integral

$$\mathcal{I}_j^{(2,2)}(r,x) = w(x)\int_{\Omega_{x,r}^{(2)}} \frac{\Delta_\delta(t-x_j)}{w(t)} P_r(x-t)dt \leq C\frac{|\sin\frac{x-x_j}{2}|^{\lambda_j}}{|\sin(\frac{x-x_j}{2}-\frac{1-r}{2})|^{\lambda_j}} \leq C'.$$

The third integral is evaluated as follows

$$\begin{aligned}
\mathcal{I}_j^{(2,3)}(r,x) &= w(x)\int_{\Omega_{x,r}^{(3)}} \frac{\Delta_\delta(t-x_j)}{w(t)} P_r(x-t)dt \\
\leq\ & w(x)\int_{[x+(1-r),x_j+2(x-x_j)]} \frac{1-r}{w(t)}|\sin\frac{x-t}{2}|^{-2}dt \\
\leq\ & C(1-r)\frac{|\sin\frac{x-x_j}{2}|^{\lambda_j}}{|\sin(\frac{x-x_j}{2}+\frac{1-r}{2})|^{\lambda_j}}\int_{[1-r,x-x_j]}|\sin\frac{y}{2}|^{-2}dy \leq C',
\end{aligned}$$

where the last integral is less than $2\cot\frac{1-r}{2}$ and it is easy to see that $(1-r)\cot\frac{1-r}{2}$ is uniformly bounded for all r such that $0 < 1-r < \delta$.
It remains to evaluate the integral

$$\begin{aligned}
\mathcal{I}_j^{(2,4)}(r,x) &= w(x)\int_{\Omega_{x,r}^{(4)}} \frac{\Delta_\delta(t-x_j)}{w(t)} P_r(x-t)dt \\
\leq\ & w(x)\int_{[x_j-2(x-x_j),x_j-1+r]} \frac{1-r}{w(t)}|\sin\frac{x-t}{2}|^{-2}dt \\
\leq\ & C(x-x_j)^{\lambda_j}\frac{1-r}{|\sin\frac{x-x_j}{2}|^2}(x-x_j)^{1-\lambda_j} \leq C'.
\end{aligned}$$

Thus for $x > x_j$ and $x \in O_j(\frac{\delta}{2}) \setminus O_j(2-2r)$ the proof of the uniform boundedness of $\mathcal{I}_j^{(2)}(r,x)$ for any $0 < r < 1$ is finished.

We skip the proof for the case $x < x_j$ because it is similar to the case considered above.

The function $\frac{1}{w(t)}\big(1-\sum_{j=1}^{s}\Delta_\delta(t-x_j)\big)$ is continuous on $\mathbb{T}$ thus $\mathcal{I}_0(r,x)$ is uniformly bounded.

Now let $|\Lambda|_* \geq 1$. We have to distinguish the points $x_j \in X$ for which $0 < \lambda_j < 1$ and the points $x_\nu \in X$ such that $\lambda_\nu > 1$. For simplicity we suppose that $\lambda_j > 1$

for all $x_j \in X$. It is easy to see that if for some points $x_j \in X$ $0 < \lambda_j < 1$ then the proof for those points can be provided in the same way as above.

Without loss in generality we can suppose that $r > \frac{3}{4}$. We have to give a proof applying Lemmas 2 and 3. By (20) we have

$$w(x)\int_{\mathbb{T}} \frac{1}{w(t)}|P_{X,\Lambda,r}(x,t)|dt \le w(x)\sum_{j=1}^{s}\int_{\mathbb{T}} \frac{1}{w(t)}|B_{r,j}(x,t)|dt.$$

It is sufficient to prove that for any $j(1 \le j \le s)$ there exists $C_j > 0$ independent of x and r such that

$$J(r,x) =: w(x)\int_{\mathbb{T}} \frac{1}{w(t)}|B_{r,j}(x,t)|dt \le C_j.$$

We write

$$\begin{aligned} J(r,x) &= \sum_{\nu=1}^{s} w(x)\int_{\mathbb{T}} \frac{1}{w(t)}\Delta_\delta(t-x_\nu)|B_{r,j}(x,t)|dt \\ &+ \quad w(x)\int_{\mathbb{T}} \frac{1}{w(t)}\big[1-\sum_{\nu=1}^{s}\Delta_\delta(t-x_\nu)\big]P_r(x-t)dt := \sum_{\nu=1}^{s} J_\nu(r,x) + J_0(r,x). \end{aligned}$$

We have to prove that for any $\nu(1 \le \nu \le s)$

$$J_\nu(r,x) \le C_\nu,$$

where $C_\nu > 0$ are independent of x and r. If $\nu \ne j$ then by Lemmas 2 and 3 the proof is reduced to the case $|\Lambda|_* = 0$ studied above. Hence, we center our attention on the case $\nu = j$. For simplicity we suppose that $|\Lambda|_* = 2N+1 (N = 0,1,\dots)$. We have to prove the inequality

$$\begin{aligned} J_j(r,x) \quad &\le \quad C\frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}\int_{\mathbb{T}} \frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)\left|\sin\frac{t-x_j}{2}\right|P_r(t-x)dt \\ &+ \quad C\frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}}\int_{\mathbb{T}} \frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)P_r(t-x)dt \le C_j. \end{aligned}$$

As in the first part of the proof we will consider the cases $1) - 3)$.

In the case 1) we have

$$\begin{aligned} J_j(r,x) \quad &\le \quad \frac{Cw(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}\int_{O_j(\delta)} \left|\sin\frac{t-x_j}{2}\right|^{k_j-\lambda_j+1} dt\ \min\Big\{\frac{2}{1-r}, \frac{1-r}{2r\sin^2\frac{\delta}{2}}\Big\} \\ &+ \quad \frac{Cw(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}}\int_{O_j(\delta)} \left|\sin\frac{t-x_j}{2}\right|^{k_j-\lambda_j} dt\ \min\left\{\frac{2}{1-r}, \frac{1-r}{2r\sin^2\frac{\delta}{2}}\right\} \\ &\le \quad C'(1-r)w(x)\delta^{2+k_j-\lambda_j}\min\left\{\frac{2}{(1-r)^2}, \frac{1}{2r\sin^2\delta}\right\}^{\frac{k_j+3}{2}} \\ &+ \quad C'(1-r)w(x)\delta^{1+k_j-\lambda_j}\min\left\{\frac{1}{(1-r)^2}, \frac{1}{2r\sin^2\delta}\right\}^{\frac{k_j+2}{2}} \quad \le C_j, \end{aligned}$$

where $C_j > 0$ is independent of $r(0 < r < 1)$ and $x \in \mathbb{T}\setminus O_j(2\delta)$.

We skip the proof in the case 2) because it is similar to the analogous case provided above.

In the case 3) it is sufficient to consider that $1-r<\frac{\delta}{8}$. If $x\in O_j(2-2r)$ and $1-r<\frac{\delta}{8}$ then we have

$$J_j(r,x)\le \frac{Cw(x)}{(1-r)^{k_j+1}}\left\{\int_{O_j(4-4r)}+\int_{O^*_{j,r}}\right\}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)|\sin\frac{t-x_j}{2}|P_r(t-x)dt$$
$$+\quad C\frac{w(x)}{(1-r)^{k_j}}\left\{\int_{O_j(4-4r)}+\int_{O^*_{j,r}}\right\}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)P_r(t-x)dt,$$

where $O^*_{j,r}=O_j(2\delta)\setminus O_j(4-4r)$. The integrals over $O^*_{j,r}$ are less than or equal to

$$\frac{C'w(x)}{(1-r)^{k_j}}\int_{O^*_{j,r}}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)\left|\sin\frac{t-x_j}{2}\right|\left|\sin\frac{t-x}{2}\right|^{-2}dt$$
$$+\quad C'\frac{w(x)}{(1-r)^{k_j-1}}\int_{O^*_{j,r}}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)\left|\sin\frac{t-x}{2}\right|^{-2}dt.$$

We have $\sin\frac{t-x}{2}=\sin\frac{t-x_j}{2}\cos\frac{x_j-x}{2}+\cos\frac{t-x_j}{2}\sin\frac{x_j-x}{2}$. On the other hand

$$\frac{|\sin\frac{x_j-x}{2}|}{|\sin\frac{t-x_j}{2}|}=\frac{|\sin\frac{x_j-x}{2}|}{2|\sin\frac{t-x_j}{4}\cos\frac{t-x_j}{4}|}\le\frac{1}{2}$$

and $|\cos\frac{x_j-x}{2}|\ge|\cos\frac{t-x_j}{2}|$. Thus

$$|\sin\frac{t-x}{2}|^{-2}\le 4|\sin\frac{t-x_j}{2}|^{-2}|\cos\frac{x_j-x}{2}|^{-2}\le 8|\sin\frac{t-x_j}{2}|^{-2}.$$

By (1) and (2) we will have that

$$\frac{w(x)}{(1-r)^{k_j}}\int_{O^*_{j,r}}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)\left|\sin\frac{t-x_j}{2}\right|^{-1}dt$$
$$+\quad\frac{w(x)}{(1-r)^{k_j-1}}\int_{O^*_{j,r}}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)\left|\sin\frac{t-x_j}{2}\right|^{-2}dt$$
$$\le\quad\frac{Cw(x)}{(1-r)^{k_j}}\int_{O^*_{j,r}}\Delta_\delta(t-x_j)\left|\sin\frac{t-x_j}{2}\right|^{k_j-\lambda_j-1}dt$$
$$+\quad\frac{Cw(x)}{(1-r)^{k_j-1}}\int_{O^*_{j,r}}\Delta_\delta(t-x_j)\left|\sin\frac{t-x_j}{2}\right|^{k_j-\lambda_j-2}dt$$
$$\le\quad\frac{C'w(x)}{(1-r)^{k_j}}(1-r)^{k_j-\lambda_j}+\frac{C'w(x)}{(1-r)^{k_j-1}}(1-r)^{k_j-\lambda_j-1}\le C''$$

for any $x\in O_j(2-2r)$ and $1-r<\frac{\delta}{8}$.

It is easy to see that the integrals over $O_j(4-4r)$ are less than or equal to

$$\frac{Cw(x)}{(1-r)^{k_j+2}}\int_{O_j(4-4r)}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)\left|\sin\frac{t-x_j}{2}\right|dt$$
$$+\quad\frac{Cw(x)}{(1-r)^{k_j+1}}\int_{O_j(4-4r)}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)dt$$
$$\le\quad\frac{C'w(x)}{(1-r)^{k_j+2}}(1-r)^{2+k_j-\lambda_j}+\frac{C'w(x)}{(1-r)^{k_j+1}}(1-r)^{1+k_j-\lambda_j}\le C^*,$$

where $C'' > 0, C^* > 0$ are independent of $x \in O_j(2-2r)$ and $1-r < \frac{\delta}{4}$. Thus we have proved that

$$J_j(r,x) \le C'_j \quad \text{for any} \quad x \in O_j(2-2r), \tag{28}$$

where $C'_j > 0$ is independent of $r (0 < r < 1)$.

If $x \in O_j(\frac{\delta}{2}) \setminus O_j(2-2r)$ then we write

$$\begin{aligned} J_j(r,x) \le {} & \frac{Cw(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}} \int_{O_j(1-r)} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) \left| \sin\frac{t-x_j}{2} \right| P_r(t-x)dt \\ + {} & \frac{Cw(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}} \int_{O_j(1-r)} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) P_r(t-x)dt \\ + {} & \frac{Cw(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}} \int_{1-r\le|t-x_j|\le 2|x-x_j|} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) \left| \sin\frac{t-x_j}{2} \right| P_r(t-x)dt \\ + {} & \frac{Cw(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}} \int_{1-r\le|t-x_j|\le 2|x-x_j|} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) P_r(t-x)dt \\ + {} & \frac{Cw(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}} \int_{2|x-x_j|\le|t-x_j|\le 2\delta} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) \left| \sin\frac{t-x_j}{2} \right| P_r(t-x)dt \\ + {} & \frac{Cw(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}} \int_{2|x-x_j|\le|t-x_j|\le 2\delta} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) P_r(t-x)dt \\ := {} & \sum_{l=1}^{6} J_j^{(l)}(r,x). \end{aligned}$$

To get the inequality for the sum of the first four terms we use the assertion of the theorem for the case $|\Lambda|_* = 0$ which was proved above. Observe that $|\Lambda|_* = 0$ for the weight $\frac{w(x)}{|\omega(x)|}$. Thus it follows that

$$\begin{aligned} & J_j^{(1)}(r,x) + J_j^{(2)}(r,x) + J_j^{(3)}(r,x) + J_j^{(4)}(r,x) \\ \le {} & \frac{C(1-r)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}} |\omega(x)| + \frac{C}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}} |\omega(x)| \\ + {} & \frac{C|\sin(x-x_j)|}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}} |\omega(x)| + \frac{C}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}} |\omega(x)| \le C'_j, \end{aligned}$$

where $C'_j > 0$ is independent of $x \in O_j(\frac{\delta}{2}) \setminus O_j(2-2r)$ and $r (0 < r < 1)$. Similarly we get the inequality for the term $J_j^{(6)}(r,x)$.

To estimate the term $J_j^{(5)}(r,x)$ suppose that $x > x_j$ and write

$$\begin{aligned} & \int_{2|x-x_j|\le|t-x_j|\le 2\delta} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) \left| \sin\frac{t-x_j}{2} \right| P_r(t-x)dt \\ = {} & \int_{x_j+2(x-x_j)}^{x_j+2\delta} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) \sin\frac{t-x_j}{2} P_r(t-x)dt \\ + {} & \int_{x_j-2\delta}^{x_j-2(x-x_j)} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) \sin\frac{x_j-t}{2} P_r(t-x)dt. \end{aligned}$$

If $t\in[x_j-2\delta,x_j-2(x-x_j)]$ then it follows that $x-t>x_j-t$. Thus

$$\begin{aligned}
&\frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}\int_{x_j-2\delta}^{x_j-2(x-x_j)}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)\sin\frac{x_j-t}{2}P_r(t-x)dt\\
\le\ &\frac{C'(1-r)|\sin(x-x_j)|^{\lambda_j}}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}\int_{x_j-2\delta}^{x_j-2(x-x_j)}[\sin\frac{x_j-t}{2}]^{k_j-\lambda_j+1}[\sin\frac{x-t}{2}]^{-2}dt\\
\le\ &\frac{C''(1-r)|\sin(x-x_j)|^{\lambda_j}}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}(x-x_j)^{k_j-\lambda_j}\le C_j',
\end{aligned}$$

where $C_j'>0$ is independent of $x\in O_j(\frac{\delta}{2})\setminus O_j(2-2r)$ and $r(0<r<1)$.

On the other hand if $t\in[x_j+2(x-x_j),x_j+2\delta]$ then $\sin\frac{t-x_j}{2}>0$, hence, having in mind that $0<1+k_j-\lambda_j<1$ we can write $[\sin\frac{t-x_j}{2}]^{1+k_j-\lambda_j}\le[\sin\frac{t-x}{2}\cos\frac{x-x_j}{2}]^{1+k_j-\lambda_j}+[\cos\frac{t-x}{2}\sin\frac{x-x_j}{2}]^{1+k_j-\lambda_j}$. Thus

$$\begin{aligned}
&\int_{x_j+2(x-x_j)}^{x_j+2\delta}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)\sin\frac{t-x_j}{2}P_r(t-x)dt\\
\le\ &C'(1-r)\int_{x_j+2(x-x_j)}^{x_j+2\delta}[\sin\frac{t-x}{2}\cos\frac{x-x_j}{2}]^{1+k_j-\lambda_j}[\sin\frac{t-x}{2}]^{-2}dt\\
+\ &C'(1-r)\int_{x_j+2(x-x_j)}^{x_j+2\delta}[\cos\frac{t-x}{2}\sin\frac{x-x_j}{2}]^{1+k_j-\lambda_j}[\sin\frac{t-x}{2}]^{-2}dt.
\end{aligned}$$

The last inequality yields

$$\begin{aligned}
&\frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}\int_{x_j+2(x-x_j)}^{x_j+2\delta}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)\sin\frac{x_j-t}{2}P_r(t-x)dt\\
\le\ &\frac{C'(1-r)|\sin(x-x_j)|^{\lambda_j}}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}\int_{x_j+2(x-x_j)}^{x_j+2\delta}[\sin\frac{t-x}{2}]^{1+k_j-\lambda_j}[\sin\frac{x-t}{2}]^{-2}dt\\
+\ &\frac{C'(1-r)|\sin(x-x_j)|^{\lambda_j}}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}\int_{x_j+2(x-x_j)}^{x_j+2\delta}[\sin\frac{x-x_j}{2}]^{1+k_j-\lambda_j}[\sin\frac{x-t}{2}]^{-2}dt\\
\le\ &\frac{C''(1-r)|\sin(x-x_j)|^{\lambda_j}}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}(x-x_j)^{k_j-\lambda_j}\\
+\ &\frac{C''(1-r)|\sin(x-x_j)|^{1+k_j}}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}(x-x_j)^{-1}\le C_j',
\end{aligned}$$

where $C_j'>0$ is independent of $x\in O_j(\frac{\delta}{2})\setminus O_j(2-2r),x>x_j$ and $r(0<r<1)$. The proof in the case $x\in O_j(\frac{\delta}{2})\setminus O_j(2-2r),x<x_j$ is completely similar and we skip it. Thus the proof of Theorem 2 is finished. □

Proof of Theorem 1. By Lemma 5 it follows that the system $\mathcal{T}_\Lambda$ is closed and minimal in the space $C(w)$. Hence, Theorem 2 and Lemma 1 yield (5) for any

$f \in C(w)$. The function $u_f(r,\theta)$ defined by (14) is an harmonic function because for a real-valued function f it is the real part of an analytic function [**7**].

□

5. Proof of Theorem 3

Proof. Assume that for some $\nu, 1 \le \nu \le s$ $\lambda_\nu = k_\nu \in \mathbb{N}$. Let $\delta > 0$ be such that $[x_\nu - \delta, x_\nu + \delta] \bigcap (X \setminus \{x_\nu\}) = \emptyset$.

By (19)-(21) it is clear that for any $f \in C(w)$

$$w(x) \int_{\mathbb{T}} f(t)[P_{X,\Lambda,r}(x,t) - B_{r,\nu}(x,t)]dt$$

converges uniformly on $[x_\nu - \delta, x_\nu + \delta]$ when $r \to 1-$. The proof of Theorem 3 is finished if we indicate a function $\varphi \in C(w)$ such that

$$w(x) \int_{\mathbb{T}} \varphi(t) B_{r,\nu}(x,t)dt$$

does not converge uniformly on $[x_\nu - \delta, x_\nu + \delta]$. Without loss of generality assume that $x_\nu = 0$. It is easy to check that there exists a function φ continuous on $\mathbb{T}$ which satisfies to the following conditions:

$$\varphi(t) = \frac{1}{T_{k_\nu - 1,\nu}(t)} \qquad \text{if} \quad t \in [-\beta\delta, \beta\delta] \qquad \text{and} \qquad 0 < \beta < 1,$$

and the function φ is linear on the intervals $[-\delta, -\beta\delta]$, $[\beta\delta, \delta]$ and vanishes out of $[-\delta, \delta]$.

The integral

$$\int_{\mathbb{T}} \varphi(t) T_{k_\nu - 1,\nu}(t)dt \neq 0.$$

By Theorem 2 it follows that

$$w(x) \int_{\mathbb{T}} \varphi(t)[B_{r,\nu}(x,t) - P_r^{(k_\nu - 1)}(x_\nu - x) T_{k_\nu - 1,\nu}(t)]dt$$

converges uniformly on $\mathbb{T}$ when $r \to 1-$. Thus to finish the proof we have to prove that if $r \to 1-$

$$w(x) P_r^{(k_\nu - 1)}(x_\nu - x) \quad \text{diverges in the uniform norm on} \quad [-\delta, \delta]. \tag{29}$$

For the proof of (29) we need following two lemmas.

Lemma 6. *For any* $m \in \mathbb{N}$

$$P_r^{(2m)}(x) = (1 - r^2) \sum_{l=0}^{m} a_l^{(m)}(r) \sin^{2l} x [\xi_r(x)]^{-(m+l+1)} + R_m(r,x),$$

where $a_l^{(m)}(r) (0 \le l \le m)$ *are polynomials in* r *such that* $(-1)^{m+l} a_l^{(m)}(1) > 0$ *and*

$$R_m(r,x) = \sum_{\nu=2}^{2m} b_\nu^{(m)}(r, \sin x, \cos x)[\xi_r(x)]^{-\nu},$$

such that $b_\nu^{(m)}(t,y,z), 2 \le \nu \le 2m-1$ *are polynomials in three variables and*

$$\lim_{r \to 1-} (1-r)^{2m+1} b_\nu^{(m)}(r, \sin x, \cos x)[\xi_r(x)]^{-\nu} = 0. \tag{30}$$

uniformly for $|x| \le \frac{1}{4}$.

LEMMA 7. *For any* $m \in \mathbb{N}$

$$P_r^{(2m-1)}(x) = (1-r^2)\sum_{l=0}^{m-1} c_l^{(m)}(r)\sin^{2l+1} x[\xi_r(x)]^{-(m+l+1)} + R^*(r,x),$$

where $c_l^{(m)}(r)(0 \le l \le m-1)$ *are polynomials in* r *such that* $(-1)^{m+l}c_l^{(m)}(1) > 0$ *and*

$$R_m^*(r,x) = \sum_{\nu=2}^{2m-1} d_\nu^{(m)}(r,\sin x,\cos x)[\xi_r(x)]^{-\nu},$$

such that $d_\nu^{(m)}(t,y,z), 2 \le \nu \le 2m-1$ *are polynomials in three variables and*

$$\lim_{r\to 1-}(1-r)^{2m} d_\nu^{(m)}(r,\sin x,\cos x)[\xi_r(x)]^{-\nu} = 0 \tag{31}$$

uniformly for $|x| \le \frac{1}{4}$.

PROOF OF LEMMA 6. The proof is provided by induction. For $m = 1$ the assertion is easily checked by calculating the second derivative of the Poisson kernel. Assume that the assertion is true for some $m \in \mathbb{N}$. Let us calculate the second derivative of the term $\sin^{2l} x[\xi_r(x)]^{-(m+l+1)}$. The derivation when $l = 0$ is easy to check so we consider the case $1 \le l \le m$. We have

$$\begin{aligned} \frac{d}{dx}\left[\sin^{2l} x[\xi_r(x)]^{-(m+l+1)}\right] &= 2l\sin^{2l-1} x\cos x[\xi_r(x)]^{-(m+l+1)} \\ - \quad &(m+l+1)2r\sin^{2l+1} x[\xi_r(x)]^{-(m+l+2)} \end{aligned}$$

and

$$\begin{aligned} &\frac{d^2}{dx^2}\left[\sin^{2l} x[\xi_r(x)]^{-(m+l+1)}\right] \\ = \quad &2l(2l-1)\sin^{2l-2} x\cos^2 x[\xi_r(x)]^{-(m+l+1)} \\ - \quad &2l(m+l+1)2r\cos x\sin^{2l} x[\xi_r(x)]^{-(m+l+2)} - 2l\sin^{2l} x[\xi_r(x)]^{-(m+l+1)} \\ - \quad &(2l+1)(m+l+1)2r\sin^{2l} x\cos x[\xi_r(x)]^{-(m+l+2)} \\ + \quad &(m+l+1)(m+l+2)(2r)^2\sin^{2l+2} x[\xi_r(x)]^{-(m+l+3)}. \end{aligned}$$

In the above equality we write

$$\sin^{2l-2} x\cos^2 x = \sin^{2l-2} x + \sin^{2l-2} x\,[\cos^2 x - 1]$$
$$\sin^{2l} x\cos x = \sin^{2l} x + \sin^{2l} x\,[\cos x - 1]$$

which permits us to write the main sum in the representation for $P_r^{(2m+2)}(x)$ without the powers of $\cos x$. Thus in the main part of the representation of $P_r^{(2m+2)}(x)$ we will have the following terms:

$$\begin{aligned} &2l(2l-1)\sin^{2l-2} x[\xi_r(x)]^{-(m+l+1)}, \qquad -2l(m+l+1)2r\sin^{2l} x[\xi_r(x)]^{-(m+l+2)}, \\ &-(2l+1)(m+l+1)2r\sin^{2l} x[\xi_r(x)]^{-(m+l+2)}, \\ &(m+l+1)(m+l+2)(2r)^2\sin^{2l+2} x[\xi_r(x)]^{-(m+l+3)} \end{aligned}$$

multiplied by $1-r^2$. We have that $a_0^{(m+1)}(r) = -a_0^{(m)}(r)(m+1)2r + 2a_1^{(m)}(r)$. Hence, $(-1)^{m+1}a_0^{(m+1)}(1) = (-1)^{m+2}2a_0^{(m)}(1)(m+1) + (-1)^{m+1}a_1^{(m)}(1) > 0$. If $1 \le l \le m$ we have that

$$a_l^{(m+1)}(r) = a_{l-1}^{(m)}(r)2l(2l-1) - a_l^{(m)}(r)(4l+1)(m+l+1)2r + a_{l+1}^{(m)}(m+l+1)(m+l+2)(2r)^2.$$

Which gives $(-1)^{m+l+1}a_l^{(m+1)}(1) > 0$. That $a_{m+1}^{(m+1)}(1) > 0$ is obvious. Observe that the coefficients of the mentioned terms are positive. It is clear that $a_0^{(1)}(1) > 0$, hence, $a_0^{(m)}(1) > 0$ for all $m \in \mathbb{N}$. The following terms

$$\begin{aligned}
&2l(2l-1)\sin^{2l-2}x[\cos^2 x - 1][\xi_r(x)]^{-(m+l+1)},\\
&-2l\sin^{2l}x[\xi_r(x)]^{-(m+l+1)},\\
&(2l+1)(m+l+1)2r\sin^{2l}x[\cos x - 1][\xi_r(x)]^{-(m+l+2)}
\end{aligned}$$

multiplied by $1 - r^2$ go to the sum $R_m(r, x)$.

It is easy to check that the second derivative of every term in the sum $R_m(r, x)$ satisfies the condition (30) if we replace m by $m + 1$. □

Lemma 7 follows immediately from Lemma 6 by derivation of the representation of $P_r^{(2m-2)}(x)$.

The proof of (29) is finished if we show that for a sufficiently small $\alpha (0 < \alpha < 1)$

$$\limsup_{r\to 1-} w(\alpha(1-r))P_r^{(k_\nu-1)}(\alpha(1-r)) \neq 0.$$

We consider that $k_\nu - 1 = 2m$. Let $\alpha(0 < \alpha < 1)$ be such that

$$\left|\sum_{l=1}^{m} a_l^{(m)}(r)\sin^{2l}(\alpha(1-r))[\xi_r(\alpha(1-r))]^{-(m+l+1)}\right| < \frac{1}{4}\left|a_0^{(m)}(1)\right|[\xi_r(\alpha(1-r))]^{-(m+1)}.$$

By Lemma 6 we obtain that

$$\begin{aligned}
&\limsup_{r\to 1-} w(\alpha(1-r))\left|P_r^{(k_\nu-1)}(\alpha(1-r))\right|\\
&\geq \limsup_{r\to 1-}\frac{1}{2}\left|a_0^{(m)}(1)\right|\sin^{2m+1}(\alpha(1-r))[\xi_r(\alpha(1-r))]^{-(m+1)} > 0
\end{aligned}$$

which proves (29). The argument of the proof in the case $k_\nu = 2m$ is similar. □

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