The Dirichlet problem in weighted norm

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Abstract. Let w be a weight functions satisfying conditions (1) and (2) and let $C(w)$ be the linear space of all complex valued functions f defined on $\mathbb{T}$ such that fw is continuous on $\mathbb{T}$ and (3) holds. We study the following classical Dirichlet problem.

For any $f \in C(w)$ find a harmonic function u_f on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ such that

$$\lim_{r\to 1-} \|u_f(r,\theta) - f(\theta)\|_{C(w)} = 0,$$

where $z = re^{i\theta}$.

1. Introduction and definitions

Set $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and let

$$w(x) = v(x) \prod_{j=1}^{s} \left| \sin\left(\frac{x - x_j}{2}\right)\right|^{\lambda_j}, \tag{1}$$

where $v(x)$ is a positive continuous function on $\mathbb{T}$ such that for some $C_0 > 0$

$$\max_{x\in\mathbb{T}} \{v(x), 1/v(x)\} \le C_0; \tag{2}$$

$X = \{x_1, x_2, \dots, x_s\} \subset \mathbb{T}$ is a set of points, and $\Lambda = \{\lambda_j\}_{l=1}^{s}$ is a collection of positive real numbers.

The linear space of all complex valued functions f defined on $\mathbb{T}$ such that fw is continuous on $\mathbb{T}$ and

$$\lim_{x\to x_j} f(x)w(x) = 0, \quad j = 1, \dots, s \tag{3}$$

will be denoted by $C(w)$. If we put

$$\|f\|_{C(w)} = \max_{x\in\mathbb{T}} |f(x)|w(x). \tag{4}$$

for any $f \in C(w)$ then it is easy to check that $C(w)$ will be a Banach space. The space of continuous complex valued functions defined on $\mathbb{T}$ with the standard norm will be denoted by $C_{\mathbb{T}}$.

We study the following classical

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Dirichlet problem. For any $f \in C(w)$ find a harmonic function u_f on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ such that

$$\lim_{r\to 1-} \|u_f(r,\theta) - f(\theta)\|_{C(w)} = 0, \tag{5}$$

where $z = re^{i\theta}$.

The Dirichlet problem in the $L^p(\psi), 1 \le p < \infty$ metric, where the weight function $\psi > 0$ has singularities was studied in [**2**].

The solution of the classical Dirichlet problem when the weight function has no singularities is represented by the convolution of f with the Poisson kernel

$$P_r(x) := \frac{1 - r^2}{1 - 2r\cos x + r^2}, \quad 0 < r < 1.$$

In our case, when the weight function has essential singularities, the solution can not be represented as a convolution. In this case modified Poisson kernels (see [**3**], [**4**]) replace the Poisson kernel.

We set $k_j := [\lambda_j]$, where $[\lambda]$ is the integer part of the number λ, $\lambda - 1 < [\lambda] \le \lambda$. Set

$$\omega(x) = \omega_{X,\Lambda}(x) := \prod_{j=1}^{s} \sin^{k_j}\left(\frac{x - x_j}{2}\right). \tag{6}$$

if $|\Lambda| := \sum_{j=1}^{s} k_j \ge 1$ and

$$\omega(x) = \omega_{X,\Lambda}(x) :\equiv 1 \qquad \text{if} \quad |\Lambda| = 0. \tag{7}$$

If $|\Lambda| > 0$ and $|\Lambda| = 2n - 1$, where $n = 1, \dots$ we denote by $T_{j,l}(x)$ the trigonometric polynomials of degree n such that

$$T_{j,l}^{(m)}(x_i) = \delta_{l,m}\delta_{i,j} \quad 1 \le i, j \le s;\ 0 \le m \le k_i - 1;\ 0 \le l \le k_j - 1. \tag{8}$$

If in the above formula $k_j = 0$ then no polynomials $T_{j,l}$ are defined.

For the uniqueness of the solution when $|\Lambda| = 2n$ for $n = 1, 2, \dots$ we put an additional condition on the trigonometric polynomials $T_{j,l}(x)$. That condition is formulated in terms of the leading coefficients of the trigonometric polynomial $\omega_{X,\Lambda}(x)$ defined by (6)

$$\omega_{X,\Lambda}(x) = a_n \cos nx + b_n \sin nx + \cdots. \tag{9}$$

We set that

$$\frac{b_n^{(j,l)}}{a_n^{(j,l)}} = -\frac{a_n}{b_n} \quad \text{and} \quad a_n^{(j,l)} = 0 \quad \text{if} \quad b_n = 0, \tag{10}$$

where

$$T_{j,l}(x) = a_n^{(j,l)} \cos nx + b_n^{(j,l)} \sin nx + \cdots \quad (1 \le j \le s, 0 \le l \le k_j - 1).$$

The modified Poisson kernels ([**2**], (1.9)) are defined as follows:

$$P_{X,\Lambda,r}(x,t) := P_r(t - x) - \sum_{j=1}^{s} \sum_{l=0}^{k_j - 1} P_r^{(l)}(x_j - x) T_{j,l}(t) \tag{11}$$

if $|\Lambda| > 0$ and

$$P_{X,\Lambda,r}(x,t) := P_r(t - x) \quad \text{if} \quad |\Lambda| = 0, \tag{12}$$

where $P_r^{(l)}(x) := \frac{d^l}{dx^l} P_r(x)$. Note that if $k_j = 0$ then the term with the index j is absent in the formula (11). Set

$$O_j(\rho) = \{t \in \mathbb{T} : |t - x_j| < \rho\}, \qquad \text{where} \quad 1 \leq j \leq s \quad \text{and} \quad \rho > 0.$$

Further in the text constants will be denoted by C, C_j, C_j' and they may be different in different inequalities.

We prove the following main theorem.

THEOREM 1.1. *Let $\Lambda \bigcap \mathbb{Z} = \emptyset$ and let w be a weight function, where w satisfies the conditions (1) and (2). Then there exists a unique harmonic function u_f on the unit disk D such that (5) holds. Moreover,*

$$u_f(r, \theta) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) P_{X,\Lambda,r}(\theta, t)\, dt, \tag{13}$$

where the kernel $P_{X,\Lambda,r}$ is defined by (11).

The proof of the above theorem is based on the following result.

THEOREM 1.2. *For any weight function w, where w satisfies the conditions (1), (2) and $\Lambda \bigcap \mathbb{Z} = \emptyset$ there exists $C > 0$ such that*

$$\sup_{0<r<1} \sup_{x\in\mathbb{T}} w(x) \int_{\mathbb{T}} \frac{1}{w(t)} |P_{X,\Lambda,r}(x,t)| dt \leq C. \tag{14}$$

Further we will use the following terminology. A system of elements $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ in a Banach space B will be called closed system if any element of B can be arbitrarily approximated by a finite linear combination of elements of Φ. We will say that Φ is complete with respect to the dual space B^* if the condition

$$\phi^*(\varphi_n) = 0, \quad \text{for all} \quad n \in \mathbb{N},$$

where $\phi^* \in B^*$ yields that ϕ^* is the trivial element of the space B^*. The system $\Phi = \{\varphi_n\}_{n=1}^{\infty} \subset B$ is called a minimal system if there exists $\Phi^* = \{\phi_n^*\}_{n=1}^{\infty} \subset B^*$ such that

$$\phi_n^*(\varphi_k) = \delta_{nk} \qquad\qquad n, k \in \mathbb{N}, \tag{15}$$

where δ_{nk} is the Kronecker symbol. We will say that a system of elements $\Phi = \{\varphi_n\}_{n=1}^{\infty} \subset B$ is an $A-$basis of the Banach space B if Φ is closed and minimal in B and for any $x \in B$

$$\lim_{r\longrightarrow 1-} \|x - \sum_{n=1}^{\infty} r^n \phi_n^*(x)\varphi_n\|_B = 0,$$

where $\Phi^* = \{\phi_n^*\}_{n=1}^{\infty} \subset B^*$ is the uniquely defined system in the dual space for which the condition (15) holds. We will say that the system Φ^* is the conjugate system of Φ. For the convenience of the reader we will formulate the analogue of Banach's theorem for the $A-$*bases*. We will not bring the proof because it is similar with some technical modifications to the proof of Banach's original proof [**1**]. Some references about summation bases can be found in [**6**] and [**2**].

LEMMA 1.1. *Let $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ is a closed and minimal system in a separable Banach space B. Then Φ is an $A-$basis of B if and only if there exists a constant $C > 0$ such that for any $x \in B$*

$$\sup_{0<r<1} \|\sum_{n=1}^{\infty} r^n \phi_n^*(x)\varphi_n\|_B \leq C\|x\|_B. \tag{16}$$

2. Auxiliary results

In the proof of Theorem 2 we are going to decompose the kernel $P_{X,\Lambda,r}(x,t)$ into a sum of kernels $B_{r,j}(x,t)$ $(1 \le j \le s)$. For that purpose we use the identity

$$\sum_{j=1}^{s} T_{j,0}(t) \equiv 1, \tag{17}$$

where it is supposed that $T_{j,0} \equiv 0$, if $k_j = 0$.

By (11) and (17) we have

$$P_{X,\Lambda,r}(x,t) = \sum_{j=1}^{s} B_{r,j}(x,t), \tag{18}$$

where $B_{r,j}(x,t) \equiv 0$ if $k_j = 0$, and

$$B_{r,j}(x,t) = P_r(t-x)T_{j,0}(t) - \sum_{l=0}^{k_j-1} P_r^{(l)}(x_j - x)T_{j,l}(t) \tag{19}$$

if $k_j > 0$. We set

$$\xi_r(t) = 1 - 2r\cos t + r^2. \tag{20}$$

Recall some lemmas from [**3**] which would be applied for the proof of our main result.

Lemma 2.1. *Let $\Lambda = 2N+1 (N = 0, 1, \dots)$. Then for every $j (1 \le j \le s)$*

$$B_{r,j}(x,t) = P_r(t-x)\omega(t)\big[G_r^*(x)\sin\frac{t-x_j}{2} + G_r^{**}(x)\cos\frac{t-x_j}{2}\big]$$

and there is a $C > 0$ independent of r and x such that

$$|G_r^*(x)| \le C[\xi_r(x-x_j)]^{-\frac{k_j+1}{2}}$$

and

$$|G_r^{**}(x)| \le C[\xi_r(x-x_j)]^{-\frac{k_j}{2}}.$$

Lemma 2.2. *Let $\Lambda = 2N (N = 1, 2 \dots)$. Then for every $j (1 \le j \le s)$*

$$B_{r,j}(x,t) = P_r(t-x)\omega(t)\big[G_r^*(x)\sin(t-x_j) + G_r^{**}(x)\cos(t-x_j) + G_r^{***}(x)\big]$$

and there is a $C > 0$ independent of r and x such that

$$|G_r^*(x)| \le C[\xi_r(x-x_j)]^{-\frac{k_j+1}{2}},$$

$$|G_r^{**}(x)| \le C[\xi_r(x-x_j)]^{-\frac{k_j}{2}}$$

and

$$|G_r^{***}(x)| \le C[\xi_r(x-x_j)]^{-\frac{k_j}{2}}.$$

Let δ_x be the Dirac measure concentrated at a given point $x \in \mathbb{T}$. We consider the finite dimensional subspace of the dual space $C_{\mathbb{T}}^*$ generated by the Dirac measures $\delta_{x_j}, 1 \le j \le s$ which will be denoted by $\mathcal{M}_X$. From the Hahn-Banach theorem we obtain the following description of the dual space $C^*(w)$ of $C(w)$.

LEMMA 2.3. *Let w be a weight function, where w satisfies the conditions (1), (2). Then $\tau \in C^*(w)$ if and only if there exists a unique class of equivalences $E_\tau \in C^*_{\mathbb{T}}/\mathcal{M}_X$ of complex Borel measures such that*

$$\tau(f) = \int_{\mathbb{T}} f(t)w(t)d\mu(t) \qquad \forall f \in C(w) \quad \textit{and} \quad \forall \mu \in E_\tau,$$

and

$$\|\tau\|_{C^*(w)} = \|E_\tau\|_{C^*/\mathcal{M}_X}.$$

We take a system of functions $\mathcal{T}_\Lambda$ which in the space $C(w)$ will replace the trigonometric system. Let $\mathbb{Z}^*_\Lambda = \{k \in \mathbb{Z} : \ k = -n, n, -n-1, n+1, \dots\}$ if $|\Lambda| = 2n-1$, and $\mathbb{Z}^*_\Lambda = \{k \in \mathbb{Z} : \ k = -n-1, n+1, \dots\}$ if $|\Lambda| = 2n$.

Set

$$\mathcal{T}_\Lambda = \{e^{ikx} : \ k \in \mathbb{Z}^*_\Lambda\} \qquad \text{if} \quad |\Lambda| = 2n-1,$$

where $n = 1, \dots$; and if $|\Lambda| = 2n$ we put

$$\mathcal{T}_\Lambda = \{a_n \cos nx + b_n \sin nx, e^{ikx} : \ k \in \mathbb{Z}^*_\Lambda\},$$

where the numbers a_n, b_n are the senior coefficients of the polynomial (9).

LEMMA 2.4. *Let w be a weight function, where w satisfies the conditions (1) and (2). Then the system $\mathcal{T}_\Lambda$ is closed and minimal in $C(w)$ with the conjugate system $\{E_k\}_{k \in \mathbb{Z}^*_\Lambda}$ if $|\Lambda| = 2n-1$ and with the conjugate system $\{E_n, E_k\}_{k \in \mathbb{Z}^*_\Lambda}$ if $|\Lambda| = 2n$, where $E_k \in C^*/\mathcal{M}_X$. Moreover, absolutely continuous complex Borel measures $dg_k \in E_k$ for $k \in \mathbb{Z}^*_\Lambda$ are defined by the equations*

$$dg_k(x) = \frac{1}{2\pi w(x)} \left(e^{ikx} - \sum_{j=1}^{s} \sum_{l=0}^{k_{j-1}} \frac{d^{\,l}}{dt^l} e^{ikt} \Big|_{t=x_j} T_{j,l}(x) \right) dx,$$

when $|\Lambda| = 2n-1$ and if $|\Lambda| = 2n$

$$dg_n(x) = \frac{1}{\pi(a_n^2 + b_n^2)w(x)} \Big(a_n \cos nx + b_n \sin nx - \sum_{j=1}^{s} \sum_{l=0}^{k_j-1} \frac{d^{\,l}}{dt^l} (a_n \cos nt + b_n \sin nt) \Big|_{t=x_j} T_{j,l}(x) \Big) dx, \tag{21}$$

and for $k \in \mathbb{Z}^*_\Lambda$

$$dg_k(x) = \frac{1}{w(x)2\pi} \Big(e^{ikx} - \sum_{j=1}^{s} \sum_{l=0}^{k_j-1} \frac{d^l}{dt^l} e^{ikt} \Big|_{t=x_j} T_{j,l}(x) \Big) dx. \tag{22}$$

PROOF. We will bring the proof for the case $|\Lambda| = 2n$. When $|\Lambda| = 2n-1$ the proof is similar. Suppose that for some $\phi^* \in C^*(w)$

$$\phi^*(a_n \cos nx + b_n \sin nx) = 0$$

and

$$\phi^*(e^{ikx}) = 0 \quad \text{for all} \quad k \in \mathbb{Z}^*_\Lambda.$$

Then by Lemma 2.3 there exists a unique class of equivalences of Borel measures $E_{\phi^*} \in C^*/\mathcal{M}_X$ such that for all $\mu \in E_{\phi^*}$

$$\phi^*(a_n \cos nx + b_n \sin nx) = \int_{\mathbb{T}} (a_n \cos nt + b_n \sin nt) w(t) d\mu(t) = 0$$

and

$$\phi^*(e^{ikx}) = \int_{\mathbb{T}} e^{-ikt} w(t) d\mu(t) = 0 \qquad \forall \mu \in E_{\phi^*} \quad \text{and} \quad \forall k \in \mathbb{Z}_\Lambda^*.$$

We put

$$\alpha_n(\mu) = \frac{1}{\pi(a_n^2 + b_n^2)} \int_{\mathbb{T}} (b_n \cos nt - a_n \sin nt) w(t) d\mu(t)$$

and

$$\alpha_m(\mu) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-imt} w(t) d\mu(t) \quad \text{for} \quad |m| \leq n-1.$$

Hence, by the closedness of the trigonometrical system in $C_{\mathbb{T}}$ we obtain that for any $\mu \in E_\phi$

$$w(t)d\mu(t) = \big[\alpha_n(\mu)\big(b_n \cos nt - a_n \sin nt\big) + \sum_{|m| \leq n-1} \alpha_{-m}(\mu) e^{imt}\big] dt.$$

Hence, if $\mu_0(t) \in E_\phi$ is such that $\mu_0(\{x_j\}) = 0, 1 \leq j \leq s$ then by (1), (2) and (6) we obtain that

$$\alpha_n(\mu_0)\big(b_n \cos nt - a_n \sin nt\big) + \sum_{|m| \leq n-1} \alpha_m(\mu_0) e^{imt} = C \cdot \omega_{X,\Lambda}(t),$$

where $C \in \mathbb{C}$. From the last equality and (9) immediately follows that $\alpha_n(\mu_0) = 0$. Which yields $C = 0$ and consequently $\alpha_m(\mu_0) = 0$ for all $|m| \leq n-1$. Thus $E_{\phi^*} = \mathcal{M}_X$ which proves that the system $\mathcal{T}_\Lambda$ is closed in $C(w)$. One can easily check that the absolutely continuous Borel measures (21), (22) are finite Borel measures which satisfy the conditions

$$\int_{\mathbb{T}} (a_n \cos nt + b_n \sin nt) w(t) dg_k(t) = \delta_{nk}, \text{ for } k = n \text{ and all } k \in \mathbb{Z}_\Lambda^*;$$

$$\int_{\mathbb{T}} e^{-ijt} w(t) dg_k(t) = \delta_{jk}, \text{ for } k = n \text{ and for all } j, k \in \mathbb{Z}_\Lambda^*.$$

Hence, the system is also minimal in $C(w)$. □

For any $0 < a < 1$ we define $\Delta_a \in C_{\mathbb{T}}$ as follows

$$\Delta_a(x) = \begin{cases} 1 & \text{if} \quad x \in [-\frac{a}{2}, \frac{a}{2}]; \\ \frac{2}{a}(x+a) & \text{if} \quad x \in [-a, -\frac{a}{2}); \\ -\frac{2}{a}(x-a) & \text{if} \quad x \in (\frac{a}{2}, a]; \\ 0 & \text{elsewhere.} \end{cases}$$

3. Proof of Theorem 1.2

Proof. We set $\delta = \min_{i \neq j}\{\frac{1}{2}, \frac{1}{4}|x_i - x_j|\}$.

For the convenience of the reader at first let us consider the case $|\Lambda| = 0$. By (12) the inequality (14) can be written in the following form:

$$\mathcal{I}(r,x) := w(x) \int_{\mathbb{T}} \frac{1}{w(t)} P_r(x-t) dt \leq C \qquad \text{for any} \quad x \in \mathbb{T}. \tag{23}$$

To prove (23) we write

$$\mathcal{I}(r,x) = \sum_{j=1}^{s} w(x) \int_{\mathbb{T}} \frac{1}{w(t)} \Delta_\delta(t - x_j) P_r(x-t) dt$$

$$+w(x)\int_{\mathbb{T}}\frac{1}{w(t)}\big[1-\sum_{j=1}^{s}\Delta_\delta(t-x_j)\big]P_r(x-t)dt := \sum_{j=1}^{s}\mathcal{I}_j(r,x)+\mathcal{I}_0(r,x).$$

Fix any $j (1\le j\le s)$ and consider three cases:
1) $x\in\mathbb{T}\setminus O_j(2\delta)$;
2) $x\in O_j(2\delta)\setminus O_j(\frac{\delta}{2})$;
3) $x\in O_j(\frac{\delta}{2})$.

In the case 1) the well known estimates for the Poisson kernel yield

$$\mathcal{I}_j(r,x)\le w(x)\int_{O_j(\delta)}\frac{1}{w(t)}dt\,\min\left\{\frac{2}{1-r},\frac{1-r}{2r\sin^2\delta}\right\}. \tag{24}$$

Recall that $0<\lambda_j<1$. Hence, by (1) and (2) we obtain that for some $C_j>0$

$$\mathcal{I}_j(r,x)\le C_j \tag{25}$$

for any $0<r<1$.

In the case 2) the estimate (25) is trivial if $1-r\ge\frac{\delta}{4}$. If $1-r<\frac{\delta}{4}$ then we write

$$\mathcal{I}_j(r,x)=w(x)\left\{\int_{O_j(\frac{\delta}{4})}+\int_{O_j(2\delta)\setminus O_j(\frac{\delta}{4})}\right\}\frac{1}{w(t)}\Delta_\delta(t-x_j)P_r(x-t)dt.$$

Afterwards conditions (1), (2) yield that the function $\frac{w(x)}{w(t)}$ is bounded uniformly on the set

$$\Pi_j(\delta)=\left\{(x,t)\in\mathbb{T}^2:\frac{\delta}{2}<|x-x_j|<2\delta\quad\&\quad\frac{\delta}{4}\le|t-x_j|\le2\delta\right\}.$$

Thus the second integral on the right hand of the above equality is bounded. To finish the proof for the case 2) we write

$$w(x)\int_{O_j(\frac{\delta}{4})}\frac{\Delta_\delta(t-x_j)}{w(t)}P_r(x-t)dt\le w(x)\int_{O_j(\frac{\delta}{4})}\frac{1}{w(t)}dt\,\frac{1-r}{2\sin^2\frac{\delta}{8}}\le C_j'$$

for some $C_j'>0$.

In the case 3) the estimate (25) is trivial if $1-r\ge\frac{\delta}{4}$. If $x\in O_j(2-2r)$ and $1-r<\frac{\delta}{4}$ then we have

$$\mathcal{I}_j(r,x)=w(x)\left\{\int_{O_j(1-r)}+\int_{O_j(2\delta)\setminus O_j(1-r)}\right\}\frac{1}{w(t)}\Delta_\delta(t-x_j)P_r(x-t)dt.$$

By (1) and (2) we have that the function $\frac{w(x)}{w(t)}$ is bounded by a $C>0$ independent of any t from the set $1-r\le|t-x_j|\le2\delta$. Thus the second integral on the right hand of the above equality is bounded. Afterwards we write

$$w(x)\int_{O_j(1-r)}\frac{\Delta_\delta(t-x_j)}{w(t)}P_r(x-t)dt\le\frac{w(x)}{1-r}\int_{O_j(1-r)}\frac{1}{w(t)}dt\le C'$$

for some $C'>0$.

If $x\in O_j(\frac{\delta}{2})\setminus O_j(2-2r)$ then we write

$$\begin{aligned}\mathcal{I}_j(r,x) &= w(x)\left\{\int_{O_j(1-r)}+\int_{O_j(2|x-x_j|)\setminus O_j(1-r)}+\int_{O_j(2\delta)\setminus O_j(2|x-x_j|)}\right\}\cdots dt\\ &:= \mathcal{I}_j^{(1)}(r,x)+\mathcal{I}_j^{(2)}(r,x)+\mathcal{I}_j^{(3)}(r,x).\end{aligned}$$

As above by (1), (2) we have that $\frac{w(x)}{w(t)}$ is bounded by an absolute constant for any t from the set $O_j(2\delta)\setminus O_j(2|x-x_j|)$. Thus $\mathcal{I}_j^{(3)}(r,x)$ is bounded.

Afterwards we write

$$\begin{aligned}\mathcal{I}_j^{(1)}(r,x) &= w(x)\int_{O_j(1-r)}\frac{\Delta_\delta(t-x_j)}{w(t)}P_r(x-t)dt\\ &\le \frac{w(x)}{2\sin^2\frac{|x-x_j|}{4}}(1-r)\int_{O_j(1-r)}\frac{1}{w(t)}dt\\ &\le C\frac{w(x)}{2\sin^2\frac{|x-x_j|}{4}}(1-r)^{2-\lambda_j}\le C',\quad \forall\, x\in O_j(\frac{\delta}{2})\setminus O_j(2-2r),\end{aligned}$$

where $C'>0$.

To evaluate $\mathcal{I}_j^{(2)}(r,x)$ we set $\xi=x-x_j$ and

$$\Upsilon_\xi(a):=\{\tau\in\mathbb{T}:|\tau-\xi|<a\},\quad a>0. \tag{26}$$

We recall that $1-r<\frac{\delta}{4}$ and check that

$$\mathcal{I}_j^{(2,1)}(r,\xi+x_j):=w(\xi+x_j)\int_{\Upsilon_\xi(1-r)}\frac{\Delta_\delta(\tau)}{w(\tau+x_j)}P_r(\xi-\tau)d\tau\le C$$

for any ξ such that $|\xi|<\frac{\delta}{2}$, where $C>0$. Afterwards we set

$$\Omega_\xi=\{\tau\in\mathbb{T}:1-r\le|\tau|\le 2\xi\,\&\,1-r\le|\xi-\tau|\} \tag{27}$$

and write for $\xi>0$

$$\begin{aligned}\mathcal{I}_j^{(2,2)}(r,\xi+x_j) &:= w(\xi+x_j)\int_{\Omega_\xi}\frac{\Delta_\delta(\tau)}{w(\tau+x_j)}P_r(\xi-\tau)d\tau\\ &= w(\xi+x_j)\left\{\int_{\Omega'_\xi}+\int_{\Omega''_\xi}\right\}\frac{\Delta_\delta(\tau)}{w(\tau+x_j)}P_r(\xi-\tau)d\tau,\end{aligned}$$

where

$$\Omega'_\xi := \left\{\tau\in\mathbb{T}:1-r\le|\tau|\le 2\xi\,\&\,|\xi-\tau|\ge\frac{3}{4}\xi\right\}; \tag{28}$$

$$\Omega''_\xi := \left\{\tau\in\mathbb{T}:1-r\le|\tau|\le 2\xi\,\&\,1-r\le|\xi-\tau|<\frac{3}{4}\xi\right\}. \tag{29}$$

Then we derive

$$\begin{aligned}& w(\xi+x_j)\int_{\Omega''_\xi}\frac{\Delta_\delta(\tau)}{w(\tau+x_j)}P_r(\xi-\tau)d\tau\\ \le\;& C\xi^{\lambda_j}\left\{\int_{-2\xi}^{-(1-r)}+\int_{(1-r)}^{\xi-(1-r)}+\int_{\xi+(1-r)}^{2\xi}\right\}\frac{1-r}{|\xi-\tau|^2\,|\tau|^{\lambda_j}}d\tau\\ :=\;& \mathcal{I}_j^{(2,2,1)}(r,\xi+x_j)+\mathcal{I}_j^{(2,2,2)}(r,\xi+x_j)+\mathcal{I}_j^{(2,2,3)}(r,\xi+x_j)\end{aligned}$$

for some $C>0$. Afterwards we obtain

$$\mathcal{I}_j^{(2,2,1)}(r,\xi+x_j)\le C(1-r)^{-1}\xi^{\lambda_j}\xi^{-\lambda_j+1}\le C';$$

$$\mathcal{I}_j^{(2,2,3)}(r,\xi+x_j)\le C(1-r)\int_{1-r}^{\xi}u^{-2}du\le C',$$

uniformly for some $C'>0$.

If $k_\xi \geq 3$ is the natural number for which $(1-r)(k_\xi - 1) \leq \xi < (1-r)k_\xi$ then we derive

$$\mathcal{I}_j^{(2,2,2)}(r, \xi + x_j) \leq C(1-r)\xi^{\lambda_j}\left\{\int_{(1-r)}^{\frac{k_\xi-1}{2}(1-r)} + \int_{\frac{k_\xi-1}{2}(1-r)}^{\xi-(1-r)}\right\}\frac{1}{|\xi-\tau|^2\,|\tau|^{\lambda_j}}d\tau$$

$$\leq \quad \frac{C(1-r)\xi^{\lambda_j}}{[(1-r)(k_\xi-1)]^2}\left[\frac{(1-r)(k_\xi-1)}{2}\right]^{1-\lambda_j} + C(1-r)\int_{(1-r)}^{+\infty} u^{-2}du \leq C',$$

where $C' > 0$. Thus $w(\xi + x_j)\int_{\Omega_\xi''}\frac{\Delta_\delta(\tau)}{w(\tau+x_j)}P_r(\xi-\tau)d\tau$ is uniformly bounded.

On the other hand

$$\begin{aligned}
& w(\xi + x_j)\int_{\Omega_\xi'}\frac{\Delta_\delta(\tau)}{w(\tau+x_j)}P_r(\xi-\tau)d\tau \\
= \quad & w(\xi + x_j)\int_{-2\xi}^{-(1-r)}\frac{\Delta_\delta(\tau)}{w(\tau+x_j)}P_r(\xi-\tau)d\tau \\
+ \quad & w(\xi + x_j)\int_{1-r}^{\frac{1}{4}\xi}\frac{\Delta_\delta(\tau)}{w(\tau+x_j)}P_r(\xi-\tau)d\tau \\
+ \quad & w(\xi + x_j)\int_{\frac{7}{4}\xi}^{2\xi}\frac{\Delta_\delta(\tau)}{w(\tau+x_j)}P_r(\xi-\tau)d\tau \\
\leq \quad & C\xi^{\lambda_j}\left(\int_{-2\xi}^{-(1-r)} + \int_{1-r}^{\frac{1}{4}\xi} + \int_{\frac{7}{4}\xi}^{2\xi}\right)\frac{1-r}{|\xi-\tau|^2\,\tau^{\lambda_j}}d\tau \leq C',
\end{aligned}$$

uniformly for some $C' > 0$. Thus $\mathcal{I}_j^{(2)}(r,x) = \mathcal{I}_j^{(2,1)}(r,\xi+x_j) + \mathcal{I}_j^{(2,2)}(r,\xi+x_j)$ is uniformly bounded for any $\xi > 0$ such that $|\xi| < \frac{\delta}{2}$. The case $\xi < 0$ is checked in a similar way. Thus we finish the proof of the inequality $\mathcal{I}_j(r,x) < C$ uniformly for any $x \in \mathbb{T}$ and $0 < r < 1$.

The function $\frac{1}{w(t)}\big(1 - \sum_{j=1}^s \Delta_\delta(t-x_j)\big)$ is continuous on $\mathbb{T}$ thus $\mathcal{I}_0(r,x)$ is uniformly bounded.

Now let $|\Lambda| \geq 1$. Without loss in generality we can suppose that $r > \frac{3}{4}$. We have to give a similar proof applying Lemmas 2.1 and 2.2. By (18) we have

$$w(x)\int_{\mathbb{T}}\frac{1}{w(t)}|P_{X,\Lambda,r}(x,t)|dt \leq w(x)\sum_{\nu=1}^s\int_{\mathbb{T}}\frac{1}{w(t)}|B_{r,\nu}(x,t)|dt.$$

It is sufficient to prove that for any $j (1 \leq j \leq s)$ such that $\lambda_j > 1$ there exists $C_j > 0$ independent of x and r such that

$$J(r,x) := w(x)\int_{\mathbb{T}}\frac{1}{w(t)}|B_{r,j}(x,t)|dt \leq C_j.$$

We write

$$J(r,x) = \sum_{\nu=1}^s w(x)\int_{\mathbb{T}}\frac{1}{w(t)}\Delta_\delta(t-x_\nu)|B_{r,\nu}(x,t)|dt$$

$$+w(x)\int_{\mathbb{T}}\frac{1}{w(t)}\big[1 - \sum_{\nu=1}^s\Delta_\delta(t-x_\nu)\big]P_r(x-t)dt := \sum_{\nu=1}^s J_\nu(r,x) + J_0(r,x).$$

We have to prove that for any $\nu(1 \leq \nu \leq s)$ $J_\nu(r,x) \leq C_\nu$, where $C_\nu > 0$ are independent of x and r. According Lemmas 2.1 and 2.2 the case $\nu = j$ is technically more complicated. Hence, our objective will be to prove the inequality

$$\begin{aligned} J_j(r,x) &\leq C\frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}\int_{\mathbb{T}}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)\Big|\sin(\frac{t-x_j}{2})\Big|P_r(t-x)dt \\ &+ C\frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}}\int_{\mathbb{T}}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)P_r(t-x)dt \leq C_j. \end{aligned}$$

As in the first part of the proof we will consider the cases 1) – 3).

In the case 1) we derive

$$\begin{aligned} J_j(r,x) &\leq C\frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}\int_{O_j(\delta)}\Big|\sin(\frac{t-x_j}{2})\Big|^{k_j-\lambda_j+1}dt\,\min\left\{\frac{2}{1-r},\frac{1-r}{2r\sin^2\delta}\right\} \\ &+ C\frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}}\int_{O_j(\delta)}\Big|\sin(\frac{t-x_j}{2})\Big|^{k_j-\lambda_j}dt\,\min\left\{\frac{2}{1-r},\frac{1-r}{2r\sin^2\delta}\right\} \\ &\leq C'(1-r)w(x)\delta^{2+k_j-\lambda_j}\min\left\{\frac{2}{(1-r)^2},\frac{1}{2r\sin^2\delta}\right\}^{\frac{k_j+3}{2}} \\ &+ C'(1-r)w(x)\delta^{1+k_j-\lambda_j}\min\left\{\frac{1}{(1-r)^2},\frac{1}{2r\sin^2\delta}\right\}^{\frac{k_j+2}{2}} \leq C_j, \end{aligned}$$

where $C_j > 0$ is independent of $r(0 < r < 1)$ and $x \in \mathbb{T} \setminus O_j(2\delta)$.

We skip the proof in the case 2) because it is similar to the analogous case provided above.

In the case 3) if $1 - r \geq \frac{\delta}{4}$ then we have

$$\begin{aligned} J_j(r,x) &\leq C(1-r)\frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}\delta^{2+k_j-\lambda_j}\min\left\{\frac{2}{(1-r)^2},\frac{1}{2r\sin^2\delta}\right\} \\ &+ C(1-r)\frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}}\delta^{1+k_j-\lambda_j}\min\left\{\frac{2}{(1-r)^2},\frac{1}{2r\sin^2\delta}\right\} \\ &\leq C'\frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}}\delta^{1+k_j-\lambda_j} + C'\frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}}\delta^{k_j-\lambda_j} \leq C_j, \end{aligned}$$

where $C_j > 0$ is independent of $r(0 < r < 1 - \frac{\delta}{4})$ and $x \in O_j(\frac{\delta}{2})$.

If $x \in O_j(2-2r)$ and $1 - r < \frac{\delta}{4}$ then we have

$$\begin{aligned} J_j(r,x) &\leq C\frac{w(x)}{(1-r)^{k_j+2}}\left\{\int_{O_j(1-r)}+\int_{O_j(2\delta)\setminus O_j(1-r)}\right\}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)\Big|\sin(\frac{t-x_j}{2})\Big|dt \\ &+ C\frac{w(x)}{(1-r)^{k_j+1}}\left\{\int_{O_j(1-r)}+\int_{O_j(2\delta)\setminus O_j(1-r)}\right\}\frac{|\omega(t)|}{w(t)}\Delta_\delta(t-x_j)dt. \end{aligned}$$

By (1) and (2) we will have that

$$\begin{aligned}
& \frac{w(x)}{(1-r)^{k_j+2}} \int_{O_j(2\delta)\setminus O_j(1-r)} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) \Big| \sin(\frac{t-x_j}{2}) \Big| dt \\
+ & \frac{w(x)}{(1-r)^{k_j+1}} \int_{O_j(2\delta)\setminus O_j(1-r)} |\omega(t)| \Delta_\delta(t-x_j) dt \\
\leq & \frac{Cw(x)}{(1-r)^{k_j+2}} \int_{O_j(2\delta)\setminus O_j(1-r)} \Delta_\delta(t-x_j) \Big| \sin(\frac{t-x_j}{2}) \Big|^{1+k_j-\lambda_j} dt \\
+ & \frac{Cw(x)}{(1-r)^{k_j+1}} \int_{O_j(2\delta)\setminus O_j(1-r)} \Delta_\delta(t-x_j) \Big| \sin(\frac{t-x_j}{2}) \Big|^{k_j-\lambda_j} dt \\
\leq & \frac{Cw(x)}{(1-r)^{k_j+2}} \delta^{2+k_j-\lambda_j} + \frac{Cw(x)}{(1-r)^{k_j+1}} \delta^{1+k_j-\lambda_j} \leq C'
\end{aligned}$$

for any $x \in O_j(2-2r)$ and $1-r < \frac{\delta}{4}$.

Afterwards we write

$$\begin{aligned}
& \frac{w(x)}{(1-r)^{k_j+2}} \int_{O_j(1-r)} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) \Big| \sin(\frac{t-x_j}{2}) \Big| dt \\
+ & \frac{w(x)}{(1-r)^{k_j+1}} \int_{O_j(1-r)} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) dt \\
+ & \frac{w(x)}{(1-r)^{k_j+2}} (1-r)^{2+k_j-\lambda_j} + \frac{w(x)}{(1-r)^{k_j+1}} (1-r)^{1+k_j-\lambda_j} \leq C'
\end{aligned}$$

where $C' > 0$ is independent of $x \in O_j(2-2r)$ and $1-r < \frac{\delta}{4}$. Thus we have proved that

$$J_j(r,x) \leq C'_j \quad \text{for any} \quad x \in O_j(2-2r), \tag{30}$$

where $C'_j > 0$ is independent of $r (0 < r < 1)$.

If $x \in O_j(\frac{\delta}{2}) \setminus O_j(2-2r)$ then we write

$$\begin{aligned}
J_j(r,x) \leq & C \frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}} \int_{O_j(1-r)} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) \Big| \sin(\frac{t-x_j}{2}) \Big| P_r(t-x) dt \\
+ & C \frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}} \int_{O_j(1-r)} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) P_r(t-x) dt \\
+ & C \frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}} \int_{O_j(2|x-x_j|)\setminus O_j(1-r)} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) \Big| \sin(\frac{t-x_j}{2}) \Big| P_r(t-x) dt \\
+ & C \frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}} \int_{O_j(2|x-x_j|)\setminus O_j(1-r)} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) P_r(t-x) dt \\
+ & C \frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}} \int_{O_j(2\delta)\setminus O_j(2|x-x_j|)} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) \Big| \sin(\frac{t-x_j}{2}) \Big| P_r(t-x) dt \\
+ & C \frac{w(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}} \int_{O_j(2\delta)\setminus O_j(2|x-x_j|)} \frac{|\omega(t)|}{w(t)} \Delta_\delta(t-x_j) P_r(t-x) dt := \sum_{l=1}^{6} J_j^{(l)}(r,x)
\end{aligned}$$

Afterwards we write

$$\begin{aligned}
& J_j^{(5)}(r,x) + J_j^{(6)}(r,x) \\
\le\ & \frac{C(1-r)w(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}} \delta^{k_j-\lambda_j+2} \min\left\{\frac{2}{(1-r)^2}, \frac{1}{2r\sin^2\delta}\right\} \\
+\ & \frac{C(1-r)w(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}} \delta^{k_j-\lambda_j+1} dt \ \min\left\{\frac{2}{(1-r)^2}, \frac{1}{2r\sin^2\delta}\right\} \\
\le\ & C'(1-r)\left|\sin(\frac{x-x_j}{2})\right|^{\lambda_j-k_j-1} \delta^{k_j-\lambda_j} \\
+\ & C'(1-r)\left|\sin(\frac{x-x_j}{2})\right|^{\lambda_j-k_j} \delta^{k_j-\lambda_j-1} \le C_j',
\end{aligned}$$

where $C_j' > 0$ is independent of $x \in O_j(\frac{\delta}{2}) \setminus O_j(2-2r)$ and $r (0<r<1)$.

Then we evaluate

$$\begin{aligned}
& J_j^{(1)}(r,x) + J_j^{(2)}(r,x) \\
\le\ & \frac{C(1-r)w(x)}{[\xi_r(x-x_j)]^{\frac{k_j+1}{2}}} (1-r)^{k_j-\lambda_j+2} \min\left\{\frac{2}{(1-r)^2}, \frac{1}{2r\sin^2\delta}\right\} \\
+\ & \frac{C(1-r)w(x)}{[\xi_r(x-x_j)]^{\frac{k_j}{2}}} (1-r)^{k_j-\lambda_j+1} dt \ \min\left\{\frac{2}{(1-r)^2}, \frac{1}{2r\sin^2\delta}\right\} \le C_j',
\end{aligned}$$

where $C_j' > 0$ is independent of $x \in O_j(\frac{\delta}{2}) \setminus O_j(2-2r)$ and $r (0<r<1)$.

To evaluate $J_j^{(3)}(r,x) + J_j^{(4)}(r,x)$ we set $\zeta = x - x_j$ and derive that

$$\begin{aligned}
J_j^{(3,1)}(r,\zeta+x_j) :\ &= \frac{w(\zeta+x_j)}{[\xi_r(\zeta)]^{\frac{k_j+1}{2}}} \int_{\Upsilon_\zeta(1-r)} \frac{|\omega(\tau+x_j)|}{w(\tau+x_j)} \Delta_\delta(\tau) \left|\sin(\frac{\tau}{2})\right| P_r(\tau-\zeta) d\tau \\
&\le \frac{C|\zeta|^{\lambda_j}}{(1-r)^{k_j+2}} \int_{\zeta-1+r}^{\zeta+1-r} |\tau|^{-\lambda_j+k_j+1} \le C';
\end{aligned}$$

and

$$\begin{aligned}
J_j^{(4,1)}(r,\zeta+x_j) :\ &= \frac{w(\zeta+x_j)}{[\xi_r(\zeta)]^{\frac{k_j}{2}}} \int_{\Upsilon_\xi(1-r)} \frac{|\omega(\tau+x_j)|}{w(\tau+x_j)} \Delta_\delta(\tau) P_r(\tau-\zeta) d\tau \\
&\le \frac{C|\zeta|^{\lambda_j}}{(1-r)^{k_j+1}} \int_{\zeta-1+r}^{\zeta+1-r} |\tau|^{-\lambda_j+k_j} \le C'
\end{aligned}$$

for all ζ such that $|\zeta| < \frac{\delta}{2}$, where $\Upsilon_\xi(1-r)$ is defined by (26) and $C' > 0$. Afterwards we suppose that $\zeta > 0$ and set

$$\begin{aligned}
& J_j^{(3,2)}(r,\zeta+x_j) := \frac{w(\zeta+x_j)}{[\xi_r(\zeta)]^{\frac{k_j+1}{2}}} \int_{\Omega_\xi} \frac{|\omega(\tau+x_j)|}{w(\tau+x_j)} \Delta_\delta(\tau) \left|\sin(\frac{\tau}{2})\right| P_r(\tau-\zeta) d\tau \\
=\ & \frac{w(\zeta+x_j)}{[\xi_r(\zeta)]^{\frac{k_j+1}{2}}} \left\{\int_{\Omega_\xi'} + \int_{\Omega_\xi''}\right\} \frac{|\omega(\tau+x_j)|}{w(\tau+x_j)} \Delta_\delta(\tau) \left|\sin(\frac{\tau}{2})\right| P_r(\tau-\zeta) d\tau,
\end{aligned}$$

and

$$\begin{aligned} J_j^{(4,2)}(r,\zeta+x_j) &:= \frac{w(\zeta+x_j)}{[\xi_r(\zeta)]^{\frac{k_j}{2}}}\int_{\Omega_\xi}\frac{|\omega(\tau+x_j)|}{w(\tau+x_j)}\Delta_\delta(\tau)P_r(\tau-\zeta)d\tau \\ &= \frac{w(\zeta+x_j)}{[\xi_r(\zeta)]^{\frac{k_j}{2}}}\left\{\int_{\Omega'_\xi}+\int_{\Omega''_\xi}\right\}\frac{|\omega(\tau+x_j)|}{w(\tau+x_j)}\Delta_\delta(\tau)P_r(\tau-\zeta)d\tau, \end{aligned}$$

where $\Omega_\xi, \Omega'_\xi, \Omega''_\xi$ are defined by (27), (28).

Afterwards, we obtain that

$$\begin{aligned} & \frac{w(\zeta+x_j)}{[\xi_r(\zeta)]^{\frac{k_j+1}{2}}}\int_{\Omega''_\xi}\frac{|\omega(\tau+x_j)|}{w(\tau+x_j)}\Delta_\delta(\tau)\left|\sin(\frac{\tau}{2})\right|P_r(\tau-\zeta)d\tau \\ + \; & \frac{w(\zeta+x_j)}{[\xi_r(\zeta)]^{\frac{k_j}{2}}}\int_{\Omega''_\xi}\frac{|\omega(\tau+x_j)|}{w(\tau+x_j)}\Delta_\delta(\tau)P_r(\tau-\zeta)d\tau \\ \le \; & C\left\{\int_{-2\zeta}^{-(1-r)}+\int_{(1-r)}^{\zeta-(1-r)}+\int_{\zeta+(1-r)}^{2\zeta}\right\}\left[\frac{\zeta^{\lambda_j-k_j-1}(1-r)}{|\zeta-\tau|^2\,\tau^{\lambda_j-k_j-1}}+\frac{\zeta^{\lambda_j-k_j}(1-r)}{|\zeta-\tau|^2\,\tau^{\lambda_j-k_j}}\right]d\tau \\ := \; & J_j^{(4,2,1)}(r,\zeta+x_j)+J_j^{(4,2,2)}(r,\zeta+x_j)+J_j^{(4,2,3)}(r,\zeta+x_j) \end{aligned}$$

Afterwards we derive

$$J_j^{(4,2,1)}(r,\zeta+x_j)\le C(1-r)^{-1}\zeta^{\lambda_j-k_j-1}\zeta^{-\lambda_j+k_j+2}+C(1-r)^{-1}\zeta^{\lambda_j-k_j}\zeta^{-\lambda_j+k_j+1}\le C';$$

$$J_j^{(4,2,3)}(r,\zeta+x_j)\le C(1-r)\int_{(1-r)}^{\zeta}u^{-2}du\le C';$$

uniformly for some $C'>0$.

Again denoting by $k_\zeta\ge 3$ the natural number for which $(1-r)(k_\zeta-1)\le\zeta<(1-r)k_\zeta$ we obtain

$$\begin{aligned} & J_j^{(4,2,2)}(r,\zeta+x_j) \\ \le \; & C(1-r)\left\{\int_{(1-r)}^{\frac{k_\xi-1}{2}(1-r)}+\int_{\frac{k_\xi-1}{2}(1-r)}^{\xi-(1-r)}\right\}\left[\frac{\zeta^{\lambda_j-k_j-1}}{|\zeta-\tau|^2\,\tau^{\lambda_j-k_j-1}}+\frac{\zeta^{\lambda_j-k_j}}{|\zeta-\tau|^2\,\tau^{\lambda_j-k_j}}\right]d\tau \\ \le \; & \frac{C(1-r)\zeta^{\lambda_j-k_j-1}}{[(1-r)(k_\xi-1)]^2}\left[\frac{(1-r)(k_\xi-1)}{2}\right]^{k_j-\lambda_j+2} \\ + \; & \frac{C(1-r)\zeta^{\lambda_j-k_j}}{[(1-r)(k_\xi-1)]^2}\left[\frac{(1-r)(k_\xi-1)}{2}\right]^{k_j-\lambda_j+1}+C(1-r)\int_{(1-r)}^{+\infty}u^{-2}du\le C', \end{aligned}$$

where $C'>0$. Thus

$$\begin{aligned} & \frac{w(\zeta+x_j)}{[\xi_r(\zeta)]^{\frac{k_j+1}{2}}}\int_{\Omega''_\xi}\frac{|\omega(\tau+x_j)|}{w(\tau+x_j)}\Delta_\delta(\tau)\left|\sin(\frac{\tau}{2})\right|P_r(\tau-\zeta)d\tau \\ + \; & \frac{w(\zeta+x_j)}{[\xi_r(\zeta)]^{\frac{k_j}{2}}}\int_{\Omega''_\xi}\frac{|\omega(\tau+x_j)|}{w(\tau+x_j)}\Delta_\delta(\tau)P_r(\tau-\zeta)d\tau \end{aligned}$$

is uniformly bounded.

We skip the proof of the inequality

$$\frac{w(\zeta+x_j)}{[\xi_r(\zeta)]^{\frac{k_j+1}{2}}}\int_{\Omega'_\xi}\frac{|\omega(\tau+x_j)|}{w(\tau+x_j)}\Delta_\delta(\tau)\left|\sin(\frac{\tau}{2})\right|P_r(\tau-\zeta)d\tau$$
$$+\quad\frac{w(\zeta+x_j)}{[\xi_r(\zeta)]^{\frac{k_j}{2}}}\int_{\Omega'_\xi}\frac{|\omega(\tau+x_j)|}{w(\tau+x_j)}\Delta_\delta(\tau)P_r(\tau-\zeta)d\tau\le C$$

for some $C>0$ and any $0<\zeta<\frac{\delta}{2}$ and $0<r<1$ because it is provided in a similar way.

The proof for the case $\zeta<0$ is analogous. Hence we proved that

$$J_j(r,x)\le C'_j\quad\text{for any}\quad x\in O_j(\frac{\delta}{2})\setminus O_j(2-2r),$$

where $C'_j>0$ is independent of $r(0<r<1)$. Thus the inequality

$$J_j(r,x)\le C\quad\text{for any}\quad x\in\mathbb{T},$$

where $C>0$ is independent of $r(0<r<1)$ is proved.

Observing that the function $\frac{1}{w(t)}\left(1-\sum_{j=1}^{s}\Delta_\delta(t-x_j)\right)$ is continuous on $\mathbb{T}$ we derive that $J_0(r,x)$ is uniformly bounded.

□

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