$\varrho(w)$- NORMAL POINT SYSTEMS

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INTRODUCTION

For $w \equiv 1$ the original definition of ϱ-normality given by L. Fejér is the following: an X point system or matrix is ϱ - normal or normal on $[a,b] \subset \mathbf{R}$, if for every $x \in [a,b]$,

$$1 - c_k(x - x_k) \geq \varrho > 0.$$

Here $c_k = \frac{\omega''}{\omega'}(x_k), x_k = x_{k,n} \in X, k = 1, ..., n$, and $\omega(x) = \prod_{k=1}^{n}(x - x_k)$. On normal point systems the kernel of the operator $H_n(f)(x) = \sum_{k=1}^{n}(1 - c_k(x - x_k))l_k^2(x)f(x_k)$ is positive (nonnegative) and bounded. ($l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}$.) In this case L. Fejér and G. Grünwald proved some convergence theorems in connection with Lagrange-, Hermite- and Hermite-Fejér interpolation [1], [2]. The classical examples for normal systems, as it is well-known, are the root-systems of Jacobi polynomials with parameters $\alpha, \beta \in (-1, 0)$. Our aim is to extend these results to some wilder classes of functions using the best weighted polynomial approximation and the connected tools which are given by the developments of the last years.

A motivation of the new definition is the following: with the usual notation of the Hermite interpolatory polynomial

$$H_n(f, f')(x) = \sum_{k=1}^{n}(1 - c_k(x - x_k))l_k^2(x)f(x_k) + \sum_{k=1}^{n}(x - x_k)l_k^2(x)f'(x_k),$$

we can write that

$$(f(x) - H_n(f, f')(x))w(x) = (f(x) - p(x))w(x) + w(x)H_n(f - p, f' - p')(x) =$$

$$(A) \qquad = (f(x) - p(x))w(x) + w(x)\sum_{k=1}^{n}\frac{1 - C_k(x - x_k)}{w(x_k)}l_k^2(x)((fw)(x_k) - (pw)(x_k)) +$$

$$+ w(x)\sum_{k=1}^{n}\frac{x - x_k}{w(x_k)}l_k^2(x)((fw)'(x_k) - (pw)'(x_k)),$$

where

$$C_k = \frac{\omega''(x_k)}{\omega'(x_k)} + \frac{w'(x_k)}{w(x_k)}.$$

(w is some weight function which is differentiable and positive on (a,b).)

We will call the first sum in the upper expression as weighted Hermite-Fejér interpolatory polynomial ($H_{w,n}(f,x)$ see Def. 1.), which is equal to the original Hermite-Fejér interpolatory polynomial ($H_n(f,x)$) for $w \equiv 1$. This operator has the hoped good properties and is strongly connected whith some previous investigations of D. L. Berman and P. Vértesi [3], namely if the basic point system of the original Hermite-Fejér method is the system of the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ completed by the two endpoints of the interval in question, by -1 and 1, then for some f for which $f(-1)=f(1)=0$ we get that

$$H_{n+2}(f,x)=(1-x^2)^2H_{(1-x^2)^2,n}\left(\frac{f(x)}{(1-x^2)^2},x\right).$$

The first attempt to define weighted normal point systems is from I. Joó in 1975 [4]. During the seventies a lot of results was given on normal systems but in that times weighted interpolation was not so crucial question as nowadays is.

DEFINITIONS, EXAMPLES

Definition 1 $(a,b)\subset \mathbf{R}, w$ *is a differentiable, positive weight function on* (a,b), *and* f *is a function on* (a,b). *Then for a point system* $x_k=x_{k,n}\in X\subset(a,b), k=1,\dots,n,$ *the weighted Hermite-Fejér interpolatory polynomial of* f *whith degree* n *is*

$$H_{w,n}(f,x)=\sum_{k=1}^{n}(1-C_k(x-x_k))l_k^2(x)f(x_k), \tag{1}$$

where

$$C_k=C_k^{(n)}=\frac{\omega''(x_k)}{\omega'(x_k)}+\frac{w'(x_k)}{w(x_k)}. \tag{2}$$

We have to note that this operator is called as "Hermite-Fejér", because

$$w(x)H_{w,n}(f,x)|_{x=x_k}=w(x_k)f(x_k),$$

$$(w(x)H_{w,n}(f,x))'|_{x=x_k}=0, k=1,\dots,n.$$

Notation

w is an "admissible" [5] weight function on $(a,b)\subset \mathbf{R}$, $a_m(w),b_m(w)$ are the so-called Mhaskar-Rahmanov-Saff-numbers, that is $(a_m(w),b_m(w))\subset(a,b)$ is that interval where the norm of a weighted polynomial with degree m, $p_m(x)w(x)$, "lives". After these preliminaries

$$I_n:=(a_{2n-1}(w),b_{2n-1}(w)).$$

Definition 2 $(a,b)\subset\mathbf{R}, w$ *is a positive differentiable weight function on* (a,b), *then an* $X=\{x_{1,n},\dots,x_{n,n}, n\in\mathbf{N}\}\subset(a,b)$ *is a* $\varrho(w)$ *-normal point system for some* $\varrho\in(0,1]$, *if for any* $n\in\mathbf{N}$

$$w(x)\sum_{k=1}^{n}\frac{(1-C_k(x-x_k))l_k^2(x)}{w(x_k)}\le 1;\quad x\in I_n, \tag{3}$$

$$1-C_k(x-x_k)\ge\varrho>0;\quad x\in I_n. \tag{4}$$

A point system is called $w-$*normal if instead of* (4) *we have that*

$$1-C_k(x-x_k)>0;\quad x\in I_n. \tag{5}$$

Remarks

(1) If in Definition 2. we have (3) and (4) for certain b_k-s instead of C_k-s, then we will get the form of $b_k=C_k$ directly because $\frac{w(x)(1-b_k(x-x_k)l_k^2(x)}{w(x_k)}$ has a local maximum in x_k, and so

$$0=\left(\frac{w(x)(1-b_k(x-x_k))l_k^2(x)}{w(x_k)}\right)'(x_k)=\frac{1}{w(x_k)}\left(w'(x_k)+w(x_k)\left(-b_k+\frac{\omega''}{\omega'}(x_k)\right)\right).$$

(2) If $\left(\frac{1}{w}\right)^{(2m)} \geq 0$ on $(a,b), m = 1, 2\ldots,$ then

$$w(x)\sum_{k=1}^{n}\frac{(1-C_k(x-x_k))l_k^2(x)}{w(x_k)} \leq 1; \quad (x \in I_n)$$

is automatically valid because

$$\frac{1}{w}(x_k) = \left(\sum_{k=1}^{n}\frac{(1-C_k(x-x_k))l_k^2(x)}{w(x_k)}\right)|_{x=x_k}$$

and

$$\left(\frac{1}{w}\right)'(x_k) = \left(\sum_{k=1}^{n}\frac{(1-C_k(x-x_k))l_k^2(x)}{w(x_k)}\right)'|_{x=x_k}$$

So by the well-known relation (valid for any function f)

$$\frac{1}{w}(x) - H_n(\frac{1}{w}\left(\frac{1}{w}\right)', x) = \frac{1}{(2n)!}\left(\frac{1}{w}\right)^{(2n)}(\xi)\omega^2(x),$$

which is nonnegative by the conditions.

(3) From the definition of $\varrho(w)$-normality, follows that

$$0 < w(x)\sum_{k=1}^{n}\varrho\frac{l_k^2(x)}{w(x_k)} \leq w(x)\sum_{k=1}^{n}\frac{(1-C_k(x-x_k))l_k^2(x)}{w(x_k)} \leq 1,$$

that is

$$0 < w(x)\sum_{k=1}^{n}\frac{l_k^2(x)}{w(x_k)} \leq \frac{1}{\varrho} \tag{6}$$

Examples

(1) $(a,b) = (-1,1), w \equiv 1$, the root systems of $p_n^{(\alpha,\beta)}, \alpha, \beta \in (-1,0)$ are naturally ϱ-normal ($\varrho = \min\{-\alpha,-\beta\}$) in the new sense too.

(2) $(a,b) = (-1,1), w(x) = (1-x)^\alpha(1+x)^\beta, \alpha,\beta > 0$. In this case the root system of $p_n^{(\alpha-\mu,\beta-\nu)}$ is $(min(\mu,\nu))(w)$-normal, where $p_n^{(\alpha,\beta)}$ are the Jacobi polynomials and $\mu,\nu \in (0,1)$ arbitrary. By second remark and [6] we only need to control the sign of the line of (4) in Definition 2 in the end points of the interval. To computing the expression $\frac{\omega''}{\omega'}(x_k)$, one can use the differential equation of Jacobi polynomials and so in 1 :

$$1 - \left(\frac{\beta}{1+x_k} - \frac{\alpha}{1-x_k} + \frac{\alpha-\mu-\beta+\nu+x_k(\alpha+\beta-\mu-\nu+2)}{1-x_k^2}(1-x_k)\right) =$$

$$\frac{1}{1+x_k}(\mu-\nu+1+x_k(\mu+\nu-1)) \geq \mu.$$

similarly in -1 :

$$1 - C_k(-1-x_x) = \frac{1}{1-x_k}(\mu-\nu+1+x_k(1-\mu-\nu)) \geq \nu.$$

(3) $(a,b) = \mathbf{R}, w(x) = e^{-x^2}$ In this case the root system of Hermite polynomials is $1(w)$-normal, namely

$$1 - C_k(x-x_k) \equiv 1, \quad x \in \mathbf{R}$$

(we can use second remark).

(4) $(a,b)=(0,\infty), w(x)=x^{\alpha}e^{-x}, \alpha\geq 0$. The positivity of the derivatives in question is valid [6], so by controling the positivity of the line in Definition 2, we get that the root systems of Laguerre polynomials with parameter $\alpha-\mu, \mu\in(0,1)$ are $\mu(w)-$normal.

In present paper definitions were given in a rather general form, (however these definitions can be generalised further to weight functions with zeros in the interior of the interval) but now we deal with the Jacobi case only. Further more we have to mention that our third example is investigated in [7] by S. Szabó, because from the differential equation of Hermite polynomials turns out that $C_k=0, k\in\mathbf{N}$, and so

$$H_{w,n}(f,x)=\sum_{k=1}^{n} l_k^2(x)f(x_k)=Y_n(W^2,f,x),$$

which is the so-called Grünwald operator, for which a convergence theorem is proved there.

Definition 3 $I=(a,b)\subset\mathbf{R}$.

$$C_w(I)=\{f\in C(I)|\lim_{x\longrightarrow a} f(x)w(x)=\lim_{x\longrightarrow b} f(x)w(x)=0, \tag{7}$$

$$C_w((-1,1))=C_w. \tag{8}$$

Notations

In the followings we will use the next notations:

(1) $\|f\|_I$ means the sup$-$ norm of f on I, if $I=(-1,1)$, $\|f\|_I=\|f\|$.

(2) $p_n^{(w)}(x)$ is the orthonormal polynomial to w with degree n,

(3) $\varphi(x)=\sqrt{1-x^2}, \varphi^2(x)=q(x)$,

(4) $E_n^w(f)$ is the best weighted approximation by polynomial with degree n, that is

$$E_n^w(f):=\min_{p_n\in\Pi_n}\|(f-p_n)w\| \tag{9}$$

(5)

$$\tilde{E}_n^w(f):=\min_{p_n\in\Pi_n}\|(f-qp_n)w\| \tag{10}$$

(6) The Ditzian-Totik weighted modulus of smoothness of f is

$$\Omega_1^{\varphi}(f,t)_{w,\infty}=\sup_{h\in[0,t]}\left\|\left(f\left(x+\frac{h\varphi(x)}{2}\right)-f\left(x-\frac{h\varphi(x)}{2}\right)\right)w(x)\right\|_{I_h}, \tag{11}$$

where

$$I_h=[-1+2h^2,1-2h^2],\quad h\in[0,1]. \tag{12}$$

(7) The usual modulus of smoothness is $\omega(f,t)=\sup_{|x-y|\leq t}\{|f(x)-f(y)|\}$.

RESULTS

Theorem 1 $I=(-1,1), w(x)=(1-x)^{\alpha}(1+x)^{\beta}, \alpha,\beta\geq 0$. *If* $X\subset(-1,1)$ *a* $\varrho(w)$ *-normal point system with some* $0<\varrho\leq 1$, *and if* $f\in C_w$ *is a differentiable function on* $(-1,1)$, *with*

$$\lim_{|x|\longrightarrow 1}(fw)'(x)=0,$$

then

$$\lim_{n\longrightarrow\infty}\|(f-H_n(f,f'))w\|=0.$$

Proof. Since $\frac{(fw)'}{w} \in C_w, E_n^w\left(\frac{(fw)'}{w}\right) \longrightarrow 0$, if $n \longrightarrow \infty$. Let u_n be defined by

$$\left\|\left(\frac{(fw)'}{w} - u_n\right) w\right\| = E_n^w\left(\frac{(fw)'}{w}\right).$$

In this case

$$2E_n^w\left(\frac{(fw)'}{w}\right) \geq \int_{-1}^{x} |(fw)'(y) - (u_n w)(y)|dy \geq \left|f(x)w(x) - \int_{-1}^{x}(u_n w)(y)dy\right| =$$

$$\left|f(x)w(x) - \int_{-1}^{x}\sum_{k=0}^{n} b_k^{(n)}(p_k^{(w)}w)(y)dy\right| =$$

$$\left|f(x)w(x) - b_0^{(n)}p_0^{(w)}\int_{-1}^{x} w(y)dy + (r_{n+1}w)(x)\right|.$$

Here $r_{n+1} \in \Pi_{n+1}$ is a polynomial with degree $n+1$, namely using Rodrigues' formula [9 (4.3.1)] we get that

$$\int_{-1}^{x} p_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx$$

$=$

$$c(n)p_{n-1}^{\alpha+1,\beta+1}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1} = r_{n+1}(x)(1-x)^{\alpha}(1+x)^{\beta}. \tag{13}$$

First of all using that $f \in C_w$ and the orthogonality of $p_k^{(w)}$ on $(-1,1)$ we will prove that

$$|b_0^{(n)}| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Indeed if $x = 1$, we get

$$2E_n^w \geq \left|0 - b_0^{(n)}p_0^{(w)}\int_{-1}^{1} w(y)dy + 0\right|,$$

which was to be proved. Summarizing these results we get that

$$\|(fw)(x) - (r_{n+1}w)(x)\| \leq C\left\{E_n^w\left(\frac{(fw)'}{w}\right) + |b_0^{(n)}p_0^{(w)}|\|w\|_1\right\} \leq CE_n^w\left(\frac{(fw)'}{w}\right) \tag{14}$$

and

$$\|(fw)'(x) - (r_{n+1}w)'(x)\| \leq C\left\{\|(fw)' - u_n w\| + |b_0^{(w)}|\|w\|_1\right\} \leq CE_n^w\left(\frac{(fw)'}{w}\right) \tag{15}$$

Thus using the reconstruction property of the Hermite interpolation:

$$\|(f - H_n(f,f')w\| \leq \|(f - r_{n+1})w\| + \left\|w(x)\sum_{k=1}^{n}\frac{1 - C_k(x-x_k)}{w(x_k)}l_k^2(x)((fw)(x_k) - (r_{n+1}w)(x_k))\right\| +$$

$$\left\|w(x)\sum_{k=1}^{n}\frac{l_k^2(x)(x-x_k)}{w(x_k)}((fw)'(x_k) - (r_{n+1}w)'(x_k))\right\| \leq CE_n^w\left(\frac{(fw)'}{w}\right)(1 + \frac{1}{\varrho}),$$

where the definition of $\varrho(w)$-normality and Remark (3). were used.

($\|\cdot\|_1$ is the usual L_1-norm on $(-1,1)$.)

Theorem 2 *$I = (-1,1), w = (1-x)^{\alpha}(1+x)^{\beta}, \alpha, \beta \geq 0,\ f \in C_w$. If X is a $\varrho(w)-$normal point system then*

$$\|(f - H_{w,M}(f))w\| \longrightarrow 0 \text{ if } M \longrightarrow \infty.$$

For the proof we need some lemmas:

Lemma 1 *If $f \in C_w$, then*

$$\lim_{n \longrightarrow \infty} \tilde{E}_n^w(f) = 0.$$

Proof.Let $\varepsilon > 0$ arbitrary, $\delta = \delta(\varepsilon)$ such that $|f(x)w(x)| < \varepsilon$ for $|x| > 1 - \delta$. Let

$$g_\varepsilon(x) = \Psi_\varepsilon \frac{f}{q}(x),$$

where $\Psi_\varepsilon(x)$ is a continuous function such that

$$\Psi_\varepsilon(x) = \begin{cases} 0 & \text{if x} \in (-1, -1 + \frac{\delta}{2}) \text{ or } x \in (1 - \frac{\delta}{2}, 1) \\ 1 & \text{if } x \in [-1 + \delta, 1 - \delta] \\ \text{linear} & \text{otherwise} \end{cases}$$

Since g_ε is continuous on $(-1, 1)$, thence for any ε there exists a polynomial $p_{n(\varepsilon)}$ such that $\|g_\varepsilon - p_{n(\varepsilon)}\| < \varepsilon$. Thus

$$\|(f - qp_{n(\varepsilon)})w\| \le \|(f - qg_\varepsilon)w\| + \|(qg_\varepsilon - qp_{n(\varepsilon)})w\| \le \varepsilon(1 + \|qw\|).$$

Lemma 2 *If $x \in I_h$, then*

$$\frac{w(x)}{w\left(x \pm \frac{h\varphi(x)}{2}\right)} \sim 1$$

($f(x) \sim g(x)$ means that there exists positive constants C_1, C_2, such that $|f(x)| \le C_1|g(x)|$, and $|g(x)| \le C_2|f(x)|$.)

Proof. Let $x \le 0$. (For $x \ge 0$ the proof is similar.) Because on the given interval

$$\frac{w(x)}{w\left(x \pm \frac{h\varphi(x)}{2}\right)} \sim \left(\frac{1+x}{1 + x \pm \frac{h\varphi(x)}{2}}\right)^\beta,$$

where $\beta \ge 0$, it is enough to investigate

$$f_\pm(x) = \frac{1}{1 \pm \frac{h}{2}\sqrt{\frac{1-x}{1+x}}}.$$

Since f_+ is increasing, f_- is decreasing, we get that

$$\frac{2}{3} \le f_+(-1 + 2h^2) \le f_+(x) \le f_+(0) \le 1,$$

$$1 \le f_-(0) \le f_-(x) \le f_-(-1 + 2h^2) \le 2.$$

Lemma 3 *Let $\gamma \in (0, \frac{1}{2}], -1 < a < b < 1$,*

$$g(x) = g_{a,b,\gamma}(x) = \begin{cases} (x-a)^\gamma (b-x)^{\gamma+1} & \textit{if } x \in (a, b) \\ 0 & \textit{if } x \in [-1, 1] - (a, b) \end{cases}$$

In this case there exists a polynomial sequence $\{u_m\}_{m=m_0}^\infty (= \{u_m\}) \subset \Pi_m$, such that

$$\|g'(x)(x-a) - u_m(x)(x-a)w(x)\| \le \varepsilon_1(m), \tag{16}$$

$$\|g(x) - \int_{-1}^{x} u_m(y)w(y)dy\| \le \varepsilon_2(m). \tag{17}$$

Moreover if u_m *has the form*

$$u_m = \sum_{k=0}^{m} b_k^{(m)} p_k^{(w)},$$

then

$$|b_0^{(m)}| \le \varepsilon_3(m), \tag{18}$$

where

$$\lim_{m \longrightarrow \infty} \varepsilon_i(m) = 0,$$

independently of $a, b,$ $(i = 1, 2, 3)$.

($\{u_m\}$ naturally depends on a and b.)

Proof. Let

$$k(x) = \begin{cases} \frac{g'(x)(x-a)}{(wq)(x)} & \text{if if } x \in (a,b) \\ 0 & \text{if } x \in [-1,1] \setminus (a,b) \end{cases}$$

We will estimate a Ditzian - Totik weighted modulus of smoothness of $k(x)$.

$$\Omega_1^{\varphi}(k,t)_{qw,\infty} \le \sup_{h \in [0,t]} \left\{ \left\| \frac{(qw)(x)}{(qw)\left(x + \frac{h\varphi(x)}{2}\right)} \left(g'\left(x + \frac{h\varphi(x)}{2}\right)\left(x + \frac{h\varphi(x)}{2} - a\right) - \right.\right.\right.$$

$$\left.\left. g'\left(x - \frac{h\varphi(x)}{2}\right)\left(x - \frac{h\varphi(x)}{2} - a\right)\right)\right\|_{[a,b]} +$$

$$\left\| g'\left(x - \frac{h\varphi(x)}{2}\right)\left(x - \frac{h\varphi(x)}{2} - a\right)\left(\frac{(qw)(x)}{(qw)\left(x + \frac{h\varphi(x)}{2}\right)} - \frac{(qw)(x)}{(qw)\left(x - \frac{h\varphi(x)}{2}\right)}\right)\right\|_{[a,b]} +$$

$$\left. \sup_{\substack{x \in I_h, \\ x - \frac{h\varphi(x)}{2} \notin [a,b]}} \left| k\left(x + \frac{h\varphi(x)}{2}\right)(qw)(x)\right| + \sup_{\substack{x \in I_h, \\ x + \frac{h\varphi(x)}{2} \notin [a,b]}} \left| k\left(x - \frac{h\varphi(x)}{2}\right)(qw)(x)\right| \right\} =$$

$$= I + II + III + IV.$$

By Lemma 2., the first, the third and the forth terms can be estimated with the original modulus of continuity of $g'(x)(x-a)$:

$$I \le Ct^{\gamma}.$$

To estimate II we have to distinguish two cases: If $x - \frac{h\varphi(x)}{2} \in (a, a+h)$, then we can use Lemma 2. again and here $g'(x)(x-a) \le Ct^{\gamma}$. If $x - \frac{h\varphi(x)}{2} \in (a+h, b)$, then for some $\eta \in (-h, h)$

$$\left| \frac{(qw)(x)}{(qw)\left(x + \frac{h\varphi(x)}{2}\right)} - \frac{(qw)(x)}{(qw)\left(x - \frac{h\varphi(x)}{2}\right)} \right| \le C \left| \frac{(qw)(x)}{(qw)\left(x + \frac{\eta\varphi(x)}{2}\right)} \right| \left| \frac{(qw)'\left(x + \frac{\eta\varphi(x)}{2}\right)}{(qw)\left(x + \frac{\eta\varphi(x)}{2}\right)} h\varphi(x) \right| \le$$

$$C \frac{h\varphi(x)}{\left(\varphi\left(x + \frac{\eta\varphi(x)}{2}\right)\right)^2} \le C \frac{h}{\varphi\left(x + \frac{\eta\varphi(x)}{2}\right)} \le Ch^{\frac{1}{2}}.$$

We have the same two cases around b,and so the estimations are the same there as well, that is

$$II \le Ct^{\min(\gamma, \frac{1}{2})} \le Ct^{\gamma},$$

and so

$$\Omega_1^{\varphi}(k,t)_{qw,\infty} = O\left(t^{\gamma}\right) \tag{19}$$

According to Corollary 8.2.2 in [8] we can choose a polynomial sequence $\{p_n\}$, such that

$$\|(k(x)-p_n(x))(qw)(x)\| = O\left(n^{-\gamma}\right). \tag{20}$$

Denoted by

$$Q_{n+2}(x) = p_n(x)q(x),$$

we will estimate the expression:

$$\left| g(x) - \int_{-1}^{x} \frac{Q_{n+2}(y)w(y) - Q_{n+2}(a)w(a)}{y-a} dy \right|.$$

We have to distinguish two cases again: $x \in [-1, a+n^{-3}], x \in (a+n^{-3}, 1]$.

First case:

$$g(x) = O\left(n^{-3\gamma}\right) = O\left(n^{-\gamma}\right), \text{ if } x \in [-1, a+n^{-3}]. \tag{21}$$

and on the same interval

$$\left| \int_{-1}^{x} \frac{Q_{n+2}(y)w(y) - Q_{n+2}(a)w(a)}{y-a} dy \right| \leq \left| \int_{-1}^{a-n^{-3}} \frac{Q_{n+2}(y)w(y) - Q_{n+2}(a)w(a)}{y-a} dy \right| +$$

$$\left| \int_{a-n^{-3}}^{x} \frac{Q_{n+2}(y)w(y) - Q_{n+2}(a)w(a)}{y-a} dy \right| = I + II.$$

Because of (20) and (21)

$$I \leq C \int_{-1}^{a-n^{-3}} \frac{n^{-\gamma}}{|y-a|} dy = O\left(n^{-\gamma}\log n\right).$$

For investigation of II we have to estimate $\|(Q_{n+2}w)'\|$. Applying [8, (8.1.3.)] to $W = \varphi w$, we get that

$$\|(Q_{n+2}w)'\| \leq C\left(\|p_n' qw\|_{I_{n^{-1}}} + \|p_n q' w\|_{I_{n^{-1}}} + \left\| p_n \left(q\frac{w'}{w} \right) w \right\|_{I_{n^{-1}}} \right) \leq$$

$$C\left(n\|p_n \varphi(x) w\|_{I_{n^{-1}}} + \left\| p_n \frac{q'}{q} qw \right\|_{I_{n^{-1}}} + \left\| p_n q \frac{w'}{w} w \right\|_{I_{n^{-1}}} \right) \leq$$

$$C\|p_n qw\| \left(n \left\| \frac{1}{\varphi(x)} \right\|_{I_{n^{-1}}} + \left\| \frac{q'}{q} \right\|_{I_{n^{-1}}} + \left\| \frac{w'}{w} \right\|_{I_{n^{-1}}} \right) \leq Cn^{2-\gamma},$$

that is

$$II \leq Cn^{-3}n^{2-\gamma} = o\left(n^{-1}\right),$$

and it yields that

$$\left| g(x) - \int_{-1}^{x} \frac{(Q_{n+2}w)(y) - (Q_{n+2}w)(a)}{y-a} dy \right| = O\left(n^{-\gamma}\log n\right), \text{ if } x \in [-1, a+n^{-3}]. \tag{22}$$

Second case, $x \in (a+n^{-3}, 1]$. Using the definition of $k(x)$ we can write

$$\left| g(x) - \int_{-1}^{x} \frac{(Q_{n+2}w)(y) - (Q_{n+2}w)(a)}{y-a} dy \right| = \left| \int_{-1}^{x} \left(g'(y) - \frac{(Q_{n+2}w)(y) - (Q_{n+2}w)(a)}{y-a} \right) dy \right| \leq$$

$$\int_{-1}^{a+n^{-3}}\left|\frac{k(y)(qw)(y)-((Q_{n+2}w)(y)-(Q_{n+2}w)(a))}{y-a}\right|dy+$$
$$\int_{a+n^{-3}}^{x}\left|\frac{k(y)(qw)(y)-((Q_{n+2}w)(y)-(Q_{n+2}w)(a))}{y-a}\right|dy\leq$$
$$Cn^{-\gamma}\log n+C\int_{a+n^{-3}}^{x}\left|\frac{n^{-\gamma}}{y-a}\right|dy=O\left(n^{-\gamma}\log n\right). \tag{23}$$

In the upper estimation we used (22) for the first term and (19) for the second term (cf. [2, (103) ,(104)]). Thus (22) and (23) yields that

$$\left|g(x)-\int_{-1}^{x}\frac{(Q_{n+2}w)(y)-(Q_{n+2}w)(a)}{y-a}dy\right|=O\left(n^{-\gamma}\log n\right)\ \text{ for every } x\in(-1,1). \tag{24}$$

Now we almost have the statement of the lemma, but it speaks about polynomials and we have not polynomials after the integral sign yet. These reasons induce us continuing the investigations: Let

$$A(y)=A_{a,b,n}(y)=\begin{cases}\frac{(Q_{n+2}w)(y)-(Q_{n+2}w)(a)}{y-a} & \text{if } a+n^{-3}<y<b-n^{-3}\\ 0 & \text{if } y\in(-1,c(a,n))\bigcup(c(b,n),1)\\ l_1(y) & \text{if } y\in[c(a,n),a+n^{-3}]\\ l_2(y) & \text{if } y\in[b-n^{-3},c(b,n)]\end{cases}$$

Here $c(a,n)=\max\left(a-n^{-3},\frac{a-1}{2}\right)$, $c(b,n)=\min\left(b+n^{-3},\frac{1+b}{2}\right)$, $l_i(y)$are linear such that $A(y)$ be continuous. Because $\frac{A}{w}\in C_w$, similarly to the $k(x)$, we have to estimate the modulus of smoothness of $\frac{A}{w}$:

$$\Omega_1^{\varphi}\left(\frac{A}{w},t\right)_{w,\infty}\leq\sup_{h\in[0,t]}\left\|\frac{w(y)}{w\left(y+\frac{h\varphi(y)}{2}\right)}\left|A\left(y+\frac{h\varphi(y)}{2}\right)-A\left(y-\frac{h\varphi(y)}{2}\right)\right|\right\|_{I_h}+$$
$$\sup_{h\in[0,t]}\left\|\left|A\left(y-\frac{h\varphi(y)}{2}\right)\right|\left|\frac{w(y)}{w\left(y+\frac{h\varphi(y)}{2}\right)}-\frac{w(y)}{w\left(y-\frac{h\varphi(y)}{2}\right)}\right|\right\|_{I_h}=\sup_{h\in[0,t]}\|I\|_{I_h}+\sup_{h\in[0,t]}\|II\|_{I_h}.$$

We have to distinguish several cases:

$I, y-\frac{h\varphi(y)}{2}, y+\frac{h\varphi(y)}{2}\in[a+n^{-3},b-n^{-3}]$:

Applying Lemma 2 we get that

$$I\leq\left(|(Q_{n+2}w)(a)|+\left|(Q_{n+2}w)\left(y+\frac{h\varphi(y)}{2}\right)\right|\right)\left(\frac{1}{y-\frac{h\varphi(y)}{2}-a}-\frac{1}{y+\frac{h\varphi(y)}{2}-a}\right)+$$
$$\frac{1}{y-\frac{h\varphi(y)}{2}-a}\left|(Q_{n+2}w)\left(y+\frac{h\varphi(y)}{2}\right)-(Q_{n+2}w)\left(y-\frac{h\varphi(y)}{2}\right)\right|\leq$$
$$C_1\frac{h}{n^{-6}}+C_2\frac{1}{n^{-3}}n^2h\leq Cn^6h,$$

where the estimation on $\|(Qw)'\|$ and the boundedness of φ and $Q_{n+2}w$ were used. C is independent of a,b.

$I, A\left(y\pm\frac{h\varphi(y)}{2}\right)=l_i\left(y\pm\frac{h\varphi(y)}{2}\right)$:

$$I\leq Ch\varphi(y)\|l_i'\|\leq Chn^5,$$

which estimation shows the behavior of the function around a, at it may be much less around b.

By setting new terms (the values of the function at the joining points) around the joining points we get the same estimations .

$II, y-\frac{h\varphi(y)}{2}\in(c(a,n)+h, c(b,n)-h)$:

As in the estimation of $k(x)$, we get that

$$II\le C\frac{hn^2}{\varphi(y)\left(y+\eta\frac{\varphi(y)}{2}\right)}\le Cn^2h^{\frac{1}{2}},$$

where $\eta\in[-h,h]$.

$II, y-\frac{h\varphi(y)}{2}\in I_h\bigcap([-1,c(a,n)+h]\bigcup[c(b,n)-h,1])$:

In this case around a we get a weeker estimation than around b again. Thus using Lemma 2. we get that

$$II\le Ch\|l_1^{'}\|\le Cn^5h.$$

That is

$$\Omega_1^{\varphi}\left(\frac{A}{w},t\right)_{w,\infty}\le Cn^6\sqrt{t}, \tag{25}$$

which estimation is independent of a and b and it means that there exists a polinomial sequence $\{u_m\}$, $u_m\in\Pi_m$, such that

$$\|A(y)-(u_mw)(y)\|\le Cn^6m^{-\frac{1}{2}}=\varepsilon(n,m). \tag{26}$$

With this sequence , using (19) and (25) (cf.[2,(100)] too) we get that

$$\|g^{'}(x)(x-a)-(u_mw)(x)(x-a)\|\le\|g^{'}(x)(x-a)-A(x)(x-a)\|+$$
$$\|A(x)(x-a)-(u_mw)(x)(x-a)\|\le C\left(n^{-\gamma}+\varepsilon(n,m)\right) \tag{27}$$

and using (23) and (25) :

$$\left|g(x)-\int_{-1}^{x}(u_mw)(y)dy\right|\le\left|g(x)-\int_{-1}^{x}\frac{(Q_{n+2}w)(y)-(Q_{n+2}w)(a)}{y-a}dy\right|+$$
$$\left|\int_{-1}^{x}\frac{(Q_{n+2}w)(y)-(Q_{n+2}w)(a)}{y-a}dy-A(y)\right|+$$
$$\int_{-1}^{x}|A(y)-(u_nw)(y)|dy\le C\left(n^{-\gamma}\log n+n^{-\gamma}\log n+\varepsilon(n,m)\right) \tag{28}$$

We can see now that when $m=n^{13}$ (say), then (27) and (28) tend to 0 if m tends to infinity. Further more investigating (28) in case $x=1$

$$\left|0-\int_{-1}^{1}(u_mw)(y)dy\right|=|b_0^{(m)}|\int_{-1}^{1}w\le C\left(n^{-\gamma}\log n+\varepsilon(n,m)\right) \tag{29}$$

Similarly to the previous discussion $|b_0^{(m)}|\longrightarrow 0$ when $m\longrightarrow\infty$. Herewith Lemma 3 is proved.

Proof of Theorem 2. Using the polynomial $p_{n(\varepsilon)}$ defined in Lemma 1., for $N>2n+3$,

$$\|(f-H_{N,w}(f))w\|\le\|(f-p_nq)w\|+\left\|w(x)\sum_{k=1}^{N}\frac{(1-C_k(x-x_k))l_k^2(x)}{w(x_k)}((f-p_nq)w)(x_k)\right\|+$$
$$\left\|w(x)\sum_{k=1}^{N}\frac{l_k^2(x)}{w(x_k)}|x-x_k|(p_nqw)^{'}(x_k)\right\|.$$

Because of $\varrho(w)$-normality the first and the second terms are less the $C\tilde{E}_n^w(f)$. For the third term we will apply that

$$\lim_{|x|\longrightarrow 1}(p_nqw)^{'}=0.$$

Namely for an arbitrary $\varepsilon>0$ an $I_\delta=[-1+\delta,1-\delta],\delta=\delta(\varepsilon,n)$ can be given such that if $x_k\notin I_\delta$, then $|(p_nqw)^{'}(x_k)|<\varepsilon$. Then, using Remark (3)

$$\left\|w(x)\sum_{k=1}^N\frac{l_k^2(x)}{w(x_k)}|x-x_k|(p_nqw)^{'}(x_k)\right\|\leq\left\|w(x)\sum_{\substack{k\\x_k\in I_\delta}}\frac{l_k^2(x)}{w(x_k)}|x-x_k|(p_nqw)^{'}(x_k)\right\|+$$

$$\left\|w(x)\sum_{\substack{k\\x_k\notin I_\delta}}\frac{l_k^2(x)}{w(x_k)}|x-x_k|(p_nqw)^{'}(x_k)\right\|\leq$$

$$\|(p_nqw)^{'}\|\left\|w(x)\sum_{\substack{k\\x_k\in I_\delta}}\frac{l_k^2(x)}{w(x_k)}|x-x_k|\right\|+\frac{2}{\varrho}\varepsilon.$$

It means that for proving Theorem 2 we only need to show that

$$\left\|w(x)\sum_{\substack{k\\x_k\in I_\delta}}\frac{l_k^2(x)}{w(x_k)}|x-x_k|\right\|\longrightarrow 0\text{ as }N\longrightarrow\infty. \tag{30}$$

Let's see now $g(x)$ defined in Lemma 3! Let $\gamma=\frac{\varrho}{2}$, and $b=1-\frac{\delta}{2}$ and $a\in(-1,b)$ otherwise is arbitrary. In this case

$$w(a)H_M\left(\frac{g}{w},\left(\frac{g}{w}\right)^{'},a\right)=$$

$$w(a)\sum_{\substack{k\\x_k\in[a,b]}}\frac{l_k^2(a)}{w(x_k)}(x_k-a)^{\frac{\varrho}{2}}(b-x_k)^{\frac{\varrho}{2}+1}\left((1-C_k(a-x_k))-\frac{\varrho}{2}+\left(\frac{\varrho}{2}+1\right)\frac{x_k-a}{b-x_k}\right)\geq$$

$$\frac{\varrho}{2}w(a)\sum_{\substack{k\\x_k\in[a,1-\delta]}}\frac{l_k^2(a)}{w(x_k)}(x_k-a)^{\frac{\varrho}{2}}(b-x_k)^{\frac{\varrho}{2}+1}\geq\frac{\varrho}{2}\left(\frac{\delta}{2}\right)^{\varrho+1}w(a)\sum_{\substack{k\\x_k\in[a,1-\delta]}}\frac{l_k^2(a)}{w(x_k)}(x_k-a)^{\frac{\varrho}{2}}\geq$$

$$\geq C\left(\frac{\varrho}{2},\delta\right)w(a)\sum_{\substack{k\\x_k\in[a,1-\delta]}}\frac{l_k^2(a)}{w(x_k)}(x_k-a)>0 \tag{31}$$

In the upper estimation besides the definition of $w-\varrho-$normality we used the facts that $\left(\frac{\varrho}{2}+1\right)\frac{x_k-a}{b-x_k}>0$, and $0<\frac{x_k-a}{2},\frac{\varrho}{2}<1$. Thus introducing the notation

$$U_{m+1}(x):=\frac{\int_{-1}^x(u_mw)(y)dy}{w(x)},$$

from (31) and from the definition of $g(x)$ with $C=C^{-1}\left(\frac{\varrho}{2},\delta\right)$ and for $M>2m$ we get that

$$w(a)\sum_{\substack{k\\x_k\in[a,1-\delta]}}\frac{l_k^2(a)}{w(x_k)}(x_k-a)\leq C\left|\left(\frac{g(a)}{w(a)}-H_M\left(\frac{g}{w},\left(\frac{g}{w}\right)^{'},a\right)\right)w(a)\right|\leq$$

$$\left|g(a)-\int_{-1}^{a}(u_m w)(x)dx\right|+\left|\int_{-1}^{a}(u_m w)(x)dx-w(a)H_M\left(U_{m+1},U_{m+1}^{'},a\right)\right|+$$

$$\left|w(a)H_M\left(U_{m+1},U_{m+1}^{'},a\right)-w(a)H_M\left(\frac{g}{w},\left(\frac{g}{w}\right)^{'},a\right)\right|=I+II+III.$$

Here by (17), $I\longrightarrow 0$, if $M\longrightarrow\infty$. Further by Remark (3) and Lemma 3 III. tends to 0 when M tends to infinity. Indeed

$$III\le w(a)\sum_{k=1}^{M}\frac{(1-C_k(a-x_k))l_k^2(a)}{w(x_k)}\left|g(x_k)-\int_{-1}^{x_k}(u_m w)(y)dy\right|+$$

$$w(a)\sum_{k=1}^{M}\frac{l_k^2(a)}{w(x_k)}|(a-x_k)g^{'}(x_k)-(a-x_k)(u_m w)(x_k)|\le$$

$$\left\|g(x)-\int_{-1}^{x}(u_m w)(y)dy\right\|+\frac{1}{\varrho}\|(a-x)g^{'}(x)-(a-x)(u_m w)(x)\|\longrightarrow 0.$$

To prove that $II\longrightarrow 0$, we recall the Rodrigues' formula again (compare with (13)):

$$\int_{-1}^{x}(u_m(y)-b_0^{(m)}p_0^{(w)})w(y)dy=(r_{m+1}w)(x), \tag{32}$$

where r_{m+1} is a polynomial with degree $m+1$. It is clear that the operator of Hermite interpolation is linear and bounded on $\varrho(w)-$normal systems in weighted norm (see (3) and (6)) and has a reconstruction property on polynomials. So denoting by

$$V(x):=\frac{\int_{-1}^{x}w(y)dy}{w(x)},$$

using (A) we can estimate term by term as it follows

$$II=\left|b_0^{(m)}p_0^{(w)}\left(\int_{-1}^{a}w(y)dy-w(a)H_M\left(V,V^{'},a\right)\right)\right|\le C(\varrho)p_0^{(w)}\|w\|_\infty|b_0^{(m)}|.$$

Thus the following was proved for $-1<a<1-\frac{\delta}{2}$:

$$w(a)\sum_{\substack{1\le k\le M\\ x_k\in(a,1-\delta)}}\frac{l_k^2(a)}{w(x_k)}(x_k-a)\longrightarrow 0\quad(M\longrightarrow\infty). \tag{33}$$

Let us see the function $g(x)$ in Lemma 3 again, but with

$$b=-1+\frac{\delta}{2}<a<1,$$

that is

$$G(x)=\begin{cases}(a-x)^{\gamma}(x-b)^{\gamma+1} & x\in[a,b]\\ 0 & \text{otherwise}\end{cases} \tag{34}$$

By the same chain of ideas the following can be proved:

$$w(a)\sum_{\substack{1\le k\le M\\ x_k\in(-1+\delta,a)}}\frac{l_k^2(a)}{w(x_k)}(a-x_k)\longrightarrow 0\quad(M\longrightarrow\infty). \tag{35}$$

According to (33) and (35) if $a\in(-1,1)$, then

$$w(a)\sum_{\substack{1\le k\le M\\ x_k\in(-1+\delta,1-\delta)}}\frac{l_k^2(a)}{w(x_k)}|x_k-a|\longrightarrow 0\quad(M\longrightarrow\infty). \tag{36}$$

(30) and (36) show that for an arbitrary ε we can choose δ, and m and then M such that

$$\|(f - H_{w,M}(f))w\| \longrightarrow 0 \text{ uniformly, if } M \longrightarrow \infty.$$

The case of infinite intervall is planned to investigate in a following paper.

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