A Contribution to the Grünwald - Marcinkiewicz Theorem

Ágota P. Horváth, Péter Vértesi

February 7, 2012

Abstract

This paper is a certain generalization of the GRÜNWALD - MARCINKIEWICZ theorem revealing its connection to a process defined by S. N. Bernstein.

1 Introduction

1.1. We begin with some definitions and notations. $\tilde{C}$ stands for the space of 2π-periodic continuous functions, $\mathcal{T}_m$ denotes the space of trigonometric polynomials of degree at most m of form $\frac{a_0}{2} + \sum_{k=1}^{m}(a_k \cos k\vartheta + b_k \sin k\vartheta)$, a_k, b_k reals. If $\Theta = \{\vartheta_{km}, k = 0, \dots, 2m, m = 1, 2, \dots\} \subset [0, 2\pi)$ is an interpolatory matrix with

$$0 \leq \vartheta_{0m} < \vartheta_{1m} < \cdots < \vartheta_{2m,m} < 2\pi, \tag{1}$$

the uniquely defined m^{th} trigonometric interpolatory polynomial for $f \in \tilde{C}$ is

$$T_m(f, \Theta, \vartheta) = \sum_{k=0}^{2m} f(\vartheta_{km}) t_{km}(\Theta, \vartheta), \tag{2}$$

where the uniquely defined fundamental trigonometric polynomials of degree exactly m satisfies the conditions

$$t_{km}(\Theta, \vartheta_{jm}) = \delta_{kj}, \quad 0 \leq k, j, \leq 2m \tag{3}$$

In 1914 G. Faber [**?**] proved that for arbitrary fixed interpolatory matrix Θ

$$\Lambda_m(\Theta) = \|\lambda_m(\Theta, \vartheta)\| = \|\sum_{k=0}^{2m} |t_{km}(\Theta, \vartheta)|\| \geq c \log m. \tag{4}$$

(Above $\|\cdot\|$ is the usual sup-norm on $[0, 2\pi)$; here and later $c, c_1, \dots$ are positive constants which may denote different values even in subsequent formulae.)

Key words: interpolation, operator norm, divergence, Grünwald-Marcinkiewicz theorem, Bernstein operator.
2000 MS Classification: 41A05

1.2. The relation (4) yields that for any fixed interpolatory matrix Θ one can find a function $f \in \tilde{C}$ for which

$$\limsup_{m\to\infty} \|T_m(f,\Theta,\vartheta)\| = \infty \tag{5}$$

(cf. [**?**, Vol. III; Chapter II, § 3]).

Considering pointwise convergence, the situation is not better. Let us take the "best" interpolatory matrix

$$E = \{\vartheta_{km} = \frac{2k\pi}{2m+1}, k = 0,\dots,2m, m = 1,2,\dots\} \tag{6}$$

for which

$$\Lambda_m(E) = \frac{2}{\pi}\log m + O(1)$$

is the smallest possible among all interpolatory matrices. One can prove

GRÜNWALD–MARCINKIEWICZ THEOREM.
There exists a function $f \in \tilde{C}$ for which

$$\limsup_{m\to\infty} |T_m(f,E,\vartheta)| = \infty \tag{7}$$

for every $\vartheta \in [0,2\pi)$.

(See G. Grünwald [**?**] and J. Marcinkiewicz [**?**].)

1.3. However, if we raise the degree, we can define a convergent interpolatory process. Namely, as L. Fejér did, if we take the trigonometric interpolatory polynomial of degree $2m$

$$H_m(f,\vartheta) := \frac{1}{(2m+1)^2}\sum_{k=0}^{2m} f(\vartheta_{km})\left(\frac{\sin\frac{2m+1}{2}(\vartheta-\vartheta_{km})}{\sin\frac{\vartheta-\vartheta_{km}}{2}}\right)^2, \tag{8}$$

where from now on ϑ_{km} are defined by (6), we have

$$\begin{cases} H_m(f,\vartheta_{km}) = f(\vartheta_{km}), & 0 \le k \le 2m \\ H_m^{'}(f,\vartheta_{km}) = 0, & 0 \le k \le 2m \end{cases}, \tag{9}$$

and

$$\lim_{n\to\infty} \|H_m(f,\vartheta) - f(\vartheta)\| = 0$$

for any $f \in \tilde{C}$ ([**?**], Part 17). Notice that $\left(\frac{\sin\frac{2m+1}{2}(\vartheta-\vartheta_{km})}{\sin\frac{\vartheta-\vartheta_{km}}{2}}\right)^2$ is the square of the fundamental functions $t_{km}(E,\vartheta)$.

Remark:
The interpolatory property of (8) was noticed by D. Jackson; however, $H_m^{'}(f,\vartheta_{km}) = 0$ was noticed first by L. Fejér. (cf [**?**], Part 17)

1.4. The *bridge* between T_m and H_m was given by S. N. Bernstein [**?**] defining the trigonometric polynomial B_{mh} of degree $m+h$, $0 \le h \le m$, as follows. Using again the equidistant nodes E of (6),

$$B_{mh}(f,\vartheta) = \frac{1}{(2m+1)(2h+1)} \sum_{k=0}^{2m} \frac{\sin \frac{2m+1}{2}(\vartheta - \vartheta_{km}) \sin \frac{2h+1}{2}(\vartheta - \vartheta_{km})}{\sin^2 \frac{\vartheta - \vartheta_{km}}{2}} f(\vartheta_{km}). \tag{10}$$

Obviously

$$B_{m0}(f,\vartheta) = T_m(f,E,\vartheta),$$

and

$$B_{mm}(f,\vartheta) = H_m(f,\vartheta);$$

moreover

$$B_{mh}(f,\vartheta_{km}) = f(\vartheta_{km}), \quad 0 \le k \le 2m. \tag{11}$$

If the norm of the operator of $B_{mh}(f,\vartheta)$ on the normed space $\tilde{C}$ is denoted by $\Lambda_{mh}(E)$ (i.e. $\Lambda_{mh}(E) = \sup_{\substack{\|f\| \le 1 \\ f \in \tilde{C}}} \|B_{mh}(f,\vartheta)\|$), then we can write with $N = \frac{m}{h+1}$

$$\Lambda_{mh}(E) = \frac{2}{\pi} \log N + O(1). \tag{12}$$

Moreover if the sequence N is bounded as $m \to \infty$, Bernstein proved that $\|B_{mh}(f,\vartheta) - f(\vartheta)\| \to 0$ $(m \to \infty)$ for every $f \in \tilde{C}$.

1.5. Later we use the *reconstructing property* of B_{mh} which says (cf. [**?**, p. 147]) that

$$B_{mh}(t,\vartheta) = t(\vartheta) \quad \textit{whenever} \quad t \in \mathcal{T}_{m-h}.$$

2 The result

As we have seen if the sequence $\{N = m(h+1)^{-1}\}$ is bounded, the B_{mh} process uniformly tends to the function $f \in \tilde{C}$ considered. But if it is not the case one can prove the next GRÜNWALD-MARCINKIEWICZ-type statement.

Theorem 1 *Let us given the monotone increasing sequence of positive integers $\{h_k\}$ with $\lim_{k\to\infty} h_k = \infty$. Then one can define the monotone increasing sequence of positive integers $\{m_k\}$ and a function $F \in \tilde{C}$ such that*

$$\limsup_{k\to\infty} |B_{m_k h_k}(F,\vartheta)| = \infty$$

for every $\vartheta \in [0, 2\pi)$.

Remarks.
1. Although the *same* function $F \in \tilde{C}$ which is "bad" for every ϑ but the elements of the subsequences (defined by the "lim sup") generally do depend on ϑ.

2. If $0 \le h_k \le c$, one can use essentially the original proof of the Grünwald–Marcinkiewicz theorem.

3 Proof

For the proof of the theorem we need some lemmas.

As A. F. Timan ([?], Part 8.2.41, p. 506) did, we write the Bernstein operator as follows.

3.1.

Lemma 1 *Let f be a bounded function on $[0, 2\pi)$ with $\|f\| \leq 1$. Then for any f and ϑ*

$$B_{mh}(f,\vartheta) = \frac{1}{2m+1} \sum_{|\vartheta-\vartheta_{km}|\leq \frac{2\pi}{2h+1}} \frac{\sin\frac{2m+1}{2}(\vartheta-\vartheta_{km})}{\sin\frac{\vartheta-\vartheta_{km}}{2}} f(\vartheta_{km}) + O(1) \tag{13}$$

$$:= \mathfrak{L}_{mh}(f,\vartheta) + O(1),$$

where the symbol "O" doesn't depend on f, ϑ, m and h.

Proof of Lemma 1. We write

$$B_{mh}(f,\vartheta) = \frac{1}{(2m+1)(2h+1)} \sum_{|\vartheta-\vartheta_{km}|\leq \frac{2\pi}{2h+1}} \frac{\sin\frac{2m+1}{2}(\vartheta-\vartheta_{km})\sin\frac{2h+1}{2}(\vartheta-\vartheta_{km})}{\sin^2\frac{\vartheta-\vartheta_{km}}{2}} f(\vartheta_{km})$$

$$+\frac{1}{(2m+1)(2h+1)} \sum_{|\vartheta-\vartheta_{km}|> \frac{2\pi}{2h+1}} \cdots = \Sigma_1 + \Sigma_2.$$

Here using in the denominator that on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have $|\sin x| \geq \frac{2}{\pi}|x|$,

$$|\Sigma_2| \leq \frac{c}{(2m+1)(2h+1)} \sum_{k=0}^{\infty} \frac{1}{\left(\frac{2\pi}{2h+1}+\frac{2k\pi}{2m+1}\right)^2} \leq c\frac{2h+1}{2m+1} \sum_{k=0}^{\infty} \frac{1}{1+\left(\frac{2h+1}{2m+1}k\right)^2}$$

$$< c\int_0^{\infty} \frac{1}{1+x^2}dx = O(1). \tag{14}$$

The first sum can be estimated as follows.

$$\Sigma_1 = \frac{1}{2m+1} \sum_{|\vartheta-\vartheta_{km}|\leq \frac{2\pi}{2h+1}} \frac{\sin\frac{2m+1}{2}(\vartheta-\vartheta_{km})}{\sin\frac{\vartheta-\vartheta_{km}}{2}} f(\vartheta_{km}) \left(1 + \frac{\sin\frac{2h+1}{2}(\vartheta-\vartheta_{km})}{(2h+1)\sin\frac{\vartheta-\vartheta_{km}}{2}} - 1\right)$$

$$= \frac{1}{2m+1} \sum_{|\vartheta-\vartheta_{km}|\leq \frac{2\pi}{2h+1}} \frac{\sin\frac{2m+1}{2}(\vartheta-\vartheta_{km})}{\sin\frac{\vartheta-\vartheta_{km}}{2}} f(\vartheta_{km})$$

$$+\frac{1}{2m+1} \sum_{|\vartheta-\vartheta_{km}|\leq \frac{2\pi}{2h+1}} \frac{\sin\frac{2m+1}{2}(\vartheta-\vartheta_{km})}{\sin\frac{\vartheta-\vartheta_{km}}{2}} f(\vartheta_{km}) \left(\frac{\sin\frac{2h+1}{2}(\vartheta-\vartheta_{km})}{(2h+1)\sin\frac{\vartheta-\vartheta_{km}}{2}} - 1\right)$$

$$= \mathfrak{L}_{mh}(f,\vartheta) + \Sigma_3.$$

Let $\alpha_k = \frac{\vartheta - \vartheta_{km}}{2}$. Using (two times) that $\frac{\sin t}{t^2} - \frac{1}{t}$ is bounded, we have

$$\frac{1}{\sin \alpha_k}\left(\frac{\sin(2h+1)\alpha_k}{(2h+1)\sin\alpha_k} - 1\right)$$

$$= \frac{\alpha_k^2}{\sin^2\alpha_k}\left\{\left((2h+1)\left(\frac{\sin(2h+1)\alpha_k}{(2h+1)^2\alpha_k^2} - \frac{1}{(2h+1)\alpha_k}\right)\right) - \left(\frac{\sin\alpha_k}{\alpha_k^2} - \frac{1}{\alpha_k}\right)\right\}$$

$$\le c\frac{\alpha_k^2}{\sin^2\alpha_k}(2h+1)^2\alpha_k \le c(2h+1)^2\alpha_k,$$

whence

$$|\Sigma_3| \le \frac{c}{2m+1}\sum_{\substack{k_0(\vartheta)\le k\le k_0(\vartheta)+2\frac{2m+1}{2h+1}\\ |\alpha_k|\le\frac{2\pi}{2h+1}}}(2h+1)^2\alpha_k$$

$$\le c\frac{2h+1}{2m+1}\sum_{k=1}^{2\frac{2m+1}{2h+1}} 1 = O(1). \tag{15}$$

By (14) and (15) the lemma is proved.

3.2. Now we introduce some definitions and notations. Let us denote by

$$A(n) = \left\{\frac{k}{2n+1}, k = 1, \dots 2n\right\}.$$

As in [**?**, Vol. III. Ch. II. § 3], it can be easily seen that $A(n)\cap A(n+1) = \emptyset$. Indeed let us suppose that there are l and j such that $\frac{l}{2n+1} = \frac{j}{2n+3}$. We can assume that $l = 2\nu+1$, $j = 2m+1$ are odd. So $0 < \frac{2\nu+1}{2n+1} = m - \nu < 1$, which is a contradiction.

Lemma 2 *Let* $S = \left\{\frac{p_i}{q_i} : \ (p_i, q_i) = 1,\ q_i \text{ are odd }, i = 1, \dots, s\right\}$*, a set in* $[0,1]$*. Then there is an* $n > (q_1 q_2 \cdots q_s)^2$*, such that the sets* $A(n)$ *and* $A(n+1)$ *and* S *are independent, that is,* $A(n)\cap S = A(n+1)\cap S = A(n)\cap A(n+1) = \emptyset$.

Proof of Lemma 2. Let $2n+1 = \left(\prod_{i=1}^s q_i + 2\right)\left(2\prod_{i=1}^s q_i + 1\right)$. If $\frac{p_i}{q_i} = \frac{l}{2n+1}$, then $\frac{p_i\left(2\prod_{i=1}^s q_i+1\right)\left(\prod_{j=1}^s q_j+2\right)}{q_i} = l$ is an integer, which is a contradiction, because q_i is not a divisor of $2p_i$.

If $\frac{p_i}{q_i} = \frac{l}{2n+3}$, then $\frac{p_i\left(\left(\prod_{j=1}^s q_j+2\right)\left(2\prod_{i=1}^s q_i+1\right)+2\right)}{q_i} = l$, and q_i is not a divisor of $4p_i$, i.e. we get a contradiction again.

3.3. Let $p > 3$ be an integer and define J_p by

$$J_p = \left(\frac{\pi}{p}, \pi - \frac{\pi}{p}\right)\cup\left(\pi + \frac{\pi}{p}, 2\pi - \frac{\pi}{p}\right).$$

Further let u be a positive integer such that $u > e^{p^2}$. Using Lemma 2, for a fixed (!) h (see (10)), we can define an independent system of nodes and a set of the corresponding pair of indices as follows (cf. $M(u(p),h)$ and $K(u(p),h)$,respectively). Let m_1 be an arbitrary fixed positive integer, and let $\tilde{m}_1 = m_1+1$. Furthermore, let

$$S_1 := \left\{ \frac{\vartheta_{k,m_1}}{2\pi}, \frac{\vartheta_{j,\tilde{m}_1}}{2\pi}, k = 1, \dots, 2m_1; j = 1, \dots, 2\tilde{m}_1 \right\}.$$

Let us denote by $\hat{S}_1$ the set of rational numbers in S_1 in reduced form.

By Lemma 2 one can define n_1 such that $\hat{S}_1$, $A(n_1)$, $A(n_1+1)$ are independent, and so S_1, $A(n_1)$, $A(n_1+1)$ are also independent.

Let $m_2 := n_1$. Now let

$$S_2 := S_1 \cup A(n_1) \cup A(n_1+1),$$

and the reduced form of the numbers in S_2 is denoted by $\hat{S}_2$ again. By Lemma 2, as previously, let us define n_2 such that S_2, $A(n_2)$, $A(n_2+1)$ are independent. Because $\frac{1}{2n_1+1} \in \hat{S}_2$, it is clear, that $n_2 > n_1$.

Let $m_3 := n_2$ and $S_3 := S_2 \cup A(n_2) \cup A(n_2+1)$. Continuing this process, we obtain (by Lemma 2) S_t, n_t, such that S_t, $A(n_t)$, $A(n_t+1)$ are independent.

Let $m_{t+1} := n_t, t = 1, 2, \dots, u(2h+1) - 1$. Using the above sets we define a set of nodes as follows

$$M(u(p),h) = M$$
$$:= \{\vartheta_{k,m_i}, \vartheta_{j,\tilde{m}_i}; k = 1, \dots, 2m_i; j = 1, \dots, 2\tilde{m}_i; i = 1, 2, \dots, (2h+1)u\}.$$

$M(u(p),h)$ is called an independent system of nodes. The corresponding set of pairs of indices

$$K(u(p),h) = K := \{(m_i, h), i = 1, 2, \dots, (2h+1)u\}.$$

Remark. By construction, $m_2 = n_1 > (2m_1+1)^2(2m_1+3)^2$, $m_3 = n_2 > \{(2m_1+1)(2m_1+3)(2m_2+1)(2m_2+3)\}^2$ or generally,

$$m_{t+1} = n_t > \left\{ \prod_{i=1}^{t} (2m_i+1)(2m_i+3) \right\}^2.$$

(Indeed, we have to use the relation $n > \left(\prod_{i=1}^{s} q_i\right)^2$ from Lemma 2. We omit the further details.)

3.4.

Lemma 3 *With the notations above, there is a trigonometric polynomial T_p, such that $\|T_p\| < 2$ on $[0, 2\pi]$, and for any fixed $\vartheta \in J_p$ there is a pair of indices $(m,h) \in K$, such that*

$$|B_{mh}(T_p, \vartheta)| > p. \tag{16}$$

Remark. Notice that we do not say anything on the degree of T_p.

Proof of Lemma 3. Let us divide the interval $[0, 2\pi]$ to $u(2h+1)$ pieces. As above, let M be an independent system of nodes, and let us define a function φ on this system as

$$\varphi(\vartheta_{k,m_j}) = \begin{cases} (-1)^k, \text{ if } \vartheta_{k,m_j} > \frac{2\pi j}{u(2h+1)} \\ 0, \text{ otherwise} \end{cases}, \tag{17}$$

and

$$\varphi(\vartheta_{k,\tilde{m}_j}) = \begin{cases} (-1)^k, \text{ if } \vartheta_{k,\tilde{m}_j} > \frac{2\pi j}{u(2h+1)} \\ 0, \text{ otherwise} \end{cases}, \tag{18}$$

for $j = 1, 2, \ldots, (2h+1)u$. By the definition of K the values of φ are uniquely defined. We can assume that φ is continuous on $[0, 2\pi]$, and $|\varphi| \le 1$. Let T_p be a trigonometric polynomial which interpolates φ at the nodes in M. By Lemma 3 from [**?**, Vol. III; Chapter II, § 3], we can assume that $\|T_p\| < 2$. Let $\vartheta \in I_l \cap J_p$, where $I_l = I_l(h,u) = \left[\frac{2\pi(l-1)}{u(2h+1)}, \frac{2\pi l}{u(2h+1)}\right]$. It will be shown that

$$|\mathfrak{L}_{m_l h}(T_p, \vartheta)| > p \quad \text{or} \quad |\mathfrak{L}_{\tilde{m}_l h}(T_p, \vartheta)| > p. \tag{19}$$

Indeed, by (17) and Lemma 1

$$\begin{aligned} |\mathfrak{L}_{m_l,h}(T_p, \vartheta)| &= \left| \frac{\sin \frac{2m_l+1}{2}\vartheta}{2m_l+1} \sum_{|\vartheta - \vartheta_{k,m_l}| \le \frac{2\pi}{2h+1}} \frac{(-1)^k \varphi(\vartheta_{k,m_l})}{\sin \frac{\vartheta - \vartheta_{k,m_l}}{2}} \right| \\ &\ge \frac{\left|\sin \frac{2m_l+1}{2}\vartheta\right|}{2m_l+1} \sum_{\frac{2\pi}{u(2h+1)} < \vartheta_{k,m_l} - \vartheta \le \frac{2\pi}{2h+1}} \frac{1}{\sin \frac{|\vartheta - \vartheta_{k,m_l}|}{2}} \\ &\ge c \left|\sin \frac{2m_l+1}{2}\vartheta\right| \int_{\frac{1}{u(2h+1)}}^{\frac{1}{2h+1}} \frac{1}{x} dx \ge c \left|\sin \frac{2m_l+1}{2}\vartheta\right| \log u \end{aligned}$$

The calculation for $\mathfrak{L}_{\tilde{m}_l h}(T_p, \vartheta)$ is similar. Now we have to deal with the sine factor. We write

$$|\sin \vartheta| = \left|\sin\left(\frac{2m_l+3}{2}\vartheta - \frac{2m_l+1}{2}\vartheta\right)\right| \le \left|\sin \frac{2m_l+3}{2}\vartheta\right| + \left|\sin \frac{2m_l+1}{2}\vartheta\right|.$$

Using the definition of J_p, we get that $|\sin \vartheta| > \frac{2}{p}$, i.e. one of the two terms on the right-hand side has to be greater that $\frac{1}{p}$. That is, choosing $u > e^{p^2}$ (16) is proved. By Lemma 1, we obtain Lemma 3.

3.5. Now we state

Statement 1 *Let us given the monotone increasing sequence of positive integers* $\{h_k\}$ *with* $\lim_{k\to\infty} h_k = \infty$. *Then one can define a monotone increasing sequence of positive integers* $\{m_k\}$ *and the function* $g \in \tilde{C}$ *such that*

$$\limsup_{k\to\infty} |B_{m_k h_k}(g, \vartheta)| = \infty$$

for every $\vartheta \in [0, 2\pi) \setminus \{0, \pi\}$.

Proof of the statement. The argument is analogous to [**?**, Vol. III. Ch. II. § 3]. We are given the sequence $\{h_k\}$. Next we fix a sequence of real numbers $\{c_k\}$ with $0 < c_1 < c_2 < \cdots c_k < c_{k+1} < \cdots$; $\lim_{k\to\infty} c_k = \infty$.

Now we will define the sequences of integers $\{p_k\}$, $\{u_k\}$, $\{m_k\}$ and the corresponding sets $M(u_k(p_k), h_k) = M_k = \{\vartheta_{l,m(k)_i}, \vartheta_{j,\tilde{m}(k)_i}; l = 1,\dots,2m(k)_i; j = 1,\dots,2\tilde{m}(k)_i; i = 1,2,\dots,(2h_k+1)u_k\}$, and similarly $K(u_k(p_k), h_k) = K_k$ as follows.

h_1 and c_1 are given. Let $p_1 > 3$, $u_1 > e^{p_1^2}$, $m(1)_1 > c_1 h_1$, $K_1 = \{(m(1)_i, h_1)|i = 1,2,\dots,(2h_1+1)u_1\}$, are given by the construction in Section 3.3, and M_1 is the corresponding system of nodes. By Lemma 2, $m(1)_i > m(1)_{i-1}$, and M_1 is an independent system of nodes, so by Lemma 3., via φ_1, we can construct T_{p_1}, $\{\mathcal{B}_{m(1)_i h_1}(T_{p_1},\vartheta)|i = 1,2,\dots,(2h_1+1)u_1\}$, where

$$\mathcal{B}_{m(1)_i h_1}(T_{p_1},\vartheta) = B_{m(1)_i h_1}(T_{p_1},\vartheta) \quad \text{or} \quad B_{m(1)_i+1,h_1}(T_{p_1},\vartheta),$$

according to relation (16) of Lemma 3; the definition of $\mathcal{B}_{m(l)_i h_l}(T_{p_l},\vartheta)$ will be analogous.

h_2 and c_2 are given. Let $p_2 > \max\{p_1^2, D_1\}$, where

$$D_1 = \max\left\{\|B_{m(1)_i h_1}(T_{p_1})\|^2 + \|B_{m(1)_i+1,h_1}(T_{p_1})\|^2,\ i = 1,2,\dots,(2h_1+1)u_1\right\};$$

the definition of $D_2, D_3, \dots$ will be analogous. Further let $u_2 > e^{p_2^2}$, $m(2)_1 > \max\{c_2 h_2, m(1)_{(2h_1+1)u_1}, h_2 + \deg T_{p_1}\}$. Now by the construction in Section 3.3 we can define K_2 and M_2 such that $m(2)_i > m(2)_{i-1}$, and M_2 is an independent system of nodes in itself. So by Lemma 3., we can construct φ_2, and then T_{p_2} on M_2. (Let us remark that the independency was needed for the construction of φ_2, so the independency of M_1 and M_2 is not necessary.) So T_{p_2} and $\{\mathcal{B}_{m(2)_i h_2}|i = 1,2,\dots,(2h_2+1)u_2\}$ fulfil the properties in Lemma 3.

In the n^{th} step, h_n and c_n are given. Let $p_n > \max\{p_{n-1}^2, \sum_{j=1}^{n-1} D_j\}$. Further let $u_n > e^{p_n^2}$, $m(n)_1 > \max\{c_n h_n, m(n-1)_{(2h_{n-1}+1)u_{n-1}}, h_n + \max\{\deg T_{p_l}|l = 1,2,\dots,n-1\}\}$. Now by Lemma 2, we can define K_n and M_n such that $m(n)_i > m(n)_{i-1}$, and M_n is an independent system of nodes in itself, and by Lemma 3., we can construct φ_n, and then T_{p_n} on M_n, as above.

Collecting the numbers $\{m(l)_i|i = 1,2,\dots,(2h_l+1)u_l, l = 1,2,\dots\}$, we define our sequence of pairs of indices as

$$\mathcal{I} := \{(m(1)_1,h_1),\dots,(m(1)_{(2h_1+1)u_1},h_1),(m(2)_1,h_2),\dots,(m(2)_{(2h_2+1)u_2},h_2),\dots\}.$$

It is clear that

$$\frac{m(l)_i}{h_l} \ge \frac{m(l)_1}{h_l} \ge c_l, \quad 1 \le i \le (2h_l+1)u_l,$$

that is $\left\{\frac{m(l)_i}{h_l}\right\}$ tends to infinity with l.

Now let us collect again the properties of the sequences $\{p_k\}$ and $\{m(k)_i\}$, which we will use in the next step.

$$p_{k+1} > p_k^2, \tag{20}$$

$$\left.\begin{array}{c} m(k+1)_i - h_{k+1} > \max\limits_{1\le r\le k}\{\deg T_{p_r}\} \\ \forall\,(m(k+1)_i, h_{k+1}) \in K(u_{k+1}(p_{k+1}), h_{k+1}) \end{array}\right\}, \tag{21}$$

$$p_{k+1} > \max\{D_l,\ l=1,\dots,k\}. \tag{22}$$

Let us define

$$g(\vartheta) = \sum_{k=1}^{\infty} \frac{T_{p_k}(\vartheta)}{\sqrt{p_k}}. \tag{23}$$

According to (20), $g \in \tilde{C}$. If $\vartheta \in [0, 2\pi) \setminus \{0, \pi\}$, then if s is large enough, then $\vartheta \in J_{p_s}$. Let us decompose g to three parts:

$$g(\vartheta) = \sum_{k=1}^{s-1} \cdots + \frac{T_{p_s}(\vartheta)}{\sqrt{p_s}} + \sum_{k=s+1}^{\infty} \cdots = g_1(\vartheta) + \frac{T_{p_s}(\vartheta)}{\sqrt{p_s}} + g_2(\vartheta).$$

Obviously this decomposition depends on ϑ; $g_1(\vartheta) \in \tilde{C}$ and $\|g_1\| \le c$, where c does not depend on s.

Let $\vartheta \in I_{j+1}(h_s, u_s(p_s))$. Using the reconstructing property of the Bernstein operator (see Section 1.5), relation (21) yields

$$\mathcal{B}_{m(s)_j h_s}(g_1, \vartheta) = g_1(\vartheta).$$

By Lemma 3, with a proper $m(s)_j = m(s, \vartheta)$,

$$\mathcal{B}_{m(s)_j h_s}\Big(\frac{T_{p_s}}{\sqrt{p_s}}, \vartheta\Big) > \sqrt{p_s},$$

and

$$\mathcal{B}_{m(s)_j h_s}(g_2, \vartheta) \le 2\|\mathcal{B}_{m(s)_j h_s}\| \sum_{k=s+1}^{\infty} \frac{1}{\sqrt{p_k}} \le C\|\mathcal{B}_{m(s)_j h_s}\| \frac{1}{\sqrt{p_{s+1}}},$$

where C is an absolute constant. According to (22), the third term is bounded. The above estimations prove our Statement 1.

Remark. The construction shows that for every fixed ϑ the index-pairs for which $\lim_{s\to\infty} \mathcal{B}_{m(s)_j h_s}(g, \vartheta) = \infty$ ($(m_s)_j = m(s, \vartheta)$, see above) do depend on ϑ and they are from $\mathcal{I}$.

3.6. To complete our proof, we state as follows.

Lemma 4 *Let $\alpha \in [0, 2\pi)$ be arbitrary, fixed. If the sequence $\left\{\frac{m_k}{h_k}\right\}$ tends to infinity, then there is a function $\Psi \in C_{2\pi}$ such that*

$$|B_{m_k h_k}(\Psi, \vartheta)| \le c(\vartheta), \quad \forall \vartheta \in [0, 2\pi) \setminus \alpha \tag{24}$$

however

$$\limsup_{k\to\infty} |B_{m_k h_k}(\Psi, \alpha)| = \infty. \tag{25}$$

The proof of this lemma is analogous to the one in A. Zygmund [**?**, p. 46, "Remark"].

3.7. Now we complete the proof of the result stated in Part 2. By Lemma 4, we can add to $g \in \tilde{C}$ (cf. Part 3.5.) $\Psi_1 \in \tilde{C}$ and $\Psi_2 \in \tilde{C}$ diverging at 0 and π, respectively. Then $F = g + \Psi_1 + \Psi_2$ proves our Theorem 1.

Acknowledgement. The authors thank the unknown referee for the careful reading of the paper and the advices, remarks which we included in the present form of our work.

References

[1] S. N. Bernstein, Sur une classe de formules d'interpolation, *Izv. Akad. Nauk SSSR* **9** (1931) 1151-1161. in S. N. Bernstein, Collected Works, Vol. II., ANSSSR, Moscow (1954), 146-154 (Russian)

[2] G. Faber, Über die interpolatorische Darstellung stetiger Functionen, *Jahresbericht der Deutschen Math. Vereinigung* **23** (1914), 192-210.

[3] L. Fejér, Über Interpolation, *Göttinger Nachrichten* (1916), 66-91, in L. Fejér, Gesammelte Arbeiten Vol. II., Akadémiai Kiadó, Budapest, (1970) 25-48.

[4] G. Grünwald, Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome stetiger Funktionen, *Annals of Math.* **37** (1936), 908-918.

[5] J. Marcinkiewicz, Sur la divergence des polynomes d'interpolation, *Acta Sci. Math. Szeged*, **8** (1937), 131-135.

[6] T. M. Mills, P. Vértesi, An extension of the Grünwald-Marcinkiewicz theorem to the higher order Hermite-Fejér interpolation, *J. Austral Math. Bulletin* **63** (2001), 299-320.

[7] I. P. Natanson, Constructive Function Theory, Vol. I-III, Frederick Ungar, New York (1965).

[8] A. F. Timan, Theory of Approximation of Function with Real Variables, Fizmatlit, Moscow (1960) (Russian)

[9] A. Zygmund, Trigonometric Series, Vol. II., Cambridge University Press (1959)

Department of Analysis,
Budapest University of Technology and Economics
ahorvath@renyi.hu

Alfréd Rényi Institute of Mathematics,
Hungarian Academy of Sciences
veter@renyi.hu