A Contribution to the Grünwald - Marcinkiewicz Theorem

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February 7, 2012

Abstract

This paper is a certain generalization of the GRÜNWALD - MARCINKIEWICZ theorem revealing its connection to a process defined by S. N. Bernstein.

1 Introduction

1.1. We begin with some definitions and notations. $\tilde{C}$ stands for the space of $2\pi$-periodic continuous functions, $T_m$ denotes the space of trigonometric polynomials of degree at most $m$ of form $a_0 + \sum_{k=1}^{m} (a_k \cos k\vartheta + b_k \sin k\vartheta)$, $a_k, b_k$ reals. If $\Theta = \{\vartheta_{km}, k = 0, \ldots, 2m, m = 1, 2, \ldots \} \subset [0, 2\pi)$ is an interpolatory matrix with

\[ 0 \leq \vartheta_{0m} < \vartheta_{1m} < \cdots < \vartheta_{2m,m} < 2\pi, \tag{1} \]

the uniquely defined $m^{th}$ trigonometric interpolatory polynomial for $f \in \tilde{C}$ is

\[ T_m(f, \Theta, \vartheta) = \sum_{k=0}^{2m} f(\vartheta_{km})t_{km}(\Theta, \vartheta), \tag{2} \]

where the uniquely defined fundamental trigonometric polynomials of degree exactly $m$ satisfies the conditions

\[ t_{km}(\Theta, \vartheta_{jm}) = \delta_{kj}, \quad 0 \leq k, j, \leq 2m \tag{3} \]

In 1914 G. Faber [?] proved that for arbitrary fixed interpolatory matrix $\Theta$

\[ \Lambda_m(\Theta) = ||\lambda_m(\Theta, \vartheta)|| = || \sum_{k=0}^{2m} |t_{km}(\Theta, \vartheta)|| \geq c \log m. \tag{4} \]

(Above $|| \cdot ||$ is the usual sup-norm on $[0, 2\pi)$; here and later $c, c_1, \ldots$ are positive constants which may denote different values even in subsequent formulae.)

Key words: interpolation, operator norm, divergence, Grünwald-Marcinkiewicz theorem, Bernstein operator.

2000 MS Classification: 41A05
1.2. The relation (4) yields that for any fixed interpolatory matrix $\Theta$ one can find a function $f \in \tilde{C}$ for which
\[
\limsup_{m \to \infty} \|T_m(f, \Theta, \vartheta)\| = \infty
\] (5)
(cf. [?, Vol. III; Chapter II, § 3]).
Considering pointwise convergence, the situation is not better. Let us take the "best" interpolatory matrix
\[
E = \{ \vartheta_{km} = \frac{2k\pi}{2m+1}, k = 0, \ldots, 2m, m = 1, 2, \ldots \} \tag{6}
\]
for which
\[
\Lambda_m(E) = \frac{2}{\pi} \log m + O(1)
\]
is the smallest possible among all interpolatory matrices. One can prove

**GRÜNWALD–MARCINKIEWICZ THEOREM.**

There exists a function $f \in \tilde{C}$ for which
\[
\limsup_{m \to \infty} |T_m(f, E, \vartheta)| = \infty
\] (7)
for every $\vartheta \in [0, 2\pi)$.

(See G. Grünwald [?] and J. Marcinkiewicz [?].)

1.3. However, if we raise the degree, we can define a convergent interpolatory process. Namely, as L. Fejér did, if we take the trigonometric interpolatory polynomial of degree $2m$
\[
H_m(f, \vartheta) := \frac{1}{(2m+1)^2} \sum_{k=0}^{2m} f(\vartheta_{km}) \left( \frac{\sin \frac{2m+1}{2}(\vartheta - \vartheta_{km})}{\sin \frac{\vartheta - \vartheta_{km}}{2}} \right)^2, \tag{8}
\]
where from now on $\vartheta_{km}$ are defined by (6), we have
\[
\begin{cases}
H_m(f, \vartheta_{km}) = f(\vartheta_{km}), & 0 \leq k \leq 2m \\
H_m'(f, \vartheta_{km}) = 0, & 0 \leq k \leq 2m
\end{cases}
\] (9)
and
\[
\lim_{n \to \infty} \|H_m(f, \vartheta) - f(\vartheta)\| = 0
\]
for any $f \in \tilde{C}$ (??, Part 17). Notice that $\left( \frac{\sin \frac{2m+1}{2}(\vartheta - \vartheta_{km})}{\sin \frac{\vartheta - \vartheta_{km}}{2}} \right)^2$ is the square of the fundamental functions $t_{km}(E, \vartheta)$.

**Remark:**
The interpolatory property of (8) was noticed by D. Jackson; however, $H_m'(f, \vartheta_{km}) = 0$ was noticed first by L. Fejér. (cf [?], Part 17)
1.4. The bridge between $T_m$ and $H_m$ was given by S. N. Bernstein [?] defining the trigonometric polynomial $B_{ mh}$ of degree $m + h$, $0 \leq h \leq m$, as follows. Using again the equidistant nodes $E$ of (6),

$$B_{ mh}(f, \vartheta) = \frac{1}{(2m+1)(2h+1)} \sum_{k=0}^{2m} \sin \frac{2m+1}{2}(\vartheta - \vartheta_{km}) \sin \frac{2h+1}{2}(\vartheta - \vartheta_{km}) f(\vartheta_{km}).$$

(10)

Obviously $B_{ m0}(f, \vartheta) = T_m(f, E, \vartheta)$, and

$$B_{ mm}(f, \vartheta) = H_m(f, \vartheta);$$

moreover

$$B_{ mh}(f, \vartheta_{km}) = f(\vartheta_{km}), \ 0 \leq k \leq 2m. \quad (11)$$

If the norm of the operator of $B_{ mh}(f, \vartheta)$ on the normed space $\tilde{C}$ is denoted by $\Lambda_{ mh}(E)$ (i.e. $\Lambda_{ mh}(E) = \sup_{f : \|f\| \leq 1} \|B_{ mh}(f, \vartheta)\|$), then we can write with $N = \frac{m}{\pi+1}$

$$\Lambda_{ mh}(E) = \frac{2}{\pi} \log N + O(1). \quad (12)$$

Moreover if the sequence $N$ is bounded as $m \to \infty$, Bernstein proved that $\|B_{ mh}(f, \vartheta) - f(\vartheta)\| \to 0$ ($m \to \infty$) for every $f \in \tilde{C}$.

1.5. Later we use the reconstructing property of $B_{ mh}$ which says (cf. [?], p. 147) that

$$B_{ mh}(t, \vartheta) = t(\vartheta) \quad \text{whenever} \quad t \in \mathcal{T}_{m-h}.$$ 

2 The result

As we have seen if the sequence $\{N = m(h+1)^{-1}\}$ is bounded, the $B_{ mh}$ process uniformly tends to the function $f \in \tilde{C}$ considered. But if it is not the case one can prove the next GRÜNWALD-MARCINKIEWICZ-type statement.

Theorem 1 Let us given the monotone increasing sequence of positive integers $\{h_k\}$ with $\lim_{k \to \infty} h_k = \infty$. Then one can define the monotone increasing sequence of positive integers $\{m_k\}$ and a function $F \in \tilde{C}$ such that

$$\limsup_{k \to \infty} |B_{ mh_k}(F, \vartheta)| = \infty$$

for every $\vartheta \in [0, 2\pi)$.

Remarks.
1. Although the same function $F \in \tilde{C}$ which is “bad” for every $\vartheta$ but the elements of the subsequences (defined by the "lim sup") generally do depend on $\vartheta$.
2. If $0 \leq h_k \leq c$, one can use essentially the original proof of the Grünwald–Marcinkiewicz theorem.
3 Proof

For the proof of the theorem we need some lemmas.

As A. F. Timan ([?], Part 8.2.41, p. 506) did, we write the Bernstein operator as follows.

3.1.

**Lemma 1** Let $f$ be a bounded function on $[0, 2\pi)$ with $\|f\| \leq 1$. Then for any $f$ and $\vartheta$

\[ B_{mh}(f, \vartheta) = \frac{1}{2m+1} \sum_{|\vartheta - \vartheta_{km}| \leq \frac{2\pi}{2m+1}} \frac{\sin \frac{2m+1}{2}(\vartheta - \vartheta_{km})}{\sin \frac{\vartheta - \vartheta_{km}}{2}} f(\vartheta_{km}) + O(1) \quad (13) \]

\[ := L_{mh}(f, \vartheta) + O(1), \]

where the symbol "O" doesn’t depend on $f, \vartheta, m$ and $h$.

**Proof of Lemma 1.** We write

\[ B_{mh}(f, \vartheta) = \frac{1}{(2m+1)(2h+1)} \sum_{|\vartheta - \vartheta_{km}| \leq \frac{2\pi}{2m+1}} \frac{\sin \frac{2m+1}{2}(\vartheta - \vartheta_{km}) \sin \frac{2h+1}{2}(\vartheta - \vartheta_{km})}{\sin \frac{\vartheta - \vartheta_{km}}{2}} f(\vartheta_{km}) \]

\[ + \frac{1}{(2m+1)(2h+1)} \sum_{|\vartheta - \vartheta_{km}| > \frac{2\pi}{2m+1}} \cdots = \Sigma_1 + \Sigma_2. \]

Here using in the denominator that on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we have $|\sin x| \geq \frac{2}{\pi}|x|$,

\[ |\Sigma_2| \leq \frac{c}{(2m+1)(2h+1)} \sum_{k=0}^{\infty} \frac{1}{\left( \frac{2\pi}{2m+1} + \frac{2k+1}{2h+1} \right)^2} \leq \frac{c}{2m+1} \sum_{k=0}^{\infty} \frac{1}{1 + \left( \frac{2h+1}{2m+1} k \right)^2} \]

\[ < c \int_0^{\infty} \frac{1}{1 + x^2} dx = O(1). \quad (14) \]

The first sum can be estimated as follows.

\[ \Sigma_1 = \frac{1}{2m+1} \sum_{|\vartheta - \vartheta_{km}| \leq \frac{2\pi}{2m+1}} \frac{\sin \frac{2m+1}{2}(\vartheta - \vartheta_{km})}{\sin \frac{\vartheta - \vartheta_{km}}{2}} f(\vartheta_{km}) \left( 1 + \sin \frac{2h+1}{2}(\vartheta - \vartheta_{km}) \right) \]

\[ = \frac{1}{2m+1} \sum_{|\vartheta - \vartheta_{km}| \leq \frac{2\pi}{2m+1}} \frac{\sin \frac{2m+1}{2}(\vartheta - \vartheta_{km})}{\sin \frac{\vartheta - \vartheta_{km}}{2}} f(\vartheta_{km}) \]

\[ + \frac{1}{2m+1} \sum_{|\vartheta - \vartheta_{km}| \leq \frac{2\pi}{2m+1}} \frac{\sin \frac{2m+1}{2}(\vartheta - \vartheta_{km})}{\sin \frac{\vartheta - \vartheta_{km}}{2}} f(\vartheta_{km}) \left( \frac{\sin \frac{2h+1}{2}(\vartheta - \vartheta_{km})}{(2h+1) \sin \frac{\vartheta - \vartheta_{km}}{2}} - 1 \right) \]

\[ = L_{mh}(f, \vartheta) + \Sigma_3. \]
Let $\alpha_k = \frac{\vartheta - \vartheta}{2}$. Using (two times) that $\frac{\sin t}{t^2} - \frac{1}{t}$ is bounded, we have

$$\frac{1}{\sin \alpha_k} \left( \frac{\sin(2h + 1)\alpha_k}{(2h + 1)\sin \alpha_k} - 1 \right)$$

$$= \frac{\alpha_k^2}{\sin^2 \alpha_k} \left\{ \left( (2h + 1) \left( \frac{\sin(2h + 1)\alpha_k}{(2h + 1)\alpha_k^2} - \frac{1}{(2h + 1)\alpha_k} \right) \right) - \left( \frac{\sin \alpha_k}{\alpha_k^2} - \frac{1}{\alpha_k} \right) \right\}$$

$$\leq c \frac{\alpha_k^2}{\sin^2 \alpha_k} (2h + 1)^2 \alpha_k \leq c(2h + 1)^2 \alpha_k,$$

whence

$$|\Sigma_3| \leq c \frac{2m + 1}{2m} \sum_{k_0(\varrho) \leq k \leq k_0(\varrho) + 2m + 1} (2h + 1)^2 \alpha_k$$

$$\leq c \frac{2h + 1}{2m + 1} \sum_{k=1}^{2m+1} 1 = O(1). \quad (15)$$

By (14) and (15) the lemma is proved.

3.2. Now we introduce some definitions and notations. Let us denote by

$$A(n) = \left\{ \frac{k}{2n+1}, k = 1, \ldots, 2n \right\}.$$ As in [?], Vol. III. Ch. II. § 3, it can be easily seen that $A(n) \cap A(n+1) = \emptyset$. Indeed let us suppose that there are $l$ and $j$ such that $\frac{l}{2n+1} = \frac{j}{2m+1}$. We can assume that $l = 2\nu + 1, j = 2m + 1$ are odd. So $0 < \frac{2\nu+1}{2m+1} = m - \nu < 1$, which is a contradiction.

**Lemma 2** Let $S = \left\{ \frac{p_i}{q_i} : (p_i, q_i) = 1, q_i \text{ are odd}, i = 1, \ldots, s \right\}, a \text{ set in } [0, 1].$ Then there is an $n > (q_1 q_2 \cdots q_s)^2$, such that the sets $A(n)$ and $A(n+1)$ and $S$ are independent, that is, $A(n) \cap S = A(n+1) \cap S = A(n) \cap A(n+1) = \emptyset$.

**Proof of Lemma 2.** Let $2n + 1 = (\Pi_{i=1}^s q_i + 2) (2 \Pi_{i=1}^s q_i + 1)$. If $\frac{p_i}{q_i} = \frac{l}{2n+1}$,

then $\frac{l}{2n+1}(\Pi_{i=1}^s q_i + 2)(2 \Pi_{i=1}^s q_i + 1) = l$ is an integer, which is a contradiction, because $q_i$ is not a divisor of $2p_i$.

If $\frac{p_i}{q_i} = \frac{l}{2n+1}$, then $\frac{p_i((\Pi_{i=1}^s q_i + 2)(2 \Pi_{i=1}^s q_i + 1) + 2)}{q_i} = l$, and $q_i$ is not a divisor of $4p_i$, i.e. we get a contradiction again.

3.3. Let $p > 3$ be an integer and define $J_p$ by

$$J_p = \left( \frac{\pi}{p}, \pi - \frac{\pi}{p} \right) \cup \left( \pi + \frac{\pi}{p}, 2\pi - \frac{\pi}{p} \right).$$
Further let $u$ be a positive integer such that $u > e^{p^2}$. Using Lemma 2, for a fixed (!) $h$ (see (10)), we can define an independent system of nodes and a set of the corresponding pair of indices as follows (cf. $M(u(p), h)$ and $K(u(p), h)$, respectively). Let $m_1$ be an arbitrary fixed positive integer, and let $\tilde{m}_1 = m_1 + 1$. Furthermore, let

$$S_1 := \left\{ \frac{\vartheta_{k,m_1}}{2\pi}, \frac{\vartheta_{j,\tilde{m}_1}}{2\pi}, k = 1, \ldots, 2m_1; j = 1, \ldots, 2\tilde{m}_1 \right\}.$$

Let us denote by $\hat{S}_1$ the set of rational numbers in $S_1$ in reduced form.

By Lemma 2 one can define $n_1$ such that $\hat{S}_1$, $A(n_1)$, $A(n_1 + 1)$ are independent, and so $S_1$, $A(n_1)$, $A(n_1 + 1)$ are also independent.

Let $m_2 := n_1$. Now let

$$S_2 := S_1 \cup A(n_1) \cup A(n_1 + 1),$$

and the reduced form of the numbers in $S_2$ is denoted by $\hat{S}_2$ again. By Lemma 2, as previously, let us define $n_2$ such that $S_2$, $A(n_2)$, $A(n_2 + 1)$ are independent. Because $\frac{1}{2n_1 + 1} \in S_2$, it is clear, that $n_2 > n_1$.

Let $m_3 := n_2$ and $S_3 := S_2 \cup A(n_2) \cup A(n_2 + 1)$. Continuing this process, we obtain (by Lemma 2) $S_t, n_t$, such that $S_t, A(n_t), A(n_t + 1)$ are independent.

Let $m_{t+1} := n_t, t = 1, 2, \ldots, u(2h + 1) - 1$. Using the above sets we define a set of nodes as follows

$$M(u(p), h) = M$$

$$:= \{ \frac{\vartheta_{k,m_1}}{2\pi}, \frac{\vartheta_{j,\tilde{m}_1}}{2\pi}, k = 1, \ldots, 2m_1; j = 1, \ldots, 2\tilde{m}_1; i = 1, 2, \ldots, (2h + 1)u \}.$$

$M(u(p), h)$ is called an independent system of nodes. The corresponding set of pairs of indices

$$K(u(p), h) = K := \{ (m_i, h), i = 1, 2, \ldots, (2h + 1)u \}.$$

**Remark.** By construction, $m_2 = n_1 > (2m_1 + 1)^2(2m_1 + 3)^2$, $m_3 = n_2 > ((2m_1 + 1)(2m_1 + 3)(2m_2 + 1)(2m_2 + 3))^2$ or generally,

$$m_{t+1} = n_t > \left( \prod_{i=1}^t (2m_i + 1)(2m_i + 3) \right)^2.$$

(Indeed, we have to use the relation $u > (\prod_{i=1}^t q_i)^2$ from Lemma 2. We omit the further details.)

**3.4.**

**Lemma 3** With the notations above, there is a trigonometric polynomial $T_p$, such that $\|T_p\| < 2$ on $[0, 2\pi]$, and for any fixed $\vartheta \in J_p$ there is a pair of indices $(m, h) \in K$, such that

$$|B_{mh}(T_p, \vartheta)| > p. \quad (16)$$
Remark. Notice that we do not say anything on the degree of $T_p$.

Proof of Lemma 3. Let us divide the interval $[0, 2\pi]$ to $u(2h + 1)$ pieces. As above, let $M$ be an independent system of nodes, and let us define a function $\varphi$ on this system as

$$
\varphi(\vartheta_{k,m_j}) = \begin{cases} 
(-1)^k, & \text{if } \vartheta_{k,m_j} > \frac{2\pi j}{u(2h+1)} \\
0, & \text{otherwise}
\end{cases},
$$

(17)

and

$$
\varphi(\tilde{\vartheta}_{k,m_j}) = \begin{cases} 
(-1)^k, & \text{if } \tilde{\vartheta}_{k,m_j} > \frac{2\pi j}{u(2h+1)} \\
0, & \text{otherwise}
\end{cases},
$$

(18)

for $j = 1, 2, \ldots, (2h + 1)u$. By the definition of $K$ the values of $\varphi$ are uniquely defined. We can assume that $\varphi$ is continuous on $[0, 2\pi]$, and $|\varphi| \leq 1$. Let $T_p$ be a trigonometric polynomial which interpolates $\varphi$ at the nodes in $M$. By Lemma 3 from [?, Vol. III; Chapter II, § 3], we can assume that $\|T_p\| < 2$. Let $\vartheta \in I_I \cap J_p$, where $I_I = I_I(h, u) = \left[\frac{2\pi j - 1}{u(2h+1)}, \frac{2\pi j}{u(2h+1)}\right]$. It will be shown that

$$
|\mathcal{L}_{m_Ih}(T_p, \vartheta)| > p \text{ or } |\mathcal{L}_{m_Ih}(T_p, \vartheta)| > p.
$$

(19)

Indeed, by (17) and Lemma 1

$$
|\mathcal{L}_{m_Ih}(T_p, \vartheta)| = \left| \frac{\sin \frac{2m_I + 1}{2} \vartheta}{2m_I + 1} \sum_{|\vartheta - \vartheta_{k,m_j}| \leq \frac{2\pi}{u(2h+1)}} (-1)^k \varphi(\vartheta_{k,m_j}) \right|
$$

$$
\geq \left| \frac{\sin \frac{2m_I + 1}{2} \vartheta}{2m_I + 1} \right| \sum_{\frac{2\pi}{u(2h+1)} < \vartheta_{k,m_j} - \vartheta \leq \frac{2\pi}{u(2h+1)}} \frac{1}{\sin \frac{|\vartheta - \vartheta_{k,m_j}|}{2}}
$$

$$
\geq c \left| \sin \frac{2m_I + 1}{2} \vartheta \right| \int_{\frac{2\pi}{u(2h+1)}}^{\frac{2\pi}{u(2h+1)}} \frac{1}{x} \, dx \geq c \left| \sin \frac{2m_I + 1}{2} \vartheta \right| \log u
$$

The calculation for $\mathcal{L}_{m_Ih}(T_p, \vartheta)$ is similar. Now we have to deal with the sine factor. We write

$$
|\sin \vartheta| = \left| \sin \left( \frac{2m_I + 3}{2} \vartheta - \frac{2m_I + 1}{2} \vartheta \right) \right| \leq \left| \sin \frac{2m_I + 3}{2} \vartheta \right| + \left| \sin \frac{2m_I + 1}{2} \vartheta \right|.
$$

Using the definition of $J_p$, we get that $|\sin \vartheta| > \frac{\pi}{2p}$, i.e. one of the two terms on the right-hand side has to be greater than $\frac{1}{p}$. That is, choosing $u > e^{p^2}$ (16) is proved. By Lemma 1, we obtain Lemma 3.

3.5. Now we state

Statement 1 Let us given the monotone increasing sequence of positive integers $\{h_k\}$ with $\lim_{k \to \infty} h_k = \infty$. Then one can define a monotone increasing sequence of positive integers $\{m_k\}$ and the function $g \in \mathcal{C}$ such that

$$
\lim_{k \to \infty} \sup_{k} |B_{m_k h_k}(g, \vartheta)| = \infty
$$

for every $\vartheta \in [0, 2\pi) \setminus \{0, \pi\}$. 

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Proof of the statement. The argument is analogous to [2, Vol. III. Ch. II. § 3]. We are given the sequence \( \{ h_k \} \). Next we fix a sequence of real numbers \( \{ c_k \} \) with \( 0 < c_1 < c_2 < \ldots c_k < c_{k+1} < \ldots; \lim_{k \to \infty} c_k = \infty \).

Now we will define the sequences of integers \( \{ p_k \} \), \( \{ u_k \} \), \( \{ m_k \} \) and the corresponding sets \( M(u_k(p_k), h_k) = M_k = \{ \vartheta_{i(m(k), j(n(k))}; l = 1, \ldots, 2m(k); j = 1, \ldots, 2n(k); i = 1, 2, \ldots, (2h_k + 1)u_k \} \), and similarly
\[ K(u_k(p_k), h_k) = K_k \] as follows.

\( h_1 \) and \( c_1 \) are given. Let \( p_1 > 3, u_1 > e^{p_1^2}, m(1) > c_1 h_1, K_1 = \{ (m(1), h_1) \} i = 1, 2, \ldots, (2h_1 + 1)u_1 \}, \) are given by the construction in Section 3.3, and \( M_1 \) is the corresponding system of nodes. By Lemma 2, \( m(1) > m(1)_{1-1} \), and \( M_1 \) is an independent system of nodes, so by Lemma 3., via \( \varphi_1 \), we can construct \( T_{p_1} = \{ T_{m(1), h_1} \} \).

\[ B_{m(1), h_1}(T_{p_1}, \vartheta) \] or \( B_{m(1), h_1}(T_{p_1}, \vartheta) \), according to relation (16) of Lemma 3; the definition of \( B_{m(i), h_i}(T_{p_i}, \vartheta) \) will be analogous.

\( h_2 \) and \( c_2 \) are given. Let \( p_2 > \max\{ p_1^2, D_1 \} \), where
\[ D_1 = \max \{ ||B_{m(1), h_1}(T_{p_1})||^2 + ||B_{m(1), +1, h_1}(T_{p_1})||^2, i = 1, 2, \ldots, (2h_1 + 1)u_1 \}; \]
the definition of \( D_2, D_3, \ldots \) will be analogous. Further let \( u_2 > e^{p_2^2}, m(2) > \max\{ c_2 h_2, m(1)_{(2h_1 + 1)u_1}, h_2 + \deg T_{p_1} \} \). Now by the construction in Section 3.3 we can define \( K_2 \) and \( M_2 \) such that \( m(2)_{i} > m(2)_{i-1} \), and \( M_2 \) is an independent system of nodes in itself. So by Lemma 3., we can construct \( \varphi_2 \), and then \( T_{p_2} \) on \( M_2 \). (Let us remark that the independency was needed for the construction of \( \varphi_2 \), so the independency of \( M_1 \) and \( M_2 \) is not necessary.) So \( T_{p_2} \) and \( \{ B_{m(1), h_1}(T_{p_1}, \vartheta) \} \) fulfill the properties in Lemma 3.

In the \( n^\text{th} \) step, \( h_n \) and \( c_n \) are given. Let \( p_n > \max\{ p_{n-1}^2, \sum_{j=1}^{n-1} D_j \} \). Further let \( u_n > e^{p_n^2}, m(n) > \max\{ c_n h_n, m(n-1)_{(2h_{n-1} + 1)u_{n-1}}, h_n + \max\{ \deg T_{p_{n-1}} \} \} \). Now by Lemma 2, we can define \( K_n \) and \( M_n \) such that \( m(n)_{i} > m(n)_{i-1} \), and \( M_n \) is an independent system of nodes in itself, and by Lemma 3., we can construct \( \varphi_n \), and then \( T_{p_n} \) on \( M_n \), as above.

Collecting the numbers \( \{ m(l)_{i} = 1, 2, \ldots, (2h_l + 1)u_l, l = 1, 2, \ldots \}, \) we define our sequence of pairs of indices as
\[ J := \{ (m(1), h_1), \ldots, (m(1)_{(2h_1 + 1)u_1}, h_1), (m(2), h_2), \ldots, (m(2)_{(2h_2 + 1)u_2}, h_2), \ldots \}. \]

It is clear that
\[ \frac{m(l)_{i}}{h_l} \geq \frac{m(l)_{i}}{h_l} \geq c_i, \quad 1 \leq i \leq (2h_l + 1)u_l, \]
that is \( \{ \frac{m(l)_{i}}{h_l} \} \) tends to infinity with \( l \).

Now let us collect again the properties of the sequences \( \{ p_k \} \) and \( \{ m(k)_{i} \} \), which we will use in the next step.
\[ p_{k+1} > p_k^2, \quad \forall (m(k+1),h(k+1)) \in K(u_{k+1}(p_{k+1}),h_{k+1}) \}
\]
\[ m(k+1) - h_{k+1} > \max \{ \deg T_{p_r} \}_{1 \leq r \leq k}, \quad \forall (m(k+1),h_{k+1}) \in K(u_{k+1}(p_{k+1}),h_{k+1}) \}
\]
\[ p_{k+1} > \max \{ D_l, l = 1, \ldots, k \}. \]

Let us define
\[ g(\vartheta) = \sum_{k=1}^{\infty} \frac{T_{p_k}(\vartheta)}{\sqrt{p_k}}. \]

According to (20), \( g \in \tilde{C} \). If \( \vartheta \in [0, 2\pi) \setminus \{ 0, \pi \} \), then if \( s \) is large enough, then \( \vartheta \in J_{p_s} \). Let us decompose \( g \) to three parts:
\[ g(\vartheta) = \sum_{k=1}^{s-1} \cdots + \sum_{k=s+1}^{\infty} \cdots = g_1(\vartheta) + \frac{T_{p_s}(\vartheta)}{\sqrt{p_s}} + g_2(\vartheta). \]

Obviously this decomposition depends on \( \vartheta \); \( g_1(\vartheta) \in \tilde{C} \) and \( \| g_1 \| \leq c \), where \( c \) does not depend on \( s \).

Let \( \vartheta \in I_{j+1}(h_s, u_s(p_s)) \). Using the reconstructing property of the Bernstein operator (see Section 1.5), relation (21) yields
\[ \mathcal{B}_{m(s), h_s}(g_1, \vartheta) = g_1(\vartheta). \]

By Lemma 3, with a proper \( m(s) = m(s, \vartheta) \),
\[ \mathcal{B}_{m(s), h_s}(\frac{T_{p_s}}{\sqrt{p_s}}, \vartheta) > \sqrt{p_s}, \]
and
\[ \mathcal{B}_{m(s), h_s}(g_2, \vartheta) \leq 2\| \mathcal{B}_{m(s), h_s} \| \sum_{k=s+1}^{\infty} \frac{1}{\sqrt{p_k}} \leq C\| \mathcal{B}_{m(s), h_s} \| \frac{1}{\sqrt{p_{s+1}}}, \]
where \( C \) is an absolute constant. According to (22), the third term is bounded. The above estimations prove our Statement 1.

**Remark.** The construction shows that for every fixed \( \vartheta \) the index-pairs for which \( \lim_{s \to \infty} \mathcal{B}_{m(s), h_s}(g, \vartheta) = \infty \) \((m(s))_j = m(s, \vartheta), \) see above) do depend on \( \vartheta \) and they are from \( J \).

3.6. To complete our proof, we state as follows.

**Lemma 4** Let \( \alpha \in [0, 2\pi) \) be arbitrary, fixed. If the sequence \( \{ \frac{m}{n} \} \) tends to infinity, then there is a function \( \Psi \in C_{2\pi} \) such that
\[ |B_{m_k, h_k}(\vartheta, \Psi)| \leq c(\vartheta), \quad \forall \vartheta \in [0, 2\pi) \setminus \alpha \]
\[ \limsup_{k \to \infty} |B_{m_k, h_k}(\vartheta, \alpha)| = \infty. \]
The proof of this lemma is analogous to the one in A. Zygmund [? , p. 46, "Remark"].

3.7. Now we complete the proof of the result stated in Part 2. By Lemma 4, we can add to \( g \in \tilde{C} \) (cf. Part 3.5.) \( \Psi_1 \in \tilde{C} \) and \( \Psi_2 \in \tilde{C} \) diverging at 0 and \( \pi \), respectively. Then \( F = g + \Psi_1 + \Psi_2 \) proves our Theorem 1.

Acknowledgement. The authors thank the unknown referee for the careful reading of the paper and the advices, remarks which we included in the present form of our work.

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