Abel Summation in Hermite-type Weighted Spaces with Singularities

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Abstract

On the real line besides the Hermite weight (w) there is another weight function (s) with polynomial-type zeros. We will show, that if the total degree of the zeros of s is M, then $\{h_k\}_{k=M}^{\infty}$ is a basis for Abel summation in the weighted space L^p_{ws}, that is $\lim_{r\to 1-}\|f-\sum_{n=M}^{\infty} r^n a_n(f)h_n\|_{ws,p} = 0$.

1 Introduction

In the classical case, on the unite disk the Poisson integral solves two problems together: the Dirichlet problem, and the problem of Abel-summability. Until we are on the unite disk, we can handle these two questions together. On the real line the solution of the Dirichlet problem may separates to Abel-summability. At first some words on the unite disk-problem are needed.

Investigating the connection of the weighted norm of the Hardy-Littlewood maximal function with the weighted norm of the original function the following question arised by Benjamin Muckenhoupt in 1972 [10]: There is an orthonormal system ($\{\varphi_n\}$) in a space/ with respect to a weight w on $[0, 2\pi)$, and there is another weight u on the same interval. The Poisson integral of a function f is defined by $P_r(f,x) = \sum_n r^n a_n(f)\varphi_n$, where $a_n(f)$-s are the Fourier coefficients of f with respect to w. The question is the following: Under what conditions will this Poisson integral converge to the function (with $r \to 1-$) according to the weighted norm with u? B. Muckenhoupt gave the answere in two cases: in the trigonometric case (that is $w \equiv 1$), and in ultraspheric, or Gegenbauer case ($w(\theta) = \sin^{2\lambda}(\theta)$). In these cases the necessary and sufficient condition was that u had to fulfil the A_p- or the weighted A_p-condition.

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If u is an A_p-weight, then u may has only "week" zeros. The whole situation changes, when u has "strong" zeros, like $\sin^k \frac{x-x_0}{2}$. On the trigonometric system the question was generalized (in this direction) by Kazaros S. Kazarian in 1987 [7]. Developing the multiplicative completion method of R. P. Boas and H. Pollard [3], he gave a method for giving the fundamental system in the weighted space with respect to u with "strong" zeros, and for giving the modified Poisson kernel here [8] [9]. Roughly speaking the new system (with respect to u) can be get from the old one by deleting some consecutive φ_n-s, and the number of the members has to be deleted depends on the zeros of u. The characterization of the existence of the solution of Dirichlet's problem in a weighted L^p-space on the unite disk was given also by K. S. Kazarian [7].

A sufficient condition for the similar problem in the continuous case was given by K. S. Kazarian and the author in 2007 [6].

Turning to the real line we have to mention that Abel-summability for Hermite weights ($w(x) = e^{-x^2}$) was proved by B. Muckenhoupt in 1969 [12]. He showed in this paper, that to get and to solve a Dirichlet-problem here, a modified Poisson integral has to be introduced. With the original Poisson integral, the differential equation is lost, but we can discuss the Abel-summability.

In d-dimension the Poisson-, and heat-diffusion semigroups were investigated by Krzysztof Stempak and José L. Torrea (eg [13],[14]). Laguerre case is very closed to the Hermite one. In Laguerre-weighted spaces the question of multipliers, which is inconnection with the Abel-summability was investigated by George Gasper and Walter Trebels [4], and by Cristian E. Gutiérrez, Andrew Incognito and José L. Torrea [5].

The aim of the present paper is to give a sufficient condition for Abel-summability by the combination of the real line- and the unite disk-methods, when besides the Hermite weight ($w(x) = e^{-\frac{x^2}{2}}$) we have another weight ($s(x)$) with zeros. We want to get a wider class of functions to be Abel-summable than in the original Hermite case, so we will suppose that $s(x)$ has no singularities, and $s(x)$ is bounded on the whole real line.

By these investigations arised some open questions: the case of infinitely many "strong" zeros of $s(x)$, the Dirichlet problem with the modified kernel, the adequate multiplier results for the semigroups with weights like this, etc.

2 Definitions, Notations, Results

Definition 1 *We say that a locally integrable function v satisfies the A_p-property on an interval J if there exists a constant $c = c(v,p) < \infty$ such that for all intervals $I \subset J$*

$$\frac{1}{|I|}\int_I v(x)dx\left(\frac{1}{|I|}\int_I v(x)^{-\frac{q}{p}}dx\right)^{\frac{p}{q}} \le c \tag{1}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1 < p, q < \infty$.

Definition 2 *Let $X := \{x_1, x_2, \dots, x_s\}$ be a finite collection of points on the* *real axis, and let $s(x)$ be a nonnegative bounded function on $\mathbf{R}$, and which has no zeros on $\mathbf{R}\setminus X$. Let us assume further that there is a $\delta>0$ such that*

1. There are k_j nonnegative natural numbers for $j=1,\dots,s$ such that $\left(\frac{s(x)}{|x-x_j|^{k_j}}\right)^p$ fulfils the A_p-property on the ball centered at x_j with radius δ: $B(x_j,\delta)$.

2. If $\frac{1}{K}<\frac{|x-x_j|}{|y-x_j|}<K$ for a positive constant K in $B(x_j,\delta)$, then $\left|\frac{s(x)}{s(y)}\right|<C$ here. ($C=C(K)$).

3. There is a constant C such that in $B(x_j,\delta)$, ($j=1,\dots,s$).

$\sup_{0<a<\delta}\left(\int_{0<|x-x_j|<a}\left(\frac{s(x)}{|x-x_j|^{k_j}}\right)^p dx\right)^{\frac{1}{p}}\left(\int_{a<|x-x_j|<\delta}\left(\frac{|x-x_j|^{k_j-1}}{s(x)}\right)^q dx\right)^{\frac{1}{q}}<C$ *with some $C=C(p)$.*

4. There is a y_0 such that if $|x|>y_0$ then $\frac{e^{-c|x|^\alpha}}{s(x)}<C$ with $\alpha<2$ and with some $C=C(y_0,\alpha)$.

5. There is a y_0 such that for every $|x|,|y|>y_0$; $c_1x<y<c_2x$ for some $c_i>0$, there is a $C=C(y_0,c_i)$ such that $e^{-\frac{(x-y)^2}{2}}\frac{s(x)}{s(y)}<C$

Example

$$s(x)=\prod_{j=1}^{s}|x-x_j|^{k_j+\delta_j}v(x),$$

where $\frac{-1}{p}<\delta_j<\frac{1}{q}$, if $k_j>0$ and $0\le\delta_j<\frac{1}{q}$ if $k_j=0$ ($j=1,\dots,s$), and $v(x)=e^{-c|x|^\alpha}$ ($\alpha<2$) or $v(x)=(1+x^2)^{-N}$, with $\alpha_j=k_j+\delta_j : N=\sum_{j=1}^s\alpha_j$.

or simply

$$s(x)=\begin{cases}\prod_{j=1}^s|x-x_j|^{k_j+\delta_j} & \text{if } x\in(x_1-1,x_s+1)\\ 1 & \text{if } x\in(-\infty,x_1-2)\cup(x_s+2,\infty)\\ \text{continuous} & \text{elsewhere}\end{cases}$$

Remark:

1)The assumptions 2) and 5) exclude the oscillation near zero that is the functions with behavior locally as $\left(\left|\sin\frac{1}{x-x_j}\right|+\frac{1}{n}\right)^{\frac{1}{2q}}$ if $\frac{1}{x-x_j}\in(n\pi,(n+1)\pi)$ and around the infinity as $|\sin x|+\frac{1}{n}$, when $x\in(n\pi,(n+1)\pi)$.

2)A function $u^p(x)\ge 0$ has the A_p-property on $B(0,\delta)$ means that there is a constant c such that for all $a<\delta$

$$\left(\int_0^a u^p\right)^{\frac{1}{p}}\left(\int_0^a\left(\frac{1}{u}\right)^q\right)^{\frac{1}{q}}\le ca \tag{2}$$

which means that $\frac{1}{u}\in L^q_{B(0,\delta)}$, and applying the the Hölder inequality with $\frac{1}{q}$ we get that

$$\int_0^a\left(\frac{1}{xu(x)}\right)^q dx\ge\left(\int_0^a\frac{1}{x}dx\right)^q\left(\int_0^a u^p\right)^{1-q},$$

that is $\frac{1}{xu(x)}\notin L^q_{B(0,\delta)}$

According to this observation we give the following definition:

Definition 3 ([9]) *A function f has a singularity of order q with degree k at a point x_0, if x_0 has a neighbourhood $I = I(x_0, \delta)$ such that*

$$(x - x_0)^k f(x) \in L_q(I),$$

but

$$(x - x_0)^{k-1} f(x) \notin L_q(I).$$

Remark:
1) With the above definition we can say that $\frac{1}{s(x)}$ has a singularity of order q with degree k_j at the points x_j, $j = 1, \dots, s$.
2) Let us denote by $h_k(x)$ the k-th orthonormed Hermite polynomial ($k = 0, 1, \dots,$), that is $\int_{\mathbf{R}} h_k(x) h_l(x) e^{-x^2} dx = \delta_{k,l}$.
3) For the Hermite-Poisson kernel we have a nice closed form [12]:

$$P_r(x, y) := \sum_{n=0}^{\infty} r^n h_n(x) h_n(y) = \frac{1}{\sqrt{\pi}\sqrt{1 - r^2}} e^{\frac{-r^2x^2 + 2rxy - r^2y^2}{1 - r^2}} \tag{3}$$

where $0 \le r < 1$.

With this we can define the modified Poisson kernel:

Definition 4 *Let $\{x_1, \dots, x_s\}$ is a finite system of points, and $s(x)$ is a function (as in Definition 2.) such that $\frac{1}{s(x)}$ has a singularity of order q with degree k_j at the points x_j, $j = 1, \dots, s$. For this weight function the modified Poisson kernel is*

$$P_s(r, x, y) := P_r(x, y) - \sum_{j=1}^{s} \sum_{l=0}^{k_j - 1} \left. \frac{\partial^l P_r(x, y)}{\partial y^l} \right|_{y = x_j} H_{l,j}(y), \tag{4}$$

where $H_{l,j}(y)$-s are the fundamental polynomials of Hermite interpolation on x_j-s with degree k_j, that is

$$\left(H_{l,j}\right)^{(m)}(x_i) = \delta_{i,j} \delta_{l,m} \quad 1 \le i, j \le s; \;\; 0 \le l \le k_j - 1; \;\; 0 \le m \le k_i - 1 \tag{5}$$

If for some $j \in \{1, \dots, s\}$ $k_j = 0$ we have no any conditions at the point x_j, and it results a zero member in the sum.

Let us introduce the notation: $f \in L^p_w$ if $fw \in L^p(\mathbf{R})$, and for $1 \le p \le \infty$

$$\|f\|_{w,p} = \|fw\|_p \tag{6}$$

Now everything are together to formulate the first theorem

Theorem 1 *If $s(x)$ is a weight function as in Definition 2, and $w(x) = e^{-\frac{x^2}{2}}$ then*

$$\sup_{0 \le r < 1} \left\| \left(\int_{\mathbf{R}} f(y) P_s(r, x, y) e^{-y^2} dy \right) s(x) \right\|_{w,p} \le c \|fs\|_{w,p} \tag{7}$$

To give a consequence of this theorem we need the notion of A-basis.

Definition 5 *A system $\{\varphi_n\}_{n=n_0}^{\infty}$ is A-basis (that is a basis for Abel-summation) in the space L_{ws}^p $(1 \le p \le \infty)$ if for every $f \in L_{ws}^p$ there is a uniqe series $\sum_{n=n_0}^{\infty} a_n(f)\varphi_n$ for which*

$$\lim_{r\to 1-} \|f - \sum_{n=n_0}^{\infty} r^n a_n(f)\varphi_n\|_{ws,p} = 0 \tag{8}$$

With the previous notations (see eg. the remark after Definition 3) we can formulate the main result of this paper.

Theorem 2 *Let $s(x)$ be a weight function as in Definition 2, and $w(x) = e^{-\frac{x^2}{2}}$ and*

$$M := \sum_{j=1}^{s} k_j \tag{9}$$

Then $\{h_k\}_{k=M}^{\infty}$ is an A-basis in L_{ws}^p, that is the series $\sum_{k=M}^{\infty} a_k(f)h_k$ is Abel summable to f, where $f \in L_{ws}^p$, and $a_k(f)$ are the Fourier coefficients of f in L_{ws}^p.

3 Proof of Theorem 2

At first we want to prove Theorem 2, provided that Theorem 1 is valid. For this we will use a theorem of S. Banach ([2]). As it is usual a system $\{\varphi_n\}_{n=n_0}^{\infty}$ in L_{ws}^p is called complete with respect to the dual space L_{ws}^q $(\frac{1}{p} + \frac{1}{q} = 1, 1 < p, q < \infty)$ if for a $g \in L_{ws}^q$ for which $\int_{\mathbf{R}} g\varphi_n s^2 w^2 = 0$ for every $n \ge n_0$ we have that g is the zero element of the space; and we call a system $\{\varphi_n\}_{n=n_0}^{\infty}$ minimal in L_{ws}^p if there is a conjugate system $\{\varphi_n^*\}_{n=n_0}^{\infty}$ in L_{ws}^q which is biorthonormal to the original system, that is $\int_{\mathbf{R}} \varphi_n \varphi_k^* s^2 w^2 = \delta_{n,k}$.

Theorem 3 (S. Banach) *A system $\{\varphi_n\}_{n=n_0}^{\infty}$ is A-basis in the space L_{ws}^p $(1 < p < \infty)$ if and only if it is a comlete and minimal system in L_{ws}^p, and there is a constant $c = c(p)$ such that*

$$\sup_{0\le r<1} \left\| \sum_{n=n_0}^{\infty} r^n a_n(f)\varphi_n \right\|_{sw,p} \le c\|f\|_{sw,p}$$

where $a_n(f) = \varphi_n^* f = \int_{\mathbf{R}} f\varphi_n^* s^2 w^2$.

According to the theorem of S. Banach we have to prove the following lemmas:

Lemma 1 *If s, w and M are the same as in Theorem 2, then the system $H = \{h_k\}_{k=M}^{\infty}$ is complete and minimal in L_{ws}^p.*

Proof: At first we give explicitely the conjugate system of H. Let $k \ge M$, and let us constuct the Hermite interpolatory polynomial of h_k with the less degree $(p_k(x))$ which interpolates h_k at x_j in order of k_j $(j = 1, \ldots, s)$. Because the degree of p_k is at most $M-1$, we can express it as the sum of some h_l-s with $l \le M-1$: $p_k = \sum_{l=0}^{M-1} a_{l,k} h_l$ We will show that

$$h_k^* = \frac{1}{s^2}\left(h_k - \sum_{l=0}^{M-1} a_{l,k} h_l\right) =: \frac{t_k}{s^2}. \tag{10}$$

h_k^* is in L^q_{ws}, because

$$\left(\int_{\mathbf{R}} \left(\frac{t_k}{s^2} ws\right)^q\right)^{\frac{1}{q}} \le \sum_{j=1}^{s} \left(\int_{B(x_j,\delta)} \left(\frac{h_k(x) - \sum_{l=0}^{M-1} a_{l,k} h_l(x)}{s(x)} w(x)\right)^q dx\right)^{\frac{1}{q}}$$

$$+ \left(\int_{\mathbf{R}\setminus \cup_{j=1}^{s} B(x_j,\delta)} \left(\frac{h_k(x) - \sum_{l=0}^{M-1} a_{l,k} h_l(x)}{s(x)} w(x)\right)^q dx\right)^{\frac{1}{q}} = \sum_{j=1}^{s} I_j + I$$

$$I \le c(\delta) \left\| \frac{p_k w}{s} \right\|_{q, \mathbf{R}\setminus \cup_{j=1}^{s} B(x_j,\delta)} \le c(\delta, k),$$

where p_k is a polynomial with degree k, and we used the growing property of $\frac{1}{s}$ at infinity. Around the singularities we get that

$$I_j \le \left\| \left(h_k^{(k_j)}(x) - \sum_{l=0}^{M-1} a_{l,k} h_l^{(k_j)}(x)\right) w(x) \right\|_{\infty, B(x_j,\delta)}$$

$$\times \left(\int_{B(x_j,\delta)} \left(\frac{|x-x_j|^{k_j}}{s(x)}\right)^q dx\right)^{\frac{1}{q}} \le c(k, x_j)$$

Now it is shown that $h_k^* \in L^q_{ws}$. The orthonormality follows from the orthonormality of $\{h_k\}$ in L^2_w: for $k, m \ge M$

$$\int_{\mathbf{R}} h_k^* h_m w^2 s^2 = \int_{\mathbf{R}} \left(h_k - \sum_{l=0}^{M-1} a_{l,k} h_l\right) h_m w^2 = \delta_{k,m} + 0$$

In the proof of completeness we also use the completeness of the original Hermite system. Let $g \in L^q_{ws}$ such that

$$\int_{\mathbf{R}} g h_k w^2 s^2 = 0 \quad k = M, M+1, \dots$$

In this case

$$g = \frac{1}{s^2} \sum_{l=0}^{M-1} b_l h_l, \text{ and } \int_{\mathbf{R}} \left(\frac{1}{s^2} \sum_{l=0}^{M-1} b_l h_l s w\right)^q < \infty$$

It means that $\frac{\sum_{l=0}^{M-1} b_l h_l}{|x-x_j|^{k_j}}$ is in L_q around x_j ($j = 1, \dots, s$), that is the polynomial in the nominator has M roots so it must be identically zero.

Lemma 2 *The Abel sum with respect to $s(x)$ is a Poisson integral with respect to the modified Poisson kernel, that is with the previous notations:*

$$\sum_{k=M}^{\infty} r^k a_k(f) h_k(x) = \int_{\mathbf{R}} f(y) P_s(r,x,y) e^{-y^2} dy \tag{11}$$

where the right hand side is absolute and locally uniformly convergent for every $r < 1$.

Proof: At first we have to show the convergency. Let us remark that $|h_k(x)| \leq ck^{-\frac{1}{12}} e^{\frac{x^2}{2}}$ [1].

$$|a_k| \leq \int_{\mathbf{R}} |f t_k| w^2 \leq \|fsw\|_p \left\| \frac{t_k w}{s} \right\|_q ,$$

We have to give a little bit finer estimation on $\left\|\frac{t_k w}{s}\right\|_q$ as in the previous lemma. Let us choose a y_0 such that if $|x| > y_0$ then $\frac{e^{-c|x|^\alpha}}{s(x)} < K$, and $x_j \in (-y_0, y_0)$ for all $j = 1, \ldots, s$! In $(-y_0, y_0)$, as in earlier we can estimate with $c(k)\|h_k^{(k_i)} w\|_\infty < c_1(k)$, where the coefficients in h_k^* results $c(k)$ and using that $h_k' = \sqrt{2k} h_{k-1}$ [15], we can se that $c_1(k)$ grows polynomially with k. When $|x| > y_0$, we use that $w(x)h_k(x) \leq k^{-\frac{1}{12}}$ if $|x| < \sqrt{2k+1}$, and $w(x)h_k(x) \leq e^{-\gamma x^2}$ with some $\gamma > 0$, when $|x| \geq \sqrt{2k+1}$ [1]. Thus

$$\frac{|t_k| w}{s} \leq C \left| h_k(x) - \sum_{l=0}^{M-1} a_{l,k} h_l(x) \right| e^{-\frac{x^2}{2}} e^{c|x|^\alpha}$$

$$\leq C \begin{cases} k^{-\frac{1}{12}} e^{ck^{\frac{\alpha}{2}}} & \text{if } |x| < \sqrt{2k+1} \\ e^{-\gamma x^2 + c|x|} & \text{if } |x| \geq \sqrt{2k+1} \end{cases}$$

That is

$$\left(\int_{|x|>y_0} \left(\frac{t_k(x) w(x)}{s(x)} \right)^q dx \right)^{\frac{1}{q}} \leq \left(\int_{\sqrt{2k+1}>|x|>y_0} (\cdot)^q \right)^{\frac{1}{q}} + \left(\int_{\sqrt{2k+1}\leq|x|} (\cdot)^q \right)^{\frac{1}{q}}$$

$$\leq C \left(k^{-\frac{1}{12}+\frac{1}{2q}} e^{ck^{\frac{\alpha}{2}}} + C \right)$$

Because, as we have seen, $I_j \leq c(k)$, where $c(k)$ grows polynomially with k, and $\left(\int_{(|x|<y_0)\setminus \cup_{j=1}^s B(x_j,\delta} \left(\frac{t_k w}{s}\right)^q \right)^{\frac{1}{q}} \leq c(\delta) \|t_k w\|_{\infty, |x|<y_0} \leq c(k,\delta)$, where $c(k,\delta)$ also grows polynomially with k, we get that $r^k a_k h_k(x) \leq c(x,\delta) c(k) \left(r e^{ck^{\left(\frac{\alpha}{2}-1\right)}} \right)^k$, where $c(k)$ grows polynomially with k, so the sum is absolute and uniformly convergent on every bounded intervals for every fix $r < 1$.

The previous calculation results that we can apply the dominated convergence theorem, that is

$$\sum_{k=M}^{\infty} r^k a_k(f) h_k(x) = \int_{\mathbf{R}} f(y) w^2(y) \left(\sum_{k=M}^{\infty} r^k h_k(x) t_k(y) \right) dy = (*)$$

Observing that if $k < M$ then $t_k(y) = h_k(y) - \sum_{l=0}^{M-1} a_{l,k} h_l(y)$ is a polynomial with degree at most $M-1$, and with at least M roots these polynomials are identically zero, so we can write that

$$(*) = \int_{\mathbf{R}} f(y) w^2(y) \left(\sum_{k=0}^{\infty} r^k h_k(x) t_k(y) \right) dy$$

Let us rearrange the polynomials t_k as

$$t_k(y) = h_k(y) - \sum_{j=1}^{s} \sum_{l=0}^{k_j - 1} h_k^{(l)}(x_j) H_{l,j}(y),$$

where $H_{l,j}$-s are the fundamental polynomials of Hermite interpolation. Using that the series below are absolute and locally uniformly convergent together with their derivatives, we get that

$$\sum_{k=0}^{\infty} r^k h_k(x) t_k(y) = \sum_{k=0}^{\infty} r^k h_k(x) h_k(y) - \sum_{k=0}^{\infty} r^k h_k(x) \sum_{j=1}^{s} \sum_{l=0}^{k_j-1} h_k^{(l)}(x_j) H_{l,j}(y)$$

$$= P_r(x,y) - \sum_{j=1}^{s} \sum_{l=0}^{k_j-1} H_{l,j}(y) \sum_{k=0}^{\infty} r^k h_k(x) h_k^{(l)}(x_j) = P_s(r,x,y)$$

The lemma is proved, and according to the theorem of S. Banach Theorem 2 comes true if Theorem 1 is valid.

4 Proof of Theorem 1

Lemma 3 *For the norm of the Hermite-Poisson kernel we have the following estimation:*

$$(12) \qquad \sup_{0 \le r < 1} \left(\int_{\mathbf{R}} \left(w(x) \left(\int_{\mathbf{R}} (P_r(x,y) w(y))^q \, dy \right)^{\frac{1}{q}} \right)^p dx \right)^{\frac{1}{p}} \le C$$

where $C = C(p)$ is independent of r, and $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$.

Proof: Let $f \in L_{w,\infty}$ an arbitrary function. With this f

$$\sup_{0 \le r < 1} \left\| \int_{\mathbf{R}} P_r(x,y) f(y) w^2(y) dy \right\|_{w,\infty}$$

$$\le \sup_{0 \le r < 1} \sup_{x \in \mathbf{R}} \left(w(x) \int_{\mathbf{R}} P_r(x,y) w(y) dy \right) \|f\|_{w,\infty}$$

Thus we give an estimation on the weighted infinite norm of the integral of the kernel. Because the maximum in y (with fixed x) of $w(x)w(y)P_r(x,y)$ is in

$y=\frac{2rx}{1+r^2}$ and this maximum value is equal to $\frac{c}{\sqrt{1-r}}e^{-\frac{1-r^2}{2(1+r^2)}x^2}$ we can devide the integral to two parts and can estimate as it follows:

$$w(x)\int_{\mathbf{R}}P_r(x,y)w(y)dy=w(x)\int_{\left|\frac{2rx}{1+r^2}-y\right|\le\sqrt{1-r}}P_r(x,y)w(y)dy$$

$$+w(x)\int_{\left|\frac{2rx}{1+r^2}-y\right|>\sqrt{1-r}}P_r(x,y)w(y)dy=I+II$$

$$I\le\frac{c}{\sqrt{1-r}}\sqrt{1-r}e^{-\frac{1-r^2}{2(1+r^2)}x^2}<c,$$

where c is independent of x and r.

$$II\le\frac{c}{\sqrt{1-r}}\int_{\left|\frac{2rx}{1+r^2}-y\right|>\sqrt{1-r}}\left(e^{-\frac{(1+r^2)(x-y)^2+2(1-r)^2xy}{2(1-r^2)}}\right.$$

$$\left.\times\frac{(1+r^2)(x-y)-(1-r)^2x}{1-r^2}\right)\frac{1-r^2}{(1+r^2)\left(\frac{2rx}{1+r^2}-y\right)}dy$$

Estimating the two parts of this integral separately and observing that we can do the same on the two parts we get that

$$II\le c\left[e^{-\frac{(1+r^2)(x-y)^2+2(1-r)^2xy}{2(1-r^2)}}\right]_{\frac{2rx}{1+r^2}+\sqrt{1-r}}^{\infty}\le c$$

After substitution one can see that c is independent of x and r again. That is this calculation follows that

$$\sup_{0\le r<1}\left\|\int_{\mathbf{R}}P_r(x,y)f(y)w^2(y)dy\right\|_{w,\infty}\le c\|f\|_{w,\infty}\tag{13}$$

$(c\neq c(r).)$ For $p=1$ we can use the duality of the spaces, that is let $f\in L_{w,1}$:

$$\left\|\int_{\mathbf{R}}P_r(x,y)f(y)w^2(y)dy\right\|_{w,1}=\sup_{\|g\|_{w,\infty}\le1}\int_{\mathbf{R}}g(x)w^2(x)\int_{\mathbf{R}}P_r(x,y)f(y)w^2(y)dydx$$

$$=\sup_{\|g\|_{w,\infty}\le1}\int_{\mathbf{R}}f(y)w^2(y)\int_{\mathbf{R}}P_r(x,y)g(x)w^2(x)dxdy$$

$$\le\|f\|_{w,1}\left\|\int_{\mathbf{R}}P_r(x,y)g(x)w^2(x)dx\right\|_{w,\infty}\le c1\|f\|_{w,1}$$

That is

$$\sup_{0\le r<1}\left\|\int_{\mathbf{R}}P_r(x,y)f(y)w^2(y)dy\right\|_{w,1}\le c\|f\|_{w,1}\tag{14}$$

Applying the Riesz-Thorin interpolation theorem on the operator

$f \mapsto \int_{\mathbf{R}} P_r(x,y)f(y)w^2(y)dy$ in the spaces $L_{w,p}$, we get that for every $1 \le p \le \infty$

$$\sup_{0\le r<1} \left\| \int_{\mathbf{R}} P_r(x,y)f(y)w^2(y)dy \right\|_{w,p} \le c\|f\|_{w,p} \tag{15}$$

It follows that for any $1 < p, q < \infty$, for which $\frac{1}{p}+\frac{1}{q}=1$, and $0 \le r < 1$

$$\left(\int_{\mathbf{R}} \left(w(x) \left(\int_{\mathbf{R}} (P_r(x,y)w(y))^q \, dy \right)^{\frac{1}{q}} \right)^p dx \right)^{\frac{1}{p}}$$

$$= \sup_{\|g\|_{w,q}\le 1} \int_{\mathbf{R}} g(x)w(x) \left(w(x) \left(\int_{\mathbf{R}} (P_r(x,y)w(y))^q \, dy \right)^{\frac{1}{q}} \right) dx$$

$$= \sup_{\|g\|_{w,q}\le 1} \sup_{\|f\|_{w,p}\le 1} \int_{\mathbf{R}} g(x)w^2(x) \int_{\mathbf{R}} P_r(x,y)f(y)w^2(y)dydx = (*)$$

According to (15), and by the Hölder inequality

$$(*) \le c\|g\|_{w,q}\|f\|_{w,p} \le c$$

and c does not depend on r, which proves the Lemma.

Lemma 4 *For the l-th derivative of the Hermite-Poisson kernel the following formula is valid:*

$$\frac{\partial^l P_r(x,y)}{\partial y^l} = (1-r^2)^{-l} P_r(x,y) \sum_{k=0}^{l} a_{k,l}(r,y)(x-y)^k (1-r)^{\lceil \frac{l-k}{2} \rceil}$$

$$= (1-r^2)^{-l} P_r(x,y) \sum_{k=0}^{l} b_{k,l}(r,x)(x-y)^k (1-r)^{\lceil \frac{l-k}{2} \rceil} \tag{16}$$

where $a_{k,l}(r,y)$ depends on r and y polynomially and the same is valid for $b_{k,l}(r,x)$; $\lceil a \rceil = \min\{k \in \mathbf{Z} | a \le k\}$.

Proof: We will prove this Lemma by induction.

$$\frac{\partial P_r(x,y)}{\partial y} = \frac{1}{1-r^2} P_r(x,y) 2r \left((1-r)y + (x-y) \right)$$

$$= \frac{1}{1-r^2} P_r(x,y) 2r((r(x-y) + (1-r)x)$$

From the first derivative we can see that type of symmetry of the right hand side expression in the dependence of the coefficients of x or y, so we will prove only one of them, the proof of the other one is the same.

So if the result is known for some l, then

$$\frac{\partial^{l+1} P_r(x,y)}{\partial y^{l+1}} = (1-r^2)^{-l} \left(\frac{\partial P_r(x,y)}{\partial y} \sum_{k=0}^{l} a_{k,l}(r,y)(x-y)^k (1-r)^{\lceil \frac{l-k}{2} \rceil} \right.$$

$$+P_r(x,y)\frac{\partial\left(\sum_{k=0}^{l} a_{k,l}(r,y)(x-y)^k(1-r)^{\left\lceil\frac{l-k}{2}\right\rceil}\right)}{\partial y}\Bigg)$$

$$= \frac{P_r(x,y)}{(1-r^2)^{(l+1)}}\Bigg(2r\left((1-r)y+(x-y)\right)\sum_{k=0}^{l} a_{k,l}(r,y)(x-y)^k(1-r)^{\left\lceil\frac{l-k}{2}\right\rceil}$$

$$+(1-r^2)\sum_{k=0}^{l}\frac{\partial a_{k,l}(r,y)}{\partial y}(x-y)^k(1-r)^{\left\lceil\frac{l-k}{2}\right\rceil}$$

$$-(1-r^2)\sum_{k=1}^{l} a_{k,l}(r,y)k(x-y)^{k-1}(1-r)^{\left\lceil\frac{l-k}{2}\right\rceil}\Bigg)$$

$$= \sum_{k=1}^{l-1}(x-y)^k(1-r)^{\left\lceil\frac{l+1-k}{2}\right\rceil}\Big(2rya_{k,l}(r,y)(1-r)^{\varepsilon_k}+2ra_{k-1,l}(r,y)$$

$$+\frac{\partial a_{k,l}(r,y)}{\partial y}(1-r)^{\varepsilon_k}(1+r)+a_{k+1,l}(r,y)(k+1)(1+r)\Big)$$

$$+(x-y)^l(1-r)\left(2rya_{l,l}(r,y)+2la_{l-1,l}(r,y)+\frac{\partial a_{l,l}(r,y)}{\partial y}(1+r)\right)$$

$$+(x-y)^{l+1}2ra_{l,l}(r,y)$$

And so the Lemma is proved.

Remark:

Using the definition of the k-th Hermite polynomial $H_k(x)$, we can express of the l-th derivative of the Hermite-Poisson kernel as

$$\frac{\partial^l\left(\frac{1}{\sqrt{\pi}\sqrt{1-r^2}}e^{-\frac{r^2}{1-r^2}(y-x)^2}\times e^{\frac{2rx}{1+r}y}\right)}{\partial y^l}$$

$$= P_r(x,y)r^l\sum_{k=0}^{l}\binom{l}{k}\left(\frac{2x}{1+r}\right)^{l-k}\left(\frac{-1}{\sqrt{1-r^2}}\right)^k H_k\left(\frac{r(y-x)}{\sqrt{1-r^2}}\right),$$

which gives back our formula with $b_{k,l}(r,x)$.

For the proof of Theorem 1 we need one more lemma [11]:

Lemma 5 (B. Muckenhoupt) *If $1\le p\le\infty$ there is a finite constant c such that*

$$\left(\int_0^\infty\left|u(x)\int_x^\infty f(t)dt\right|^p dx\right)^{\frac{1}{p}}\le c\left(\int_0^\infty|v(x)f(x)|^p\,dx\right)^{\frac{1}{p}}$$

if and only if

$$B=\sup_{r>0}\left(\int_0^r|u(x)|^p dx\right)^{\frac{1}{p}}\left(\int_r^\infty|v(x)|^{-q}dx\right)^{\frac{1}{q}}<\infty$$

Remark:
Using Muckenhoupt's proof one can see that the previous lemma is valid with some finite d instead of ∞.

Proof of Theorem 1: Let us denote by $B(x_j,\delta)$ a neighbourhood of x_j with radius δ, such that $x_i \notin B(x_j,2\delta)$ if $i \neq j$, and the assumptions on $s(x)$ are valid in $B(x_j,2\delta)$, where $i,j=1,\ldots,s$, and by $P_r^{(l)}(x,x_i) := \left.\frac{\partial^l P_r(x,y)}{\partial y^l}\right|_{y=x_i}$. With these notations we can decompose the integral in the theorem to parts:

$$\left(\int_{\mathbf{R}} s^p(x)w^p(x)\left|\int_{\mathbf{R}} P_s(r,x,y)f(y)w^2(y)dy\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq \sum_{j=1}^{s}\left(\int_{\mathbf{R}} s^p(x)w^p(x)\left|\int_{x_j-\delta}^{x_j+\delta} P_s(r,x,y)f(y)w^2(y)dy\right|^p dx\right)^{\frac{1}{p}}$$

$$+\left(\int_{\mathbf{R}} s^p(x)w^p(x)\left|\int_{\mathbf{R}\setminus\cup_{j=1}^{s}B(x_j,\delta)} P_s(r,x,y)f(y)w^2(y)dy\right|^p dx\right)^{\frac{1}{p}} = (*)$$

Now we decompose the modified Poisson kernel to a sum. If $k_i = 0$ for an $1 \leq i \leq s$, then $\sum_{l=0}^{k_i-1}(\cdot)$ is an empty sum, which is zero, and $H_{0,i} \equiv 0$, that is, effectively both in the outer and the inner sums we have so many terms as many positive k_i-s are there.

$$(*) \leq c\sum_{j=1}^{s}\sum_{i=1}^{s}\left(\int_{\mathbf{R}} s^p(x)w^p(x)\right.$$

$$\left.\left|\int_{x_j-\delta}^{x_j+\delta}\left(P_r(x,y)H_{0,i}(y)-\sum_{l=0}^{k_i-1}P_r^{(l)}(x,x_i)H_{l,i}(y)\right)f(y)w^2(y)dy\right|^p dx\right)^{\frac{1}{p}}$$

$$+\left(\int_{\mathbf{R}} s^p(x)w^p(x)\left|\int_{\mathbf{R}\setminus\hat{\cup}_{j=1}^{s}B(x_j,\delta)} P_s(r,x,y)f(y)w^2(y)dy\right|^p dx\right)^{\frac{1}{p}} = (**)$$

Where $\hat{\cup}_{j=1}^{s}$ means that the union for that indices for which $k_j \neq 0$.

$$(**) \leq c\sum_{i=1}^{s}\left(\int_{\mathbf{R}} s^p(x)w^p(x)\right.$$

$$\left.\left|\int_{x_i-\delta}^{x_i+\delta}\left(P_r(x,y)H_{0,i}(y)-\sum_{l=0}^{k_i-1}P_r^{(l)}(x,x_i)H_{l,i}(y)\right)f(y)w^2(y)dy\right|^p dx\right)^{\frac{1}{p}}$$

$$+c\|fs\|_{w,p}\sum_{j=1}^{s}\sum_{\substack{1\leq i\leq s\\ i\neq j}}\left(\int_{\mathbf{R}} s^p(x)w^p(x)\right.$$

$$\left(\int_{x_j-\delta}^{x_j+\delta}\left(\left|P_r(x,y)H_{0,i}(y)-\sum_{l=0}^{k_i-1}P_r^{(l)}(x,x_i)H_{l,i}(y)\right|\frac{w(y)}{s(y)}\right)^q dy\right)^{\frac{p}{q}} dx\Bigg)^{\frac{1}{p}}$$

$$+c\|fs\|_{w,p}\left(\int_{\mathbf{R}} s^p(x)w^p(x)\left(\int_{\mathbf{R}\setminus\hat{\cup}_{j=1}^s B(x_j,\delta)}\left|P_s(r,x,y)\frac{w(y)}{s(y)}\right|^q dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$$

$$= c\sum_{i=1}^{s} A_i \;+\; c\|fs\|_{w,p}\,(B\; + C)$$

It has to be mentioned that in A_i and B can be an identically zero term in the integral (when $k_i = 0$), and the integral around an x_j for which $k_j = 0$ is in the term C.

Now we have to show that A_i, B and C are bounded independently of r. We have to note, that if r is less than, say $\frac{1}{2}$, then everything is trivial, so we can assume that $r > \max\left\{\frac{1}{2}, 1-\delta^4\right\}$.

At first we deal with the main part:

$\mathbf{A_i}$:

We can suppose that $k_i > 0$! In this situation we have to distinguish some cases.

Case a: If x_i separates x and y, say $x \le x_i \le y$. It means, that we are on $(-\infty, x_i] \times [x_i, x_i+\delta)$, and the computation is the same on $[x_i,\infty)\times(x_i-\delta, x_i]$.

In *Case a* we have some subcases:

I: $|x-x_i| > 2\left((1-r)\beta\log\frac{1}{1-r}\right)^{\frac{1}{2}} := 2n_\beta(r)$, where β is a suitable constant. (will be given later)

I/1: If $|y-x_i| < n_\beta(r)$, denoting by

$$g_i(r,x,y) := P_r(x,y)H_{0,i}(y) - \sum_{l=0}^{k_i-1} P^{(l)}(x,x_i)H_{l,i}(y) \tag{17}$$

we have that

$$A_i \le c\|fs\|_{w,p}\left(\int_{\mathbf{R}}\left(\int_{|y-x_i|<n_\beta(r)}\left|s(x)w(x)\frac{g_i(r,x,y)}{s(y)}w(y)\right|^q dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$$

$$\le C(x_i,\delta)\left(\int_{\mathbf{R}}\left|s(x)w(x)\frac{\partial^{k_i} g_i(r,x,y)w(y)}{\partial y^{k_i}}\right|_{y=\xi_i}\right.$$

$$\left.\left(\int_{|y-x_i|<n_\beta(r)}\left(\frac{|y-x_i|^{k_i}}{s(y)}\right)^q\right)^{\frac{1}{q}}\right|^p\Bigg)^{\frac{1}{p}} \tag{18}$$

Here $\xi_i \in (x_i, y)$ so $|x-\xi_i| > 2n_\beta(r)$. According to Lemma 4, we get that

$$\left| \frac{\partial^{k_i}(g_i(r,x,y)w(y))}{\partial y^{k_i}}\bigg|_{y=\xi_i} \right|$$

$$= \left| \sum_{m=0}^{k_i} \binom{k_i}{m} \frac{w^{(k_i-m)}(\xi_i)}{w(\xi_i)} w(\xi_i) \left(P_r(x,\xi_i) \sum_{n=0}^{m} \binom{m}{n} H_{0,i}^{(m-n)}(\xi_i) \right.\right.$$

$$\times \sum_{k=0}^{n} a_{k,n}(r,\xi_i)(x-\xi_i)^k (1-r)^{\lceil \frac{n-k}{2} \rceil - n} - P_r(x,x_i) \sum_{l=0}^{k_i-1} H_{l,i}^{(m)}(\xi_i)$$

$$\left.\left. \times \sum_{k=0}^{l} a_{k,l}(r,x_i)(x-x_i)^k (1-r)^{\lceil \frac{l-k}{2} \rceil - l} \right) \right| \tag{19}$$

Because the function $q(z) := z^k e^{-\frac{r^2}{1-r^2}(z)^2}$ attains it's maximum at $z = \sqrt{\frac{k(1-r^2)}{2r^2}}$ (where $z \geq 0$ and $k \geq 0$), and is monotone before and after this maximum place:

$$\left| (x-x_i)^{k_i} \frac{\partial^{k_i}(g_i(r,x,y)w(y))}{\partial y^{k_i}}\bigg|_{y=\xi_i} \right| \leq c(x_i,k_i) e^{-\frac{r^2}{1+r}4\beta \log \frac{1}{1-r}} \left(\log \frac{1}{1-r} \right)^{k_i}$$

$$\times \frac{1}{\sqrt{1-r}} p(x) e^{\frac{2r}{1+r} x x_i - \frac{x_i^2}{2}}, \tag{20}$$

where $p(x)$ is a polynomial, and it's coefficients depend on x_i.

So if $\beta > \frac{1+r}{8r^2}$, then using that s is bounded and has a singularity of ordedr q with degree k_i at x_i, we have that

$$A_i \leq \|fs\|_{w,p} C(x_i,k_i,\delta) \|p(x) e^{-\frac{x^2}{2} + \frac{2r}{1+r} x x_i}\|_p < C \tag{21}$$

I/2: If $|y-x_i| \geq n_\beta(r)$, then observing that if $|x-x_i| > 2n_\beta(r)$, then $|x-y| > 2n_\beta(r)$ we can esimate $|x-x_i|^{k_i}|g_i(r,x,y)|$ term by term:

$$\left| \frac{s(x)}{s(y)} w(x)w(y) g_i(r,x,y) \right| \leq c(\delta) \frac{s(x)|y-x_i|^{k_i}}{|x-x_i|^{k_i} s(y)} w(x)w(y)$$

$$\times \left(n_\beta(r)^{-k_i} \frac{1}{\sqrt{1-r}} e^{\frac{2r}{1+r}xy} |x-y|^{k_i} e^{-\frac{r^2}{1-r^2}(y-x)^2} |H_{0,i}(y)| \right.$$

$$+ \frac{1}{\sqrt{1-r}} e^{\frac{2r}{1+r} x x_i} |x-x_i|^{k_i} e^{-\frac{r^2}{1-r^2}(x-x_i)^2} |p_{3,l,i}(y)||y-x_i|^{l-k_i}$$

$$\left. \left(\sum_{l=0}^{k_i-1} \sum_{k=0}^{l} |a_{k,l}(r,x_i)| x - x_i|^k (1-r)^{\frac{-l-k}{2}} \right) \right),$$

where $|p_{3,l,i}(y)|||y-x_i|^l = |H_{l,i}(y)|$. So by the previous remark on the function $q(z)$, we can estimate the last term by

$$(n_\beta(r))^{-k_i} \frac{1}{\sqrt{1-r}} e^{\frac{2r}{1+r}xy} (n_\beta(r))^{k_i} e^{-\frac{r^2}{1+r}4\beta \log \frac{1}{1-r}} |H_{0,i}(y)| \tag{22}$$

$$+\frac{1}{\sqrt{1-r}}e^{\frac{2r}{1+r}xx_i}e^{-\frac{r^2}{1+r}4\beta\log\frac{1}{1-r}}|p_i(r,x,y)|\sup_{\substack{0\le k\le l\\ 0\le l\le k_i-1}}\left((n_\beta(r))^{k+l}(1-r)^{\frac{-l-k}{2}}\right)$$

$$\text{(23)}\le (1-r)^{\frac{4\beta r^2}{1+r}-\frac{1}{2}}\left(e^{\frac{2r}{1+r}xy}|p_i(y)|+e^{\frac{2r}{1+r}xx_i}|p_i(r,x_i,y)|\left(\log\frac{1}{1-r}\right)^{k_i-1}\right)$$

Here p_i-s are polynomials in r, and y. That is if we assume that $\beta>\frac{1+r}{8r^2}$:

$$\left(\int_{\mathbf{R}}\left(\int_{n_\beta(r)<|y-x_i|<\delta}\left(\frac{s(x)}{s(y)}w(x)w(y)g_i(r,x,y)\right)^q dy\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}$$

$$\le c(x_i,\delta)\|w(x)e^{\frac{2r}{1+r}xx_i}\frac{s(x)}{|x-x_i|^{k_i}}p(x)\|_p\|\frac{|y-x_i|^{k_i}}{s(y)}\|_q<C. \tag{24}$$

Where p is a polynomial, and in the estimation of the p-norm of the first term we had to decomposite the integral to an integral around x_i and away from x_i. With it we get again that

$$A_i\le C\|fs\|_{w,p}.$$

II: $|x-x_i|\le 2n_\beta(r)$, where β is given above.
II/1:At first we will deal with that case: $\delta\ge|y-x_i|>|x-x_i|$.
As in a previous case

$$\left|\frac{|x-x_i|^{k_i}}{|y-x_i|^{k_i-1}}w(x)w(y)g_i(r,x,y)\right|\le w(x)\left|\left.\frac{\partial^{k_i-1}(g_i(r,x,y)w(y))}{\partial y^{k_i-1}}\right|_{y=\xi_i}\right|$$

$$=\sum_{m=0}^{k_i-1}\binom{k_i-1}{m}\frac{|w^{(k_i-1-m)}|(\xi_i)}{w(\xi_i)}w(\xi_i)\left(P_r(x,\xi_i)\sum_{n=0}^{m}\binom{m}{n}|H_{0,i}^{(m-n)}|(\xi_i)\right.$$

$$\times\sum_{k=0}^{n}|a_{k,n}(r,\xi_i)||x-\xi_i|^{k+k_i}(1-r)^{\lceil\frac{n-k}{2}\rceil-n}+P_r(x,x_i)\sum_{l=0}^{k_i-1}|H_{l,i}^{(m)}|(\xi_i)$$

$$\left.\times\sum_{k=0}^{l}|a_{k,l}(r,x_i)||x-x_i|^{k+k_i}(1-r)^{\lceil\frac{l-k}{2}\rceil-l}\right)\le c \tag{25}$$

Here we used again the remark on the maximum of $q(z)$. So

$$\left(\int_{|x-x_i|\le 2n_\beta(r)}s^p(x)w^p(x)\left|\int_{|x-x_i|<|y-x_i|<\delta}f(y)g_i(r,x,y)w^2(y)dy\right|^p dx\right)^{\frac{1}{p}}\le$$

$$\left(\int_{|x-x_i|\le n_\beta(r)}\left(\frac{s(x)}{|x-x_i|^{k_i}}\right)^p\right.$$

$$\left.\times\left(\int_{|x-x_i|<|y-x_i|<\delta}|f(y)|s(y)w(y)\frac{|y-x_i|^{k_i-1}}{s(y)}dy\right)^p dx\right)^{\frac{1}{p}} \tag{26}$$

According to Lemma 5 and replacing $|x-x_i|$ or $|y-x_i|$ by z,

$$(26) \le c\|fsw\|_p \iff \sup_{0<a<\delta} \left(\int_0^a \left(\frac{\tilde{s}(z)}{z^{k_i}}\right)^p\right)^{\frac{1}{p}} \left(\int_a^\delta \left(\frac{z^{k_i-1}}{\tilde{s}(z)}\right)^q\right)^{\frac{1}{q}} \le c,$$

and the right hand side is valid by the assuption on $s(x)$.

II/2: If $|y-x_i| \le |x-x_i|$, then we have to divide the interval to infinitely many parts:

$$\left(\int_{|x-x_i|\le 2n_\beta(r)} s^p(x)w^p(x) \left|\int_{|y-x_i|<|x-x_i|} f(y)g_i(r,x,y)w^2(y)dy\right|^p dx\right)^{\frac{1}{p}}$$

$$\le \|fsw\|_p \left(\sum_{m=0}^{\infty} \int_{\frac{2n_\beta(r)}{2^{m+1}}<|x-x_i|\le\frac{2n_\beta(r)}{2^m}} \left(\frac{s(x)}{|x-x_i|^{k_i}}\right)^p\right.$$

$$\left.\left(\int_{\substack{|y-x_i|\\ \le|x-x_i|}} \left(\frac{w(x)w(y)|x-x_i|^{k_i}|g_i(r,x,y)|}{|y-x_i|^{k_i}}\right)^q \left(\frac{|y-x_i|^{k_i}}{s(y)}\right)^q dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} = (*)$$

As in Case I, (19) when $\frac{n_\beta(r)}{2^{m+1}} < |x-x_i| \le \frac{n_\beta(r)}{2^m}$ and $|y-x_i| < |x-x_i|$, we can estimate

$$\frac{w(x)w(y)|x-x_i|^{k_i}|g_i(r,x,y)|}{|y-x_i|^{k_i}} \le |x-x_i|^{k_i}w(x)\left|\left.\frac{\partial^{k_i}(g_i(r,x,y)w(y))}{\partial y^{k_i}}\right|_{y=\xi_i}\right|$$

Because of the assumption $|y-x_i| < |x-x_i| \le 2n_\beta(r)$, one term in the expression of this k_i-th derivative on the interval $\left(\frac{n_\beta(r)}{2^{m+1}}, \frac{n_\beta(r)}{2^m}\right)$ can be estimated by

$$cw(x)w(\xi_i)|x-x_i|^{k_i}(1-r)^{-\frac{k+k_i}{2}} \max\left\{P_r(x,\xi_i)|x-\xi_i|^k, P_r(x,x_i)|x-x_i|^k(1-r)\right\}$$

$$\le c\frac{1}{2^{(m-1)(k+k_i)}}\left(\log\frac{1}{1-r}\right)^{\frac{k+k_i}{2}}(1-r)^{-\frac{1}{2}+\frac{r^2\beta}{(1+r)4^m}}.$$

Using that this expression is increasing in k we have that

$$(*) \le \|fsw\|_p \left(\sum_{m=0}^{\infty} \frac{1}{2^{2(m-1)k_ip}}\left(\log\frac{1}{1-r}\right)^{k_ip}(1-r)^{-\frac{p}{2}+\frac{pr^2\beta}{(1+r)4^m}}\right.$$

$$\left.\int_{\frac{n_\beta(r)}{2^{m+1}}<|x-x_i|\le\frac{n_\beta(r)}{2^m}} \left(\frac{s(x)}{|x-x_i|^{k_i}}\right)^p \left(\int_{|y-x_i|\le|x-x_i|}\left(\frac{|y-x_i|^{k_i}}{s(y)}\right)^q dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$$

Because the A_p property is valid for $\left(\frac{s(x)}{|x-x_i|^{k_i}}\right)^p$, we have that

$$\left(\int_{|x-x_i|\le\frac{n_\beta(r)}{2^m}} \left(\frac{s(x)}{|x-x_i|^{k_i}}\right)^p dx\right)^{\frac{1}{p}} \left(\int_{|y-x_i|\le\frac{n_\beta(r)}{2^m}} \left(\frac{|y-x_i|^{k_i}}{s(y)}\right)^q dy\right)^{\frac{1}{q}} \le c\frac{n_\beta(r)}{2^m},$$

and so

$$(*) \le c\|fsw\|_p \left(\sum_{m=0}^{\infty} b_m\right)^{\frac{1}{p}}, \tag{27}$$

where

$$b_m = \frac{1}{4^{(m-1)p\left(k_i+\frac{1}{2}\right)}} \left(\log \frac{1}{1-r}\right)^{p\left(k_i+\frac{1}{2}\right)} (1-r)^{\frac{r^2\beta p}{(1+r)4^m}}$$

Let us denote by

$$c(r) := \frac{\left(k_i+\frac{1}{2}\right)(1+r)}{r^2\beta} \quad \text{and} \quad \alpha := p\left(k_i+\frac{1}{2}\right),$$

and by $M = M(r)$ that index for which

$$4^{M-1} \le \frac{1}{c(r)} \log \frac{1}{1-r} < 4^M.$$

Because the maximum of the function $\left(\log \frac{1}{1-r}\right)^{\gamma}(1-r)^{\delta}$ is at $\frac{\gamma}{\delta} = \log \frac{1}{1-r}$, we have that

$$b_M \le 4^{-M\alpha}\left(\frac{\alpha(1+r)4^{M+1}}{er^2\beta p}\right)^{\alpha} \le c\left(\frac{c(r)}{e}\right)^{\alpha} \le c$$

(If $r > \frac{1}{2}$.)

If $m < M$, then b_m is increasing with r, and so

$$b_m \le \frac{1}{4^{m\alpha}}\left(c(r)4^{M+1}\right)^{\alpha} e^{-\frac{c(r)4^{M+1}r^2\beta p}{(1+r)4^{m+1}}} = (4c(r))^{\alpha}\frac{(4^{\alpha})^{M-m}}{e^{\alpha 4^{M-m}}}.$$

If $m > M$, then b_m is decreasing with r, and so

$$b_m \le \frac{1}{4^{m\alpha}}(c(r))^{\alpha}4^{M\alpha} \le c\left(4^{\alpha}\right)^{M-m}.$$

It means that

$$\sum_{m=0}^{\infty} b_m = \sum_{m=0}^{M-1} b_m + b_M + \sum_{m=M+1}^{\infty} b_m < c \tag{28}$$

That is A_i is bounded by $c\|fs\|_{w,p}$.

Case b: If x_i does not separate x and y, (which means that we are on $(-\infty, x_i] \times (x_i-\delta, x_i] \cup [x_i, \infty) \times [x_i, x_i+\delta))$, say $x_i \le x, y$.
III: $x_i \le x \le y$.
III/1: $|x-x_i| > 2n_\beta(r)$. (In this case $|y-x_i| > 2n_\beta(r)$ as well.)
III/1.1: If $|x-y| > n_\beta(r)$, we are almost in the same situation as in *Case a* I/2; the only difference is that, when we estimate $P_r(x,y)$ (as in (22)) we have to use $n_\beta(r)$ instead of $2n_\beta(r)$, and it yields that A_i is bounded if $\beta > \frac{1+r}{2r^2}$.
III/1.2: If $|x-y| \le n_\beta(r)$: We will deal with this case later.
III/2: $|x-x_i| \le 2n_\beta(r)$. This case is coincides with II/1.
IV: $x_i \le y \le x$.

IV/1: $|x-x_i|>2n_\beta(r)$.
IV/1.1: Either $|y-x_i|\le n_\beta(r)$, or $|y-x_i|>n_\beta(r)$, if $|y-x|>n_\beta(r)$, we get back case I/1, with the same remark as in III/1.1, that is A_i is bounded if $\beta>\frac{1+r}{2r^2}$.
IV/1.2: If $|y-x|\le n_\beta(r)$, the situation is the same as in III/1.2; we will deal with these cases together.
IV/2: $|x-x_i|\le 2n_\beta(r)$. This is the same as II/2.

Now we give an estimation on A_i when $|x-x_i|>2n_\beta(r)$ and $|y-x|\le n_\beta(r)$. In this case we have that $\delta>|y-x_i|>n_\beta(r)$ and $|x-x_i|\le\frac{3}{2}\delta$.

$$\left(\int_{\substack{|x-x_i|\\ \in(2n_\beta(r),\frac{3}{2}\delta)}} w^p(x)\left|\int_{\substack{|x-y|\\ <n_\beta(r)}} f(y)w(y)s(y)P_r(x,y)w(y)\frac{s(x)}{s(y)}h_i(r,x,y)dy\right|^p dx\right)^{\frac{1}{p}}$$

where $h_i(r,x,y)=\frac{g_i(r,x,y)}{P_r(x,y)}$. Applying the Hölder inequality in y with fixed x we can estimate the previous integral by

$$\|fs\|_{w,p}\left(\int_{\mathbf{R}} w^p(x)\left(\int_{\mathbf{R}}(P_r(x,y)w(y))^q dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$$

$$\times\sup_{\substack{|x-x_i|\in(2n_\beta(r),\frac{3}{2}\delta)\\ |x-y|<n_\beta(r)}}\frac{s(x)}{s(y)}|h_i(r,x,y)|$$

In the second assumptions on $s(x)$ let us choose $K=2$. In this domain in case III.1.2 $\frac{1}{2}\le\frac{|x-x_i|}{|y-x_i|}\le 1$, and in case IV.1.2 $1\le\frac{|x-x_i|}{|y-x_i|}\le 2$, that is the function $\frac{s(x)}{s(y)}$ is bounded. We will estimate $|h_i(r,x,y)|$ term by term. Because y is around x_i, $|H_{0,i}(y)|<c(x_i)$.

$$\frac{P_r^{(l)}(x,x_i)}{P_r(x,y)}|H_{l,i}(y)|\le ce^{-\frac{r^2}{1-r^2}((x-x_i)^2-(x-y)^2)}e^{\frac{2r}{1+r}x_i(x-y)}$$

$$\times\sum_{k=0}^{l}a_{k,l}|x-x_i|^k(1-r)^{-\frac{l+k}{2}}|y-x_i|^l|\tilde{p}_l(y)|$$

If $|x-x_i|>(1-r)^{\frac{1-\gamma}{2}}$ for arbitrary $\gamma>0$, then $e^{-\frac{r^2}{1-r^2}((x-x_i)^2-(x-y)^2)}(1-r)^{-\alpha}$ is bounded for every $\alpha>0$, and so $|h_i(r,x,y)|<c(k_i,x_i,\delta)$.

If $|x-x_i|\le(1-r)^{\frac{1-\gamma}{2}}$, then

$$e^{-\frac{r^2}{1-r^2}((x-x_i)^2-(x-y)^2)}e^{\frac{2r}{1+r}x_i(x-y)}|x-x_i|^k(1-r)^{-\frac{l+k}{2}}|y-x_i|^l|\tilde{p}_l(y)|$$

$$\le c(x_i,l)e^{-\frac{r^2}{1-r^2}\beta\log\frac{1}{1-r}}(1-r)^{-\frac{l+k}{2}}\le c(x_i,l)(1-r)^{\frac{l-\gamma}{2}-\frac{l+k}{2}},$$

which is bounded by $c(x_i,k_i)$ if $\gamma<\frac{r^2\beta}{(1+r)2k_i}$. With Lemma 3 it proves the boundedness for $k_i>0$, and so $\mathbf{A_i}$ is bounded for $i=1,...,s$.

B: We can estimate thye following expression term by term:

$$\Bigg(\int_{\mathbf{R}} s^p(x) w^p(x)$$

$$\left(\int_{x_j-\delta}^{x_j+\delta}\left(\left|P_r(x,y)H_{0,i}(y)-\sum_{l=0}^{k_i-1}P_r^{(l)}(x,x_i)H_{l,i}(y)\right|\frac{w(y)}{s(y)}\right)^q dy\right)^{\frac{p}{q}} dx\Bigg)^{\frac{1}{p}}$$

$$\leq \left(\int_{\mathbf{R}} s^p(x) w^p(x)\left(\int_{x_j-\delta}^{x_j+\delta}\left|P_r(x,y)w(y)\tilde{H}_{0,i}(y)\frac{|y-x_j|^{k_j}}{s(y)}\right|^q dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$$

$$+\sum_{l=0}^{k_i-1}\Bigg(\int_{\mathbf{R}}\frac{s^p(x)}{|x-x_i|^{k_i p}} w^p(x)$$

$$\left(\int_{x_j-\delta}^{x_j+\delta}\left|w(y)P_r^{(l)}(x,x_i)|x-x_i|^{k_i}\tilde{H}_{l,i}(y)\frac{|y-x_j|^{k_j}}{s(y)}\right|^q dy\right)^{\frac{p}{q}} dx\Bigg)^{\frac{1}{p}}$$

$$= I+\sum_{l=0}^{k_i-1} I_l$$

Here we denote by $\tilde{H}_{l,i}(y)=\frac{H_{l,i}(y)}{|y-x_j|^{k_j}}$ which is a polynomial by the definition of the interpolatory polynomials. If $k_i=0$ then the above sum is empty, so we have to estimate I_l when $k_i>0$. In the integrals after the sum the expression $|P_r^{(l)}(x,x_i)|x-x_i|^{k_i}|$ $(0\leq l\leq k_i-1)$ is bounded according to Lemma 4 and the remark on the function $q(z)$. So

$$\Bigg(\int_{\mathbf{R}}\frac{s^p(x)}{|x-x_i|^{k_i p}} w^p(x)$$

$$\left(\int_{x_j-\delta}^{x_j+\delta}\left|w(y)P_r^{(l)}(x,x_i)|x-x_i|^{k_i}\tilde{H}_{l,i}(y)\frac{|y-x_j|^{k_j}}{s(y)}\right|^q dy\right)^{\frac{p}{q}} dx\Bigg)^{\frac{1}{p}}$$

$$\leq c(\delta,x_j)\left(\int_{\mathbf{R}}\frac{s(x)^p}{|x-x_i|^{k_i p}} w^p(x)dx\right)^{\frac{1}{p}}\left(\int_{x_j-\delta}^{x_j+\delta}\left(\frac{|y-x_j|^{k_j}}{s(y)}\right)^q dy\right)^{\frac{1}{q}}\leq c(\delta,x_j),$$

where we used the assumption on $s(y)$.

Now we have to deal with the first integral. The situation is almost the same as in case V. in the estimation on $\mathbf{A_i}$, the only change is to replacing i by j, and $\frac{1}{s(y)}$ by $\frac{|x-x_j|^{k_j}}{s(y)}$. So if $|x-y|>n_\beta(r)$, or $|x-y|\leq n_\beta(r)$, and there are K_i-s for which $K_1<\frac{|x-x_j|}{|y-x_j|}<K_2$, then $\frac{s(x)}{s(y)}$ is bounded around x_j, and the computations are the same

If $|x-y|\le n_\beta(r)$, and $\frac{|x-x_j|}{|y-x_j|}\ge 2$, or $\frac{|x-x_j|}{|y-x_j|}\le \frac{1}{2}$, then we have to deal with the $k_j>0$ case. As in V. $y-x_j|, |x-x_j|\le 2n_\beta(r)$.

$$I\le c(\delta,x_j,i)\left(\int_{|x-x_j|<2n_\beta(r)}\frac{s^p(x)}{|x-x_j|^{k_jp}}w^p(x)\right.$$

$$\left.\left(\int_{|y-x_j|<2n_\beta(r)}\left(w(y)P_r(x,y)|x-x_j|^{k_j}\frac{|y-x_j|^{k_j}}{s(y)}\right)^q dy\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}$$

$$\le c(\delta,x_j,i)\frac{1}{\sqrt{1-r}}n_\beta(r)^{k_j}n_\beta(r),$$

where we used the A_p-property on $\frac{s^p(x)}{|x-x_j|^{k_jp}}$. It means, that I is bounded, when $k_j>0$.

Now we finished the estimation on **B**.

C:

$$\left(\int_{\mathbf{R}}s^p(x)w^p(x)\left(\int_{\mathbf{R}\setminus\hat{\cup}_{j=1}^sB(x_j,\delta)}\left|P_s(r,x,y)\frac{w(y)}{s(y)}\right|^q dy\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}$$

$$\le\left(\int_{\mathbf{R}}s^p(x)w^p(x)\left(\int_{\mathbf{R}\setminus\cup_{j=1}^sB(x_j,\delta)}\left(P_r(x,y)\frac{w(y)}{s(y)}\right)^q dy\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}$$

$$+\left(\int_{\mathbf{R}}s^p(x)w^p(x)\left(\int_{\mathbf{R}\setminus\cup_{j=1}^sB(x_j,\delta)}\left(\left|\sum_{i=1}^s\sum_{l=0}^{k_i-1}P_r^{(l)}(x,x_i)H_{l,i}(y)\right|\frac{w(y)}{s(y)}\right)^q dy\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}$$

$$+\sum_{\substack{j\\k_j=0}}\left(\int_{\mathbf{R}}s^p(x)w^p(x)\left(\int_{B(x_j,\delta)}\left(P_r(x,y)\frac{w(y)}{s(y)}\right)^q dy\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}$$

$$+\sum_{\substack{j\\k_j=0}}\left(\int_{\mathbf{R}}s^p(x)w^p(x)\left(\int_{B(x_j,\delta)}\left(\left|\sum_{i=1}^s\sum_{l=0}^{k_i-1}P_r^{(l)}(x,x_i)H_{l,i}(y)\right|\frac{w(y)}{s(y)}\right)^q dy\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}$$

$$=I+II+III+IV$$

If $k_i=0$, then there is no i-th term in the sum in II. If $k_i>0$ then from Lemma 4 we have that for all $0\le l\le k_i-1$

$$s(x)|P_r^{(l)}(x,x_i)|w(x)\le c(x_i)\frac{s(x)}{|x-x_i|^{k_i}}e^{\frac{2r}{1+r}xx_i-\frac{x^2}{2}}\tag{29}$$

and by the assumption on $s(x)$ there is a $y_0 > \max\{|x_1|+1, |x_s|+1\}$ such that if $|y| > y_0$ for every polynomial $p(y)$ we have that

$$\left\| \frac{p(y)w(y)}{s(y)} \right\|_{q,|y|>y_0} \le c(p,s) \tag{30}$$

Applying (29) and (30) we get that

$$II \le c(x_1,\dots,x_s)) \left\| \frac{s(x)}{|x-x_i|^{k_i}} e^{\frac{2r}{1+r}xx_i - \frac{x^2}{2}} \right\|_p$$
$$\times \left(\sum_{i=1}^{s} \sum_{l=0}^{k_i-1} \left(\int_{|y|\le y_0 \setminus \cup_{j=1}^s B(x_j,\delta)} \left| \frac{H_{l,i}(y)w(y)}{s(y)} \right|^q dy \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\int_{|y|>y_0} \left| \frac{H_{l,i}(y)w(y)}{s(y)} \right|^q dy \right)^{\frac{1}{q}} \right) \le c(x_1,\dots,x_s)(c(\delta,y_0)+c(s)) \tag{31}$$

The estimation on IV is almost the same as the estimation on II, the only difference is that we have to replace the sum in (31) by

$$\left(\int_{B(x_j,\delta)} \left| \frac{H_{l,i}(y)w(y)}{s(y)} \right|^q dy \right)^{\frac{1}{q}} \le \|H_{l,i}w\|_\infty \left(\int_{B(x_j,\delta)} \left| \frac{1}{s(y)} \right|^q dy \right)^{\frac{1}{q}} < c(s)$$

To estimate I, let us observe that $\frac{s(x)}{s(y)} < K(\delta, y_0)$ on the set $M = \{|x| < cy_0\} \times \{(|y| \le cy_0) \setminus \cup_{j=1}^s B(x_j,\delta)\}$. Using Lemma 3 :

$$I \le \left(\int_{|x|<4y_0} s^p(x)w^p(x) \left(\int_{(|y|<4y_0)\setminus \cup_{j=1}^s B(x_j,\delta)} \left(P_r(x,y)\frac{w(y)}{s(y)} \right)^q dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$
$$+ \left(\int_{|x|\ge 4y_0} (\cdot) \left(\int_{|y|\ge 4y_0} (\cdot) \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$
$$+ \left(\int_{|x|\ge 4y_0} (\cdot) \left(\int_{(|y|<4y_0)\setminus \cup_{j=1}^s B(x_j,\delta)} (\cdot) \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$
$$+ \left(\int_{|x|<4y_0} (\cdot) \left(\int_{|y|\ge 4y_0} (\cdot) \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \le c(y_0) + V$$

On that part of the remainder domain, where $\frac{s(x)}{s(y)} < K(\delta, y_0)$, we can also apply Lemma 3. According to the 5th assumption on $s(x)$, when $\frac{s(x)}{s(y)} > B$ for some

B, we can suppose that $|x-y|>1$ (say). Observing that when $xy<0$ and $|x-y|>1$ then $P_r(x,y)$ is bounded independently of r, and according to the assumptions on $s(x)$ $ws \in L_p$ and $\frac{w}{s} \in L_q$ that is That part of the integral in V is bounded. We have to investigate the situation, when $\frac{s(x)}{s(y)} > K(\delta, y_0)$, and $xy>0$, say $x,y>0$.

If $y<\frac{x}{4}$, then (the only interesting case is $x>4y_0$) and $P_r(x,y)w(x) < ce^{\frac{2rxy}{1+r}-\frac{x^2}{2}} < ce^{-\frac{x^2}{4}}$, and so the integral in this part can be estimated by

$$c\left(\int_{4y_0}^{\infty} s^p w^{\frac{p}{2}} \left(\int_{y_0}^{\frac{x}{4}} \frac{w^q(y)}{s^q(y)} dy\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} < c$$

If $y>2x$ then $P_r(x,y)w(x)w(y) < ce^{\frac{2rxy}{1+r}-\frac{x^2}{2}-\frac{y^2}{2}} < ce^{-cy^2}$ $(c>0)$, and $\frac{P_r(x,y)w(y)w(x)}{s(y)} < \frac{ce^{-y^2}}{s(y)}$. Thus we can estimate this part on $x>4y_0$ by

$$c\left(\int_{4y_0}^{\infty}\left(\int_{2x}^{\infty} e^{-cqy} dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} < c$$

and on $0<x<4y_0$ we get the same.

If $\frac{x}{4} \le y \le 2x$, then by the assumption on $s(y)$, $e^{-\frac{(x-y)^2}{2}}\frac{s(x)}{s(y)}$ is bounded in this domain, and we can estimate the integral by

$$c\left(\int_0^{\infty}\left(\int_{\substack{y\in\left(\frac{x}{4},2x\right)\\ |y-x|>1}}\left(\frac{1}{\sqrt{1-r}}e^{-\frac{r^2}{2(1-r^2)}}e^{-\frac{r^2}{2(1-r^2)}(x-y)^2}e^{-\frac{1-r}{1+r}xy}\right)^q dy\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}$$

$$\le ce^{-\frac{r^2}{2(1-r^2)}}\left(\int_0^{\infty}\left(\int_{\frac{x}{4}}^{2x} e^{-\frac{1-r}{1+r}qxy} dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$$

$$\le c(1-r)^{-1}e^{-\frac{r^2}{2(1-r^2)}}\left(\int_0^{\infty} e^{-\frac{1-r}{4(1+r)}px^2} dx\right)^{\frac{1}{p}} \le \frac{c}{(1-r)^2}e^{-\frac{r^2}{2(1-r^2)}} < c$$

Finally we have to give an estimation on III, that is we have to estimate

$$\left(\int_{\mathbf{R}} s^p(x)w^p(x)\left(\int_{x_i-\delta}^{x_i+\delta}\left(P_r(x,y)\frac{w(y)}{s(y)}\right)^q dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} = J_i$$

If $\frac{s(x)}{s(y)}$ is bounded around x_i, then by Lemma 3 the result is proved. If $|x-y| > n_\beta(r)$ with $\beta = \frac{1+r}{2r^2}$, then $P_r(x,y)w(x)w(y) \le c_1(\delta,x_i)e^{c_2(\delta,x_i)x-\frac{1}{2}x^2}$, and the integral can be estimated by

$$c_1(\delta,x_i)\left(\int_{\mathbf{R}}\left(s(x)e^{c_2(\delta,x_i)x-\frac{1}{2}x^2}\right)^p dx\right)^{\frac{1}{p}}\left(\int_{\substack{y\in(x_i-\delta,x_i+\delta)\\ |x-y|>n_\beta(r)}}\left(\frac{1}{s(y)}\right)^q dy\right)^{\frac{1}{q}}$$

wich is bounded by the assumption on $s(y)$.

If $|x-y| \leq n_\beta(r)$, and $\frac{|x-x_i|}{|y-x_i|} \geq 2$, or $\frac{|x-x_i|}{|y-x_i|} \leq \frac{1}{2}$, then $|y-x_i|, |x-x_i| \leq c|x-y|$.

$$J_i \leq c(\delta, x_i)$$

$$\times \left(\int_{|x-x_i|<2n_\beta(r)} s^p(x) w^p(x) \left(\int_{\substack{|y-x_i|<2n_\beta(r) \\ |x-y|<n_\beta(r)}} \left(w(y) P_r(x,y) \frac{1}{s(y)} \right)^q dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$

$$\leq c(\delta, x_i) \sum_{m=0}^{\infty} \left(\int_{|x-x_i|<2n_\beta(r)} s^p(x) w^p(x) \right.$$

$$\left. \left(\int_{\substack{|y-x_i|<2n_\beta(r) \\ 2^{-m-1} n_\beta(r) \leq |x-y| < 2^{-m} n_\beta(r)}} \left(w(y) P_r(x,y) \frac{1}{s(y)} \right)^q dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$

$$= c(\delta, x_i) \sum_{m=0}^{\infty} b_m$$

As in earlier let $M = M(r)$ be that index, for which $1 - e^{-4^M} \leq r < 1 - e^{-4^{M+1}}$, and we will cut the sum to three parts: $\sum_{m=0}^{M-1} b_m + b_M + \sum_{m=M+1}^{\infty} b_m$.

When $2^{-m-1} n_\beta(r) \leq |x-y| < 2^{-m} n_\beta(r)$

$$P_r(x,y) \leq c(x_i)(1-r)^{\frac{4^{-m-1}-1}{2}},$$

and so by the A_p-property

$$b_m \leq c(x_i, \beta)(1-r)^{\frac{4^{-m-1}-1}{2}}$$

$$\times \left(\int_{|x-x_i|<\frac{n_\beta(r)}{2^m}} s^p(x) \left(\int_{|y-x_i|<\frac{n_\beta(r)}{2^m}} \left(\frac{1}{s(y)} \right)^q dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$

$$\leq c(x_i, \beta)(1-r)^{\frac{4^{-m-1}-1}{2}} 2^{-m} n_\beta(r) \leq c(x_i, \beta) \frac{1}{2^m} \sqrt{\log \frac{1}{1-r} (1-r)^{4^{-m-1}}}$$

That is in b_M we can estimate this function of r by its maximum, that is if $r = 1 - e^{-4^M}$, we have that

$$b_M \leq c(x_i, \beta) \frac{1}{2^M} 2^{M+1} \tag{32}$$

If $m < M$, then we get an upper estimation on b_m, when we replace r by $1 - e^{-4^M}$ again:

$$b_m \leq c(x_i, \beta) \frac{1}{2^m} 2^M \left(e^{-\frac{1}{8}} \right)^{4^{M-m}} \tag{33}$$

And if $m > M$, then we get an upper estimation on b_m again, when we replace r by $1 - e^{-4^M}$, and so we get formally the same as when $m < M$. Now

$$\sum_{m=0}^{M-1} b_m + b_M + \sum_{m=M+1}^{\infty} b_m$$

$$\leq c(x_i, \beta) \left(\sum_{l=1}^{M} 2^l \left(e^{-\frac{1}{8}} \right)^{-4^l} + 1 + \sum_{l=1}^{\infty} \frac{1}{2^l} \left(e^{-\frac{1}{8}} \right)^{-4^{-l}} \right) \leq c(x_i, \beta) \tag{34}$$

With the above estimations the boundedness of $\mathbf{C}$ is proved, and also the theorem is.

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