Biorthonormal Systems in Freud-type Weighted Spaces with Infinitely Many Zeros - An Interpolation Problem

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Abstract. In a Freud-type weighted (w) space, introducing another weight (v) with infinitely many roots, we give a complete and minimal system with respect to vw, by deleting infinitely many elements from the original orthonormal system with respect to w. The construction of the conjugate system implies an interpolation problem at infinitely many nodes. Besides the existence, we give some convergence properties of the solution.

1. Introduction

The construction of biorthonormal systems arises in several problems in both physics and mathematics. Recently physicists are interested in e.g. non-Hermitian operators ([24], [15]) and quantum Brownian motion [25], etc.; and biorthogonality is also useful for investigation of "delta estimators" in $L_p(\mathbb{R}^d, \mu)$ [27], for wavelet expansions [28], or for numerical integration on infinite intervals [11].

There are some further related problems where the main tool is giving biorthonormal systems in Banach spaces. The initial investigations of e.g. R. P. Boas and H. Pollard, A. A. Talalyan, M. Rosenblum, and B. Muckenhoupt resulted the development of e.g. A_p-weights, the theory of multiplicative completion of sets of functions, and estimations of certain norms of Poisson integrals ([2], [23], [19], [16]). Further results were given e.g. on completion ([18], [8], [7]), solving Dirichlet's problem with respect to boundary functions with singularities ([6], [5]), and constructing A-bases (basis for Abel-summability) in some Banach spaces [4].

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We are interested in constructing complete and minimal systems, that is a system $\{\varphi_n\}$ which is complete in a Banach space B, and it has a conjugate system in the dual space $\{\varphi_n^*\} \subset B^*$ such that $\varphi_n^*(\varphi_m) = \delta_{n,m}$. The reason of this interest is the following theorem of S. Banach [1]:

THEOREM X. *A system $\{\varphi_n\}_{n=n_0}^{\infty}$ is an A-basis in the space L_{vw}^p $(1 < p < \infty)$ (with some weight function vw) if and only if it is a complete and minimal system in L_{vw}^p, and there is a constant $c = c(p)$ such that*

$$\sup_{0 \le r < 1} \left\| \sum_{n=n_0}^{\infty} r^n a_n(f) \varphi_n \right\|_{vw,p} \le c \|f\|_{vw,p}$$

where $a_n(f) = \varphi_n^* f = \int_{\mathbf{R}} f \varphi_n^* v^2 w^2$.

So according to Banach's theorem, if a complete and minimal system is given in a Banach space, then for proving that this system is an A-basis, it is enough to show that the norm of the Poisson integral is bounded by the norm of the function.

In the language of weighted spaces on the real line, the common idea of the abovementioned investigations is the following: there is a complete orthonormal system $\{e_n\}$ with respect to a weight $w > 0$ (sometimes $w \equiv 1$) on a finite or infinite interval I, and v is another weight on I with some zeros. Removing some elements of $\{e_n\}$ (which omission depends on the roots of v), a complete and minimal system can be constructed in a weighted space with respect to vw which means, in other terminology, that the residual system can be multiplicatively transformed into a basis. If the number of roots M of v is finite (with multiplicity), then the biorthonormal system $\{\varphi_n, \varphi_m^*\}$ will be the following:

$$\{\varphi_n\} = \{e_n\} \setminus \{e_{k_1}, \dots, e_{k_M}\},$$

and the elements of the conjugate system will be

$$\varphi_m^* = \frac{\varphi_m - \sum_{i=1}^{M} a_{im} e_{k_i}}{v^2}.$$

Here the denominator has some zeros, so roughly speaking, the numerator has to be zero at the same points with the same multiplicity, which leads to an interpolation problem. E.g., if $\{e_n\} = \{p_n\}$ is the orthogonal polynomial system on an interval with respect to a weight then the linear combination of the first M elements, which is a polynomial of degree $M - 1$, interpolates the residual elements at the zeros of v ([4], [5], [6]). Generally, in the finite case we get a finite linear system of equations, and if it has a unique solution, the biorthonormal system is complete and minimal.

The question is the following: what can we do, if v has infinitely many zeros? Following the same chain of ideas, we have to remove infinitely many elements of the original basis such that at the roots of v the elements of the residual system can be interpolated by an infinite combination of the removed elements. (Naturally, we can not omit the first infinitely many elements of the original system.) We have to solve an infinite interpolation problem, which implies an infinite linear system of equations. That is besides the solvability of the system of equations and the unicity of the solution, the convergence of the solution (in some sense) is also a problem.

We will carry out this type of investigations on the real line, when the "outer" weight will be a Freud weight. The ideal situation would be that for an almost arbitrary system of roots (e.g. when it has no finite accumulation point) of an "inner" weight v, which does not grow too quickly at infinity, one could give a good omission system, but at present we are unable to state any result in this respect.

Supposing some polynomially uniform growth property of the choosing function, we will be able to construct a system of points which will be the zeros of v, and an omission system step by step, with which the residual system will be complete and minimal. Furthermore we can apply a finite section method ([3]) to get the numerical solution of the infinite system of equations.

2. Definitions, notations, result

At first we define Freud weights as generally as used in this paper.

DEFINITION 1 [10]. $w(x) = e^{-Q(x)}$ is a Freud weight, if $Q : \mathbb{R} \longrightarrow \mathbb{R}$ is even, continuous in $\mathbb{R}$, $Q(0) = 0$, Q'' is continuous in $[0, \infty)$, and $Q' > 0$ in $(0, \infty)$. Furthermore, assume that for some $A, B > 1$,

$$A \leq \frac{(d/dx)(xQ'(x))}{Q'(x)} \leq B, \quad x \in (0, \infty) \tag{1}$$

NOTATIONS. (1) For a Freud weight w we will denote by $p_n(w) = p_n$ the n^{th} orthonormal polynomial on the real line, with respect to w^2.

(2) If w is a weight function, then

$$f \in L^p_w \quad \text{iff} \quad fw \in L^p. \tag{2}$$

If $f \in L^p_w$ and $g \in L^q_w$, where $\frac{1}{p} + \frac{1}{q} = 1$ then let

$$\langle f, g \rangle = \int_{\mathbb{R}} fgw^2 \tag{3}$$

After the definition of the external weight we give the form of that part of the weight function which is responsible for the inner roots. The definition below is based on the Lemma 1.1 of J. Szabados [20]

DEFINITION 2. Let $X := \{x_1, x_2, \ldots\} \subset \mathbb{R}$, $0 < |x_1| \leq |x_2| \leq \ldots$ be a system of points on the real line, and let $M := \{m_1, m_2, \ldots\} \subset \mathbb{R}_+$ be a collection of positive numbers. If there exists a nonnegative number $\varrho \geq 0$ such that

$$\sum_{j=1}^{\infty} \frac{m_j}{|x_j|^{\varrho+\varepsilon}} < \infty, \quad \text{but} \quad \sum_{j=1}^{\infty} \frac{m_j}{|x_j|^{\varrho-\varepsilon}} = \infty \quad \text{for all} \quad \varepsilon > 0, \tag{4}$$

then with $\mu, d > 0$ arbitrary, let

$$v(x) = v_{X,M,\mu,d}(x) := e^{d|x|^{\varrho+\mu}} \prod_{j=1}^{\infty} \left|1 - \frac{x}{x_j}\right|^{m_j}. \tag{5}$$

After the definitions of the weights we begin to deal with the description of the functions we need for giving a good choice of points and an omission system.

REMARK. In [10], Lemma 5.1 (b) states that

$$t^A \leq \frac{tQ'(tx)}{Q'(x)} \leq t^B, \quad x \in (0,\infty), \; t \in (1,\infty), \tag{6}$$

and

$$A \leq \frac{xQ'(x)}{Q(x)} \leq B, \quad x \in (0,\infty). \tag{7}$$

Together with the definition this means that on $(0,\infty)$, $Q' > 0$; $Q(cx) \sim Q(x)(c > 0)$; $Q'(x) \sim \frac{Q(x)}{x}$, where $f(x) \sim g(x)$ means that there are positive constants C and D such that $f(x) \leq Cg(x)$ and $g(x) \leq Df(x)$. So this is the inspiration of the following definition:

DEFINITION 3. f grows "polynomially uniformly" if it is three times differentiable, f' is positive and convex on $(0,\infty)$, and there exists $x_0 > 0$ such that on (x_0,∞) the following are valid:

$$f(cx) \sim f(x) \quad (c > 0), \tag{8}$$

$$f'(x) \sim \frac{f(x)}{x}. \tag{9}$$

With this property we can define an admissible function and a system of points. In the following definition we want to summarize the properties required to ensure the solvability the abovementioned interpolation problem. The second inequality will be necessary for proving the existence of the solution, and the first one will be necessary for proving some convergence property of it. Further discussions can be found at the end of this note in the final remarks.

DEFINITION 4. w is a Freud weight, $Q = \log\frac{1}{w}$, and let us suppose for simplicity that $\frac{Q(x)}{x^3}$ is quasimonotone, that is there exists a monotone function: $m(x)$; for which $m(x) \sim \frac{Q(x)}{x^3}$ on (x_0,∞) with some x_0. Furthermore let $\frac{3}{2} < A \leq B$, $\gamma > 0$ a positive number. g is an admissible function with respect to Q and γ, if it grows polynomially uniformly on $(0,\infty)$,

$$\frac{g^{[-1]}(x)}{Q^{[-1]}(x)} = O\left(\frac{1}{x^{2\gamma}}\right), \tag{10}$$

there is an $x_0 > 0$ and $\varepsilon > 0$ such that $\frac{g^{[-1]}(x)}{(Q^{[-1]}(x))^{1-\varepsilon}}$ is decreasing on (x_0,∞); and

$$x^{\delta} \max\left\{\frac{(Q^{[-1]}(g(x)))^{\frac{1}{4}}}{(g(x))^{\frac{1}{6}}}; \frac{1}{(Q^{[-1]}(g(x)))^{\frac{1}{2}}}\right\} \longrightarrow 0 \quad \text{with a} \quad \delta > \frac{5}{4} \tag{11}$$

when $x \longrightarrow \infty$. (Here $g^{[-1]}$ denotes the inverse of the function g.)

REMARK. (1) With the same assumptions on Q and g as in the previous definition, we can formulate the inequalities (10) and (11) in a stronger form:

(*) There exists $\varepsilon > 0$ and $c, \gamma > 0$ such that

$$\frac{g^{[-1]}(Q(x))}{x^{1-\varepsilon}} \leq \frac{cx^{\varepsilon}}{Q^{2\gamma}(x)}, \quad x \geq x_0 > 0,$$

and here the leftside is decreasing.

(**)There is a $\delta > \frac{5}{4}$ such that

$$\lim_{x\to\infty} \frac{x^{\delta+\frac{1}{4}}}{Q^{2\gamma\delta}(x)} \max\left\{x^{-\frac{3}{4}}, Q^{-\frac{1}{6}}(x)\right\} = 0$$

(2) Let $A \geq \frac{9}{2}$. Then instead of $(*), (**)$ we can write that there exist $\varepsilon > 0$, $\gamma > \frac{3}{45}$ and $x_0 \geq 0$ such that for all $x \geq x_0$

$$\frac{g^{[-1]}(Q(x))}{x^{1-\varepsilon}} \leq \frac{cx^{\varepsilon}}{Q^{2\gamma}(x)},$$

and the leftside is decreasing.

(3) Let $B \leq \frac{9}{2}$. Then instead of $(**)$ we can write that there exists $\delta > \frac{5}{4}$ such that

$$\lim_{x\to\infty} \frac{x^{\delta+\frac{1}{4}}}{Q^{2\gamma\delta+\frac{1}{6}}(x)}$$

(4) Let $Q(x) = |x|^{\beta}, g(x) = x^{\alpha}$. Then we can formulate Definition 4 as follows:

If $\frac{3}{2} < \beta \leq \frac{9}{2}$, then let $\frac{15\beta}{2\beta-3} < \alpha$ and $\gamma \leq \frac{\alpha-\beta}{2\alpha\beta}$, if $\frac{9}{2} < \beta$, then let $\frac{5}{2}\beta < \alpha$ and $\gamma \leq \frac{\alpha-\beta}{2\alpha\beta}$.

DEFINITION 5. *M is an admissible system of positive numbers with respect to γ, if $0 < m_j < 1+\gamma$, and $\liminf_{j\to\infty} m_j > 0$.*

After these definitions and notations we can formulate the main theorem:

THEOREM 1. *Let w be a Freud weight on the real line with the properties given in Definition 4, and let $0 < \gamma < \frac{1}{2B}$. Furthermore let g be an admissible function with respect to Q and γ, and M an admissible system of positive numbers with respect to γ. Then there exists a system of points $X \subset \mathbb{R}$ and an "omission system" $\Psi_k = p_{l_k}(w)w$ with*

$$l_k = g(k) + O(k), \tag{12}$$

and

$$d, \mu > 0, \tag{13}$$

such that the system

$$\{\varphi_l\}_{l=1}^{\infty} := \{p_k(w)w\}_{k=0}^{\infty} \setminus \{\Psi_n\}_{n=1}^{\infty} \tag{14}$$

is complete and minimal in $L^p_{v_{X,M,\mu,d}}$*, where* $\inf_{m_j<1} \frac{1}{1-m_j} > p > \max_{\substack{j \\ \gamma - m_j < 0}} \frac{1}{\gamma - m_j + 1}$*, if there are finite many* m_j*-s for which* $\gamma - m_j < 0$*, and for* $\inf_j \frac{1}{1-m_j} > p > 1$*, if* $\gamma - m_j \geq 0$ *for all* j*.*

REMARK. With the assumptions of the theorem we will be able to give a numerical method to compute the conjugate system.

EXAMPLE. Let $Q(x) = |x|^4 \log^2(1 + x^2); g(x) = x^{16} \log(1 + x^2)$. With $\gamma = \frac{1}{12}$, the assumptions of Theorem 1 are valid.

3. Proof

As we have seen in the introduction, at first we have to solve the following infinite systems of linear equations:

$$
(15) \quad \begin{bmatrix} \Psi_1(x_1) & \Psi_2(x_1) & \dots & \Psi_n(x_1) & \dots \\ \Psi_1(x_2) & \Psi_2(x_2) & \dots & \Psi_n(x_2) & \dots \\ \vdots & \vdots & & \vdots & \\ \Psi_1(x_k) & \Psi_2(x_k) & \dots & \Psi_n(x_k) & \dots \\ \vdots & \vdots & & \vdots & \end{bmatrix} \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{km} \\ \vdots \end{bmatrix} = \begin{bmatrix} \varphi_m(x_1) \\ \varphi_m(x_2) \\ \vdots \\ \varphi_m(x_k) \\ \vdots \end{bmatrix}
$$

denoted by $Aa_m = c_m$.

In connection with this infinite linear system of equations, we have to deal with two questions: to get some solution, and to guarantee the convergence of the solution in some sense. Together with the convergence, the existence of the solution yields a biorthonormal system with respect to $\{\varphi_l\}_{l=1}^\infty$, and the uniqueness of the solution ensures the completeness of $\{\varphi_l\}_{l=1}^\infty$.

3.1. Solvability. 3.1.1. *Existence.* For the first problem we have to cite a theorem O. Toeplitz [26], [1].

THEOREM A. *The necessary and sufficient condition of the existence of a solution of an infinite linear system of equations*

$$\sum_{k=1}^{\infty} a_{ki} x_k = y_i, \quad i = 1, 2, \dots,$$

is the following: for all r *natural, and* $h_1, h_2, \dots, h_r$ *real numbers for which* $\sum_{i=1}^r h_i a_{ki} = 0, \quad k = 1, 2, \dots,$ *the equality* $\sum_{i=1}^r h_i y_i = 0$ *holds. In particular if the condition* $\sum_{i=1}^r h_i a_{ki} = 0, \quad k = 1, 2, \dots,$ *implies that* $h_1 = h_2 = \dots h_r = 0$ *the above system of equations has a solution for all* $\{y_i\}$*.*

Now we can define our point- and our omission system. For this construction and subsequently we need the following notion of the Mhaskar Rahmanov Saff number with respect to w, which shows where the sup-norm of a weighted polynomial lives [14].

DEFINITION 6. Let w be a Freud weight on the real line. $a_u = a_u(w)$, the MRS number associated with w is defined as the positive root of the equation

$$u = \frac{2}{\pi}\int_0^1 a_u t Q^{'}(a_u t)(1-t^2)^{-\frac{1}{2}}dt, \quad u > 0.$$

REMARKS. (1) According to the Mhaskar-Saff identity, for all polynomials q_n of degree n the following are valid:

$$\|q_n\|_{w,\infty} = \max_{|x|\le a_n} |q_n(x)w(x)|, \tag{16}$$

and

$$\|q_n\|_{w,\infty} > |q_n(x)w(x)| \quad \text{for all} \quad |x| > a_n. \tag{17}$$

(2) Let $p_n(w)$ be the n^{th} orthonormal polynomial with respect to w^2 again, and $x_0 = x_0(n, w)$ is that point, where $p_n(w)w$ attains its maximum modulus, that is

$$|p_n(w, x_0)w(x_0)| = \|p_n(w)w\|_\infty.$$

Then, according to a lemma of J. Szabados ([21],Lemma 2)

$$a_n\left(1 - \frac{c}{n^{\frac{2}{3}}}\right) \le x_0 \le a_n, \qquad n \in \mathbb{N},$$

with some $c > 1$ independent of n. This means that in our case, when $A > \frac{3}{2}$, x_0 is around a_n.

LEMMA 1. *Let Q and γ be as in Theorem 1, and let g be an admissible function with respect to Q and γ. Now there is a system of points $X \subset \mathbb{R}$, and an omission system $\Psi_k = p_{l_k}(w)w$ with $l_k = g(k) + O(k)$, such that*

$$|\Psi_{i-1}(x_i)| > c\|\Psi_{i-1}\|_\infty \quad i = 1, 2, \ldots \tag{18}$$

with an absolute constant c, and the determinants

$$D_n = \begin{vmatrix} \Psi_1(x_1) & \Psi_2(x_1) & \ldots & \Psi_n(x_1) \\ \Psi_1(x_2) & \Psi_2(x_2) & \ldots & \Psi_n(x_2) \\ \vdots & \vdots & & \vdots \\ \Psi_1(x_n) & \Psi_2(x_n) & \ldots & \Psi_n(x_n) \end{vmatrix} \neq 0, \quad n \in \mathbb{N}. \tag{19}$$

PROOF. Let $\Psi_0 = p_0 w$, and let $x_1 \in \mathbf{R}_+$ be an arbitrary point, say $x_1 = 1$. Further let $n_0 \in \mathbb{N}$ be a fixed number (to be given later), and g^* be a function with the properties of g, and let us denote by

$$g(k) = g^*(n_0 + k).$$

We can choose $\Psi_1 = p_{k_1}w$ such that $k_1 = g(1) + O(1)$ and $\Psi_1(x_1) \neq 0$, and $\Psi_1(x_1) \neq \|\Psi_1\|_\infty$. Now let us suppose that $x_1, \ldots, x_n$ and $\Psi_1, \ldots, \Psi_n$ were already chosen, such that $l_k = g(k) + O(k)$, $|\Psi_{k-1}(x_k)| = \|\Psi_{k-1}\|_\infty$ for $k = 2, \dots, n$ and $D_k \neq 0$ for $k = 1, \dots, n$.

At first we will give x_{n+1} such that $|\Psi_n(x_{n+1})| = \|\Psi_n\|_\infty$. So with this choice we get a not too small element in every row. It follows from [10] Lemma 5.1 that

$$a_n \sim Q^{[-1]}(n), \tag{20}$$

and so by the assumptions on g and by the second remark before the lemma, we get that

$$|x_k| \sim Q^{[-1]}(g(k)), \tag{21}$$

In the following we will show that for every $m > l_n$ among the indices $m, m+1, \dots, m+2n+1$ we can find a "good" one, that is there is a $k \in \{m, m+1, \dots, m+2n+1\}$ such that if we choose $\Psi_{n+1} = p_k w$, then $D_{n+1} \neq 0$. By (8) this means that we can choose $\Psi_{n+1} = p_{l_{n+1}} w$ such that $l_{n+1} = g(n+1) + O(n+1)$.

So let us suppose indirectly that there is an $m > l_n$ for which

$$D_{n+1} = \begin{vmatrix} \Psi_1(x_1) & \Psi_2(x_1) & \dots & \Psi_n(x_1) & p_k w(x_1) \\ \Psi_1(x_2) & \Psi_2(x_2) & \dots & \Psi_n(x_2) & p_k w(x_2) \\ \vdots & \vdots & & \vdots & \vdots \\ \Psi_1(x_n) & \Psi_2(x_n) & \dots & \Psi_n(x_n) & p_k w(x_n) \\ \Psi_1(x_{n+1}) & \Psi_2(x_{n+1}) & \dots & \Psi_n(x_{n+1}) & p_k w(x_{n+1}) \end{vmatrix} = 0 \tag{22}$$

for all $k \in \{m, m+1, \dots, m+2n+1\}$. Let us expand this determinant by the elements of the last column:

$$D_{n+1} = (-1)^{n+1} \sum_{j=1}^{n+1} (p_k w)(x_j)(-1)^j B_j = 0, \tag{23}$$

where B_j is that subdeterminant which comes when the last column and the j^{th} row are omitted ($B_{n+1} = D_n$). Denoting $A_j := (-1)^j \hat{B}_j$, where $\hat{B}_j$ are the determinants B_j divided by the product of $w(x_i)$-s we get that

$$\sum_{j=1}^{n+1} p_k(x_j) A_j = 0, \;\; k \in \{m, m+1, \dots, m+2n+1\}. \tag{24}$$

Let us recall the recurrence formula of the orthonormal polynomials with respect to the even weight w:

$$x p_{n+1} = \varrho_{n+2} p_{n+2} + \varrho_{n+1} p_n, \tag{25}$$

where $\varrho_n \sim a_n$ are constants. By this formula we get from (24) that for any $0 \leq l \leq 2n-1$,

$$\sum_{j=1}^{n+1} x_j p_{m+l+1}(x_j) A_j = \varrho_{m+l+2} \sum_{j=1}^{n+1} p_{m+l+2}(x_j) A_j + \varrho_{m+l+1} \sum_{j=1}^{n+1} p_{m+l}(x_j) A_j = 0. \tag{26}$$

By the same argument we have that

$$c_p \sum_{j=1}^{n+1} x_j^p p_{m+l+p}(x_j) A_j = 0, \quad 0 \le p \le n, \quad 0 \le l \le 2n+1-2p \tag{27}$$

that is

$$0 = \sum_{p=0}^{n} c_p \sum_{j=1}^{n+1} x_j^p p_k(x_j) A_j = \sum_{j=1}^{n+1} p_k(x_j) A_j \sum_{p=0}^{n} c_p x_j^p = \sum_{j=1}^{n+1} q_n(x_j) p_k(x_j) A_j, \tag{28}$$

where $k = m+n, m+n+1$, and q_n is a polynomial of degree n. So let us choose $q_n = q_{n,k}$ like

$$sign\ q_{n,k}(x_j) = sign\ p_k(x_j) A_j \tag{29}$$

(If at a point x_j the expression $p_k(x_j)A_j$ is zero, then we have no assumption on the sign of $q_{n,k}$ at x_j.) With this choice we get that all the terms of the above sum are zero. But we know that $A_{n+1} = (-1)^{n+1}\hat{D}_n \neq 0$ and we can suppose that $q_{n,k}(x_{n+1}) \neq 0$, that is $p_k(x_{n+1})$ must be zero for $k = m+n, m+n+1$. This is impossible, because two consecutive orthogonal polynomials cannot have zero at the same point. So the first lemma is proved.

NOTATION. We can define a modified linear systems of equations, which are equivalent with the original ones:

$$\hat{A} a_m = \hat{c}_m : \tag{30}$$

$$\begin{bmatrix} \frac{\Psi_1(x_1)}{\Psi_0(x_1)} & \frac{\Psi_2(x_1)}{\Psi_0(x_1)} & \cdots & \frac{\Psi_n(x_1)}{\Psi_0(x_1)} & \cdots \\ \frac{\Psi_1(x_2)}{\Psi_1(x_2)} & \frac{\Psi_2(x_2)}{\Psi_1(x_2)} & \cdots & \frac{\Psi_n(x_2)}{\Psi_1(x_2)} & \cdots \\ \vdots & \vdots & & \vdots & \\ \frac{\Psi_1(x_k)}{\Psi_{k-1}(x_k)} & \frac{\Psi_2(x_k)}{\Psi_{k-1}(x_k)} & \cdots & \frac{\Psi_n(x_k)}{\Psi_{k-1}(x_k)} & \cdots \\ \vdots & \vdots & & \vdots & \end{bmatrix} \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{km} \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{\varphi_m(x_1)}{\Psi_0(x_1)} \\ \frac{\varphi_m(x_2)}{\Psi_1(x_2)} \\ \vdots \\ \frac{\varphi_m(x_k)}{\Psi_{k-1}(x_k)} \\ \vdots \end{bmatrix}, \tag{31}$$

Denote the elements of $\hat{A}\hat{A}^T$ by

$$\alpha_{ij} =< {}_i\hat{A}, {}_j\hat{A} >= \sum_{k=1}^{\infty} \frac{\Psi_k(x_i)}{\Psi_{i-1}(x_i)} \frac{\Psi_k(x_j)}{\Psi_{j-1}(x_j)}, \tag{32}$$

where $< \cdot, \cdot >$ denotes the usual inner product, and ${}_i\hat{A}$ is the i^{th} row of $\hat{A}$, and $B^{(n)}$ is the principal minor of $\hat{A}\hat{A}^T$:

$$B^{(n)} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{11} & \cdots & \alpha_{1n} \end{bmatrix}, \tag{33}$$

and let

$$
\hat{c}_m^{(n)} = \begin{bmatrix} \frac{\varphi_m(x_1)}{\Psi_0(x_1)} \\ \frac{\varphi_m(x_2)}{\Psi_1(x_2)} \\ \vdots \\ \frac{\varphi_m(x_n)}{\Psi_{n-1}(x_n)} \end{bmatrix}. \tag{34}
$$

With these notations we are in a position to formulate the theorem of F. Riesz [22], which will be our basic tool for proving some convergence property of the solution.

THEOREM B. *With the notation*

$$
M^*(\frac{\varphi_m(x_1)}{\Psi_0(x_1)}, \frac{\varphi_m(x_2)}{\Psi_1(x_2)}, \ldots) = \lim_{n\to\infty} \left(- \frac{\begin{vmatrix} B^{(n)} & \hat{c}_m^{(n)} \\ \left(\hat{c}_m^{(n)}\right)^T & 0 \end{vmatrix}}{|B^{(n)}|} \right)^{\frac{1}{2}}, \tag{35}
$$

the equation $\hat{A}a_m = \hat{c}_m$ *has a solution for which*

$$
\|a_m\|_2 = \left(\sum_{k=1}^{\infty} a_{km}^2 \right)^{\frac{1}{2}} \leq M. \tag{36}
$$

iff

$$
M^*(\frac{\varphi_m(x_1)}{\Psi_0(x_1)}, \frac{\varphi_m(x_2)}{\Psi_1(x_2)}, \ldots) \leq M. \tag{37}
$$

To obtain an estimation on M^* we need some lemmas. At first we have to introduce a

NOTATION. Let

$$
f(c_0, \delta) := \frac{(6c_0^2 + 2c_0)^2}{4^\delta} \left[1 + \frac{2\sqrt{2}}{\delta - \frac{5}{4}} \exp\left((6c_0^2 + 2c_0)^2 \frac{15}{6} \right) \right], \tag{38}
$$

where $\delta > \frac{5}{4}$ is arbitrary, and let $c_0 = c_0(\delta)$ be such that $f(c_0, \delta) \in (0, 1)$.

Furthermore let $w_i = w_i^{(n)}$ be the i^{th} row of the symmetric matrix $B^{(n)}$.

LEMMA 2. *With the previous notations, if there is a* $\delta > \frac{5}{4}$ *and a* $c < c_0(\delta)$ *such that*

$$
|\alpha_{ij}| < c \max\{i, j\}^{-\delta}, \quad \textit{if} \quad i \neq j \tag{39}
$$

then there exists a $0 < q < 1$ *for which*

$$
\left| B^{(n)} \right| > q \prod_{j=1}^{n} \|w_j\|_2. \tag{40}
$$

PROOF. Suppose $i < j$. At first we will prove that assuming $|\alpha_{ij}| < cj^{-\delta}$, the cosine of the angle of the i^{th} and j^{th} rows is of order $j^{-\delta}$, that is

$$|\cos \beta_{ij}| := \frac{|< w_i, w_j >|}{\|w_i\|_2 \|w_j\|_2} < c_1 j^{-\delta}, \tag{41}$$

where $c_1 = 6c^2 + 2c$. Observe at first that $\|w_j\|_2 \geq \alpha_{jj} = \|_j \hat{A}\|_2^2 \geq 1$. Thus we get that

$$\frac{|< w_i, w_j >|}{\|w_i\|_2 \|w_j\|_2} \leq \sum_{k=1}^{i-1} \left| < {}_i\hat{A}, {}_k\hat{A} >< {}_j\hat{A}, {}_k\hat{A} > \right| + \sum_{k=i+1}^{j-1} |\cdot| + \sum_{k=j+1}^{n} |\cdot|$$

$$+ \left| < {}_i\hat{A}, {}_j\hat{A} > \right| \frac{\|_i\hat{A}\|_2^2 + \|_i\hat{A}\|_2^2}{\|w_i\|_2 \|w_j\|_2}$$

$$\leq c^2 i^{1-\delta} j^{-\delta} + c^2 j^{-\delta} \frac{i^{1-\delta}}{\delta - 1} + c^2 \frac{j^{1-2\delta}}{2\delta - 1} + cj^{-\delta} \frac{\|_i\hat{A}\|_2^2 + \|_i\hat{A}\|_2^2}{\|_i\hat{A}\|_2^2 \|_j\hat{A}\|_2^2} \leq c_1 j^{-\delta}. \tag{42}$$

In the last step we used that $\delta > 1$.

By this inequality we can prove the original one. With the notation

$$B_0^{(n)} = \left[\begin{array}{cccc} \frac{w_1^T}{\|w_1\|_2}, & \frac{w_2^T}{\|w_2\|_2}, & \ldots, & \frac{w_n^T}{\|w_n\|_2} \end{array} \right],$$

we have to show that $\left| B_0^{(n)} \right| \geq q$. Let us estimate

$$\left| \left(B_0^{(n)} \right)^T B_0^{(n)} \right| = \left| \begin{array}{cccc} 1 & \ldots & & \cos \beta_{1n} \\ \vdots & \ddots & \cos \beta_{ij} & \vdots \\ \ldots & & & \ldots \\ \vdots & & \ddots & \vdots \\ \cos \beta_{1n} & & & 1 \end{array} \right|$$

$$= \sum_{k=1}^{n-1} (-1)^{k+n} \cos \beta_{kn} \det B_k^{(n)} + \det((B_0^{(n-1)})^T B_0^{(n-1)}),$$

where

$$B_k^{(n)} = \left[\begin{array}{ccccc} 1 & \cos \beta_{12} & \ldots & & \cos \beta_{1n-1} \\ \vdots & \ddots & & \cos \beta_{ij} & \vdots \\ \cos \beta_{k-11} & & & & \cos \beta_{k-1n-1} \\ \cos \beta_{k+11} & & & & \cos \beta_{k+1n-1} \\ \vdots & & & \ddots & \vdots \\ \cos \beta_{n1} & & & \ldots & \cos \beta_{nn-1} \end{array} \right].$$

By Hadamard's inequality we get that

$$\left|\det B_k^{(n)}\right| \leq \prod_{\substack{i=1\\i\neq k}}^{n-1}\sqrt{1+\sum_{\substack{l=1\\l\neq i}}^{n-1}\cos^2\beta_{il}}\sqrt{\sum_{l=1}^{n-1}\cos^2\beta_{nl}}$$

$$\leq \prod_{\substack{i=1\\i\neq k}}^{n-1}\sqrt{1+(i-1)c_1^2 i^{-2\delta}+\sum_{k=1}^{n-1-i}\frac{1}{(i+k)^{2\delta}}}\sqrt{c_1^2\frac{n-1}{(n)^{2\delta}}}$$

$$\leq c_1\frac{\sqrt{n-1}}{(n)^{\delta}}\prod_{\substack{i=1\\i\neq k}}^{n-1}\sqrt{1+c_1^2 i^{1-2\delta}\left(1+\frac{1}{2\delta-1}\right)} \leq c_1 n^{\frac{1}{2}-\delta}\prod_{\substack{i=1\\i\neq k}}^{n}\sqrt{1+\frac{5}{3}c_1^2 i^{1-2\delta}}$$

$$\leq c_1 n^{\frac{1}{2}-\delta}\exp\left(\frac{5}{6}c_1^2\left(1+\frac{1}{2(\delta-1)}\right)\right) \leq c_1\exp\left(c_1^2\frac{15}{6}\right)n^{\frac{1}{2}-\delta} = c_2 n^{\frac{1}{2}-\delta} \tag{43}$$

By this calculation we obtain

$$\det((B_0^{(n)})^T B_0^{(n)}) \geq \det((B_0^{(n-1)})^T B_0^{(n-1)}) - (n-1)c_1c_2 n^{-\delta}n^{\frac{1}{2}-\delta}$$

$$\geq \det((B_0^{(n-2)})^T B_0^{(n-2)}) - c_1c_2(n-1)^{\frac{3}{2}-2\delta} - c_1c_2 n^{\frac{3}{2}-2\delta}$$

$$\geq \ldots \geq \det((B_0^{(2)})^T B_0^{(2)}) - c_1c_2\sum_{k=3}^{n}k^{\frac{3}{2}-2\delta}$$

$$\geq \det((B_0^{(2)})^T B_0^{(2)}) - c_1c_2\frac{1}{2^{2\delta-\frac{5}{2}}\left(2\delta-\frac{5}{2}\right)} \geq 1-\frac{c_1^2}{4^{\delta}} - c_1c_2\frac{1}{2^{2\delta-\frac{5}{2}}\left(2\delta-\frac{5}{2}\right)} \tag{44}$$

By the last inequality, the assumptions on c_0 implies that there is a $q_1 \in (0,1)$ such that $(\det B_0^{(n)})^2 > q_1$, which proves the lemma.

COROLLARY. *With the assumptions of Lemma 2 and the notations above, the following inequality is valid:*

$$-\frac{\begin{vmatrix} B^{(n)} & \hat{c}_m^{(n)} \\ \left(\hat{c}_m^{(n)}\right)^T & 0 \end{vmatrix}}{|B^{(n)}|} \leq \frac{1}{q}\|\hat{c}_m\|_2 e^{\frac{1}{2}\|\hat{c}_m\|_2^2}. \tag{45}$$

PROOF. Applying Hadamard's inequality again, and recalling that $\|w_i\|_2 \geq 1$, we get that

$$-\frac{\begin{vmatrix} B^{(n)} & \hat{c}_m^{(n)} \\ \left(\hat{c}_m^{(n)}\right)^T & 0 \end{vmatrix}}{|B^{(n)}|} \leq \frac{1}{q}\frac{\prod_{j=1}^{n}\sqrt{\sum_{i=1}^{n}\alpha_{ij}^2+\hat{c}_{m,j}^2}\sqrt{\sum_{i=1}^{n}\hat{c}_{m,i}^2}}{\prod_{j=1}^{n}\sqrt{\sum_{i=1}^{n}\alpha_{ij}^2}}$$

$$\leq \frac{1}{q}\|\hat{c}_m\|_2 \left(\prod_{j=1}^{n}\left(1+\frac{\hat{c}_{m,j}^2}{\|w_j\|_2^2}\right)\right)^{\frac{1}{2}} \leq \frac{1}{q}\|\hat{c}_m\|_2 e^{\frac{1}{2}\sum_{j=1}^{n}\hat{c}_{m,j}^2},$$

and the corollary is proved.

LEMMA 3. *With the previous notations, there is a* $\delta > \frac{5}{4}$ *and a* $c_0 = c_0(\delta)$ *such that* $f(c_0,\delta) \in (0,1)$ *(see (38) for* $f(c_0,\delta)$*), with which*

$$|\alpha_{ij}| \leq c \max\{i,j\}^{-\delta}, \quad \text{if} \quad i \neq j \tag{46}$$

for a $c < c_0$ *.*

REMARKS. (1)[10] Corollary 1.4 : If w is a Freud weight, then

$$\sup_{x\in\mathbb{R}} |p_n(w,x)|w(x)\left|1-\frac{|x|}{a_n}\right|^{\frac{1}{4}} \sim a_n^{-\frac{1}{2}} \tag{47}$$

and

$$\sup_{x\in\mathbb{R}} |p_n(w,x)|w(x) \sim n^{\frac{1}{6}} a_n^{-\frac{1}{2}}. \tag{48}$$

(2) We can easily deduce e.g. from 2.19 of [9] or 2.6 of [13], that if $x = (1+c)a_n$, then there is a $c_1 = c_1(\delta)$ such that

$$|p_n(w,x)w(x)| \leq \|p_n(w)w\|_\infty \begin{cases} e^{-c_1 n c^{\frac{3}{2}}}, & \text{if } 0 < c \leq \delta, \\ e^{-c_1 n \log(1+c)}, & \text{if } \delta < c < \infty, \end{cases}$$

that is

$$|p_n(w,x)w(x)| \leq c\frac{n^{\frac{1}{6}}}{a_n^{\frac{1}{2}}} \begin{cases} e^{-c_1 n \left(\frac{x-a_n}{a_n}\right)^{\frac{3}{2}}}, & \text{if } 0 < c \leq \delta, \\ \left(\frac{a_n}{x}\right)^{c_1 n}, & \text{if } \delta < c < \infty. \end{cases} \tag{49}$$

NOTATION. Let us denote by $I_{MRS}(p_k)$ the support of the equilibrium measure with respect to w^k, that is

$$I_{MRS}(p_k) = I_{MRS}(p_k w) = [-a_k, a_k], \tag{50}$$

PROOF. Let $c > 1$ be an arbitrary constant. We can divide the sum in (32) into some parts:

$$|\alpha_{ij}| \leq \frac{1}{|\Psi_{i-1}(x_i)||\Psi_{j-1}(x_j)|}$$

$$\times \left(\sum_{k=1}^{\frac{1}{c}j} |\Psi_k(x_i)\Psi_k(x_j)| + \sum_{k=\frac{1}{c}j+1}^{j-2} |\cdot| + \sum_{k=j-1}^{cj} |\cdot| + \sum_{k=cj+1}^{\infty} |\cdot|\right)$$

$$= \frac{1}{|\Psi_{i-1}(x_i)||\Psi_{j-1}(x_j)|}(S_1+S_2+S_3+S_4) \tag{51}$$

At first, recalling the special assumption on the denominator, considering (47) we have that

$$|\Psi_{i-1}(x_i)| \sim \|\Psi_{i-1}\|_\infty = \|p_{l_{i-1}}(w)\|_\infty \sim (l_{i-1})^{\frac{1}{6}} a_{l_{i-1}}^{-\frac{1}{2}}$$

$$\sim g(i)^{\frac{1}{6}} \left(Q^{[-1]}(g(i))\right)^{-\frac{1}{2}} \tag{52}$$

By the second remark after Lemma 3, the members in S_1 are exponentially small, because either x_i and x_j, or only x_j, are out of $cI_{MRS}(\Psi_k)$ for such k-s. According to the previous calculation we get that

$$\frac{S_1}{|\Psi_{i-1}(x_i)||\Psi_{j-1}(x_j)|} \leq c(g(i))^{-\frac{1}{6}}(Q^{[-1]}(g(i)))^{\frac{1}{2}}(g(j))^{-\frac{1}{6}}(Q^{[-1]}(g(j)))^{\frac{1}{2}}$$

$$\times \sum_{k=1}^{\frac{1}{c}j} \frac{g(k)^{\frac{1}{3}}}{Q^{[-1]}(g(k))} \left[\frac{Q^{[-1]}(g(k))}{Q^{[-1]}(g(j))}\right]^{c_1 g(k)}$$

$$= F(i,j)\frac{1}{\left[Q^{[-1]}(g(j))\right]^{M+1}} \sum_{k=1}^{\frac{1}{c}j} g(k)^{\frac{1}{3}} \left(Q^{[-1]}(g(k))\right)^{M} \left[\frac{Q^{[-1]}(g(k))}{Q^{[-1]}(g(j))}\right]^{c_1 g(k)-M-1}$$

$$\leq cF(i,j)\frac{1}{\left[Q^{[-1]}(g(j))\right]^{M+1}} = (*),$$

because in the sum the first factor grows polynomially, and the second decreases exponentially (it is less than $\left(\frac{1}{1+c}\right)^{c_1 g(k)-M-1}$), so if n_0 is large enough, the sum is convergent. Now we have to distinguish two cases according as the infinite norm of the weighted orthonormal polynomials tend to infinity with the degree of the polynomial, or it is bounded (see [10] Corollary 1.4, and the assumption in Definition 4). That is in the second case

$$(*) \leq c(g(j))^{-\frac{1}{6}}(Q^{[-1]}(g(j)))^{\frac{1}{2}} \frac{1}{\left[Q^{[-1]}(g(j))\right]^{M+1}}, \tag{53}$$

and in the first case

$$(*) \leq c(g(j))^{-\frac{1}{3}}(Q^{[-1]}g(j)) \frac{1}{\left[Q^{[-1]}(g(j))\right]^{M+1}}. \tag{54}$$

(Here all the c-s are different absolute constants.) So in both cases

$$\frac{S_1}{|\Psi_{i-1}(x_i)||\Psi_{j-1}(x_j)|} \leq cj^{-\delta} \tag{55}$$

for a $\delta > \frac{5}{4}$ with a $c < \frac{c_0}{4}$, if we choose M large enough, that is we can choose an n_0 large enough with which (46) will be valid.

In S_4 we collected that terms whose maximum points are far away from x_i and x_j. Applying [10] (1.20):

$$S_4 \leq c \sum_{k=cj+1}^{\infty} \frac{1}{\sqrt{a_{l_k}}} \frac{1}{(a_{l_k} - |x_i|)^{\frac{1}{4}}(a_{l_k} - |x_j|)^{\frac{1}{4}}} \leq c \sum_{k=cj+1}^{\infty} \frac{1}{a_{l_k}}$$

$$\leq c\int_{cj+1}^{\infty}\frac{1}{Q^{[-1]}(g(x))}dx \leq c\int_{cQ^{[-1]}(g(j))}^{\infty}\frac{\left(g^{[-1]}(Q(y))\right)'}{y}dy = (*) \tag{56}$$

Using the polynomially growing property of g and Q, and then the monotonicity of the lefthand side of (10), we get that

$$(*) \leq c\int_{cQ^{[-1]}(g(j))}^{\infty}\frac{g^{[-1]}(Q(y))}{y^{1-\varepsilon+1+\varepsilon}}dy$$

$$\leq c\frac{n_0+j}{(Q^{[-1]}(g^*(n_0+j)))^{1-\varepsilon}}\int_{cQ^{[-1]}(g^*(n_0+j))}^{\infty}\frac{1}{y^{1+\varepsilon}}dy$$

$$\leq c\frac{n_0+j}{Q^{[-1]}(g^*(n_0+j))} \tag{57}$$

This means that

$$\frac{S_4}{|\Psi_{i-1}(x_i)||\Psi_{j-1}(x_j)|} \leq c\frac{n_0+j}{Q^{[-1]}(g^*(n_0+j))}$$

$$\max\{(g^*(n_0+j))^{-\frac{1}{6}}(Q^{[-1]}(g^*(n_0+j)))^{\frac{1}{2}},(g^*(n_0+j))^{-\frac{1}{3}}(Q^{[-1]}g^*(n_0+j))\} \tag{58}$$

So it is clear that by the assumptions on g and Q, and by (11), that if n_0 is large enough, then

$$\frac{S_4}{|\Psi_{\mu_i}(x_i)||\Psi_{\mu_j}(x_j)|} \leq \frac{1}{4}c_0 j^{-\delta}. \tag{59}$$

It is possible that there are some Ψ_k-s in S_2 such that $x_j \notin I_{MRS}(\Psi_k)$, and in S_3, x_i and x_j are both in $I_{MRS}(\Psi_k)$. But we can handle the two sums similarly: there are $O(j)$ terms both in S_2 and S_3, and at most one term, the $(i-1)^{\text{th}}$ or the $(j-1)^{\text{th}}$, has a factor 1. Furthermore the distance between two consecutive maxima is more than some constant c :

$$a_{l_{k+1}} - a_{l_k} \sim \frac{Q^{[-1]}(g(k))}{k}, \tag{60}$$

where the expression above is a consequence of the following:

$$a_u' \sim \frac{a_u}{u}$$

(see [10] Lemma 5.2), and Definition 4. This implies that if $k \neq j-1$, say, we can estimate by (48)

$$|\Psi_k(x_j)| \leq c a_{l_k}^{-\frac{1}{4}}\left(a_{l_{k+1}} - a_{l_k}\right)^{-\frac{1}{4}} \leq c j^{\frac{1}{4}}(Q^{[-1]}(g(j)))^{-\frac{1}{2}}. \tag{61}$$

Let us assume at first that $\frac{1}{2c}j \leq i-1 \leq j-2$, so

$$\frac{S_2}{|\Psi_{i-1}(x_i)||\Psi_{j-1}(x_j)|} \leq c\frac{|\Psi_{i-1}(x_j)|}{|\Psi_{j-1}(x_j)|} + \sum_{\substack{k=\frac{1}{2c}j \\ k\neq i-1}}^{j-2}\frac{|\Psi_k(x_i)\Psi_k(x_j)|}{|\Psi_{i-1}(x_i)||\Psi_{j-1}(x_j)|}$$

$$\leq c\frac{j^{\frac{1}{4}}}{(g(j))^{\frac{1}{6}}}+cj\frac{j^{\frac{1}{2}}}{(g(j))^{\frac{1}{3}}}=(*), \tag{62}$$

and again by the assumptions on g and Q (11), if n_0 is large enough, then

$$(*)\leq\frac{1}{4}c_0j^{-\delta}. \tag{63}$$

If $i<\frac{1}{2c}j$, then the first term is missing, $a_{l_k}-a_{l_{i-1}}>ca_{l_k}$, and

$$\frac{S_2}{|\Psi_{i-1}(x_i)||\Psi_{j-1}(x_j)|}\leq c\frac{(Q^{[-1]}(g(i)))^{\frac{1}{2}}}{(g(i))^{\frac{1}{6}}}\frac{(Q^{[-1]}(g(j)))^{\frac{1}{2}}}{(g(j))^{\frac{1}{6}}}j\frac{j^{\frac{1}{4}}}{Q^{[-1]}(g(j))}.$$

If the first member is bounded, then

$$\frac{S_2}{|\Psi_{i-1}(x_i)||\Psi_{j-1}(x_j)|}\leq c\frac{j^{\frac{1}{4}}}{(g(j))^{\frac{1}{6}}}\frac{j}{(Q^{[-1]}(g(j)))^{\frac{1}{2}}}$$

If it can be estimated by the second, then

$$\frac{S_2}{|\Psi_{i-1}(x_i)||\Psi_{j-1}(x_j)|}\leq c\frac{j^{\frac{5}{4}}}{(g(j))^{\frac{1}{3}}}$$

so according to (11)

$$\frac{S_2}{|\Psi_{i-1}(x_i)||\Psi_{j-1}(x_j)|}\leq\frac{1}{4}c_0j^{-\delta}, \tag{64}$$

if n_0 is large enough.

We can estimate S_3 in the same way. Here the exceptional term is

$$\frac{|\Psi_{j-1}(x_i)|}{|\Psi_{i-1}(x_i)|}.$$

If $\|\Psi_{i-1}\|_\infty\geq c\|\Psi_{j-1}\|_\infty$ (e.g. if $i>\frac{1}{2c}j$ or if the infinite norm of the weighed orthonormal polynomials tend to zero), then the term above can be estimated by $c\frac{j^{\frac{1}{4}}}{(g(j))^{\frac{1}{6}}}$ as in S_2.

If $i<\frac{1}{2c}j$, and the reciprocal of the infinite norm of the weighed orthonormal polynomials is bounded, then $a_{l_{j-1}}-a_{l_{i-1}}>ca_{l_{j-1}}$, and by (52)

$$\frac{|\Psi_{j-1}(x_i)|}{|\Psi_{i-1}(x_i)|}\leq c\frac{(Q^{[-1]}(g(i)))^{\frac{1}{2}}}{(g(i))^{\frac{1}{6}}}\frac{1}{(a_{l_{j-1}}(a_{l_{j-1}}-a_{l_{i-1}}))^{\frac{1}{4}}}$$

$$\leq c\frac{1}{(Q^{[-1]}(g(j)))^{\frac{1}{2}}}\leq\frac{1}{4}c_0j^{-\delta}, \tag{65}$$

by (11), if n_0 is large enough.

The sum, without the extremal term can be estimated as in S_2, and so the lemma is proved.

In the following lemma we state that the operator $\hat{A}$ acts, and is bounded on l_2, and $\mathrm{Ran}\hat{A} = l_2$.

LEMMA 4. *With the previous notations for all $\hat{c}_m \in l_2$ there exists $a_m \in l_2$ such that $\hat{A}a_m = \hat{c}_m$, and*

$$\|a_m\|_2 \leq c\sqrt{\|\hat{c}_m\|_2} e^{c\|\hat{c}_m\|_2^2}, \tag{66}$$

and if $\hat{A}a_m = \hat{c}_m$ with some $a_m \in l_2$, then

$$\|\hat{c}_m\|_2 \leq c\|a_m\|_2, \tag{67}$$

where the c-s are different absolute constants.

PROOF. Theorem B, Lemma 2, the Corollary after Lemma 2 and Lemma 3 prove (66). For proving (67) let us consider

$$\|\hat{c}_m\|_2 = \left(\sum_{i=1}^{\infty} \hat{c}_{im}^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{\Psi_k(x_i)}{\Psi_{i-1}(x_i)} a_{km}\right)^2\right)^{\frac{1}{2}}. \tag{68}$$

Let us decompose the vector $\hat{c}_m$ into two parts:

$$\hat{c}_m = \hat{c}_m^{(1)} + \hat{c}_m^{(2)},$$

where

$$\hat{c}_{im}^{(1)} = \sum_{\substack{1\leq k<\infty \\ k\neq i-1}} \frac{\Psi_k(x_i)}{\Psi_{i-1}(x_i)} a_{km}, \quad \text{and} \quad \hat{c}_{im}^{(2)} = a_{i-1m}.$$

It is clear that

$$\|\hat{c}_m^{(2)}\|_2 \leq \|a_m\|_2. \tag{69}$$

According to (68)

$$\|\hat{c}_m^{(1)}\|_2 \leq \|a_m\|_2 \left(\sum_{i=1}^{\infty} \sum_{\substack{1\leq k<\infty \\ k\neq i-1}} \left(\frac{\Psi_k(x_i)}{\Psi_{i-1}(x_i)}\right)^2\right)^{\frac{1}{2}}$$

$$\leq c\|a_m\|_2 \left(\sum_{i=1}^{\infty} \frac{(Q^{[-1]}(g(i)))}{(g(i))^{\frac{1}{3}}} \sum_{\substack{1\leq k<\infty \\ k\neq i-1}} \Psi_k^2(x_i)\right)^{\frac{1}{2}}. \tag{70}$$

We can decompose the inner sum into three parts:

$$\sum_{\substack{1\leq k<\infty \\ k\neq i-1}} \Psi_k^2(x_i) = \sum_{1\leq k<\frac{1}{c}i} \Psi_k^2(x_i) + \sum_{\substack{\frac{1}{c}i\leq k<ci \\ k\neq i-1}} (\cdot) + \sum_{ci\leq k<\infty} (\cdot) = S_1 + S_2 + S_3$$

As we have shown in Lemma 3, S_1 is exponentially small. Also as in Lemma 3 (in the estimation of S_4)

$$S_3 \leq c \sum_{ci \leq k < \infty} \frac{1}{a_{l_k}} \leq c \frac{n_0 + i}{Q^{[-1]}(g^*(n_0 + i))}, \tag{71}$$

and as in (61)

$$S_2 \leq ci(Q^{[-1]}(g(i)))^{-\frac{1}{2}}. \tag{72}$$

So according to the previous calculation and (11) we obtain that

$$\|\hat{c}_m^{(1)}\|_2 \leq c\|a_m\|_2 \left(\sum_{i=1}^{\infty} \frac{(Q^{[-1]}(g(i)))}{(g(i))^{\frac{1}{3}}} i(Q^{[-1]}(g(i)))^{-\frac{1}{2}} \right)^{\frac{1}{2}} \leq c\|a_m\|_2, \tag{73}$$

which proves the lemma.

REMARK. It is well-known (see e.g. [12]), that if $T : H_1 \longrightarrow H_2$ is a continuous linear operator between two Hilbert spaces, then TT^* has an inverse, iff $\mathrm{Ran}T = H_2$, and in this situation $T^*(TT^*)^{-1}y$ gives the solution with the minimal norm of the linear equation $Tx = y$. Hence we get the following

COROLLARY. *$\hat{A}x = \hat{c}_m$ has a solution a_m in l_2 with the minimal norm (and it is unique with this property), and*

$$a_m = \hat{A}^T(\hat{A}\hat{A}^T)^{-1}\hat{c}_m. \tag{74}$$

3.1.2 *Unicity.* On the same chain of ideas, by changing the role of $\hat{A}$ and $\hat{A}^T$, we will prove that $\hat{A}^T\hat{A}$ has an inverse on l_2, that is $\mathrm{Ker}\hat{A} = \{0\}$ (see eg [12]). For this we need the following notations and lemma:

NOTATION. Let

$${}_k\hat{A}^T\hat{A}_l = \lambda_{kl} = \sum_{m=1}^{\infty} \frac{\Psi_k(x_m)\Psi_l(x_m)}{\Psi_{m-1}^2(x_m)} \tag{75}$$

be the elements of the matrix $\hat{A}^T\hat{A}$.

REMARK. As in the previous case,

$$\lambda_{ll} = \sum_{m=1}^{\infty} \frac{\Psi_l^2(x_m)}{\Psi_{m-1}^2(x_m)} \geq 1. \tag{76}$$

LEMMA 5. *With the previous notations, there exists $\delta > \frac{5}{4}$ and $c_0 = c_0(\delta)$ such that $f(c_0, \delta) \in (0, 1)$ (see (38) for $f(c_0, \delta)$), for which*

$$|\lambda_{kl}| \leq c \max\{k, l\}^{-\delta}, \quad \text{if} \quad k \neq l \tag{77}$$

with $c < c_0$.

PROOF. Suppose that $k < l$. We have to distinguish two cases: there exists $c > 1$ such that $ck > l$, that is $k \sim l$, or $k << l$. At first we will deal with the second case: with a $c > 1$

$$|\lambda_{kl}| \leq \sum_{m=1}^{ck} \frac{|\Psi_k(x_m)\Psi_l(x_m)|}{\Psi_{m-1}^2(x_m)} + \sum_{m=ck+1}^{\frac{1}{c}l} (\cdot) + \sum_{m=\frac{1}{c}l+1}^{cl} (\cdot) + \sum_{m=cl+1}^{\infty} (\cdot)$$

$$= S_1 + S_2 + S_3 + S_4.$$

In S_2 the first factor of the numerator is exponentially small (see (21), (49)), that is by (20), (47), (52),(11)

$$S_2 \leq c \sum_{m=ck+1}^{\frac{1}{c}l} \frac{\frac{g(k)^{\frac{1}{6}}}{\left(Q^{[-1]}g(k)\right)^{\frac{1}{2}}} \left[\frac{Q^{[-1]}g(k)}{Q^{[-1]}g(m)}\right]^{c_1 g(k)} Q^{[-1]}(g(m))}{g^{\frac{1}{3}}(m)} \frac{1}{a_{\Psi_l}^{\frac{1}{4}}(a_{\Psi_l} - |x_m|)^{\frac{1}{4}}}$$

$$\leq \frac{c}{(Q^{[-1]}(g(l)))^{\frac{1}{2}}} \sum_{m=ck+1}^{\frac{1}{c}l} \frac{\frac{g(k)^{\frac{1}{6}}}{\left(Q^{[-1]}g(k)\right)^{\frac{1}{2}}} \left[\frac{Q^{[-1]}g(k)}{Q^{[-1]}g(m)}\right]^{c_1 g(k)} Q^{[-1]}(g(m))}{g^{\frac{1}{3}}(m)}$$

$$\leq c_2 \frac{(k+n_0)^{c_3}}{(1+c)^{k+n_0}} \left(Q^{[-1]}(g(l))\right)^{-\frac{1}{2}} \leq cl^{-\delta}, \tag{78}$$

where $c < \frac{c_0}{6}$, if n_0 is large enough, and $\delta > \frac{5}{4}$.

$|\Psi_k(x_m)|$ is exponentially small in S_3, and here we can use the $0 < c < \delta$ case, so we can estimate

$$S_3 = \left|\frac{\Psi_k(x_{l+1})}{\Psi_l(x_{l+1})}\right| + \sum_{\substack{\frac{1}{c}l+1 \leq m \leq cl \\ m \neq l+1}} \frac{|\Psi_k(x_m)\Psi_l(x_m)|}{\Psi_{m-1}^2(x_m)} = S_{31} + S_{32}$$

$$S_{31} \leq \frac{g(k)^{\frac{1}{6}}}{\left(Q^{[-1]}(g(k))\right)^{\frac{1}{2}}} \frac{\left(Q^{[-1]}(g(l))\right)^{\frac{1}{2}}}{g(l)^{\frac{1}{6}}} e^{-c_1 g(k)\left(\frac{Q^{[-1]}(g(l))}{Q^{[-1]}(g(k))}-1\right)^{\frac{3}{2}}} \leq cl^{-\delta}$$

where $c < \frac{c_0}{6}$, if n_0 is large enough, and $\delta > \frac{5}{4}$ obviously, because $k < \frac{1}{c}l$. Now

$$S_{32} \frac{g(k)^{\frac{1}{6}}}{\left(Q^{[-1]}(g(k))\right)^{\frac{1}{2}}} \frac{g(l)^{\frac{1}{6}}}{\left(Q^{[-1]}(g(l))\right)^{\frac{1}{2}}}$$

$$\times \sum_{\substack{\frac{1}{c}l+1 \leq m \leq cl \\ m \neq l+1}} \frac{Q^{[-1]}(g(m))}{g(m)^{\frac{1}{3}}} e^{-c_1 g(k)\left(\frac{Q^{[-1]}(g(m))}{Q^{[-1]}(g(k))}-1\right)^{\frac{3}{2}}} = F(k,l) \sum_{\substack{\frac{1}{c}l+1 \leq m \leq cl \\ m \neq l+1}} (\cdot)$$

When the norms of the orthogonal polynomials are bounded then $F(k,l) < c$, when it tends to infinity. Then by Definition 4., $F(k,l)\frac{Q^{[-1]}(g(m))}{g(m)^{\frac{1}{3}}} < c$, that is

$$S_{32} \leq c \begin{cases} \sum_{\substack{\frac{1}{c}l+1\leq m\leq cl \\ m\neq l+1}} \frac{Q^{[-1]}(g(m))}{g(m)^{\frac{1}{3}}} e^{-c_1 g(k)\left(\frac{Q^{[-1]}(g(m))}{Q^{[-1]}(g(k))}-1\right)^{\frac{3}{2}}} & \text{(first case)}, \\ \sum_{\substack{\frac{1}{c}l+1\leq m\leq cl \\ m\neq l+1}} e^{-c_1 g(k)\left(\frac{Q^{[-1]}(g(m))}{Q^{[-1]}(g(k))}-1\right)^{\frac{3}{2}}} & \text{(second case)} \end{cases}$$

$$\leq cl^{-\delta}, \tag{79}$$

as in the previous case.

In S_4, both terms in the numerator are exponentially small, that is

$$S_4 \leq cg(k)^{\frac{1}{6}} \left[\frac{Q^{[-1]}(g(k))}{Q^{[-1]}(g(cl))}\right]^{c_1 g(k)-\frac{1}{2}} g(l)^{\frac{1}{6}} \left[\frac{Q^{[-1]}(g(l))}{Q^{[-1]}(g(cl))}\right]^{c_1 g(l)-\frac{1}{2}} \sum_{m=cl+1}^{\infty} \frac{1}{g(m)^{\frac{1}{3}}},$$

where the sum is convergent by (11). So

$$S_4 \leq cg(l)^{\frac{1}{6}} \left(\frac{1}{1+c}\right)^{g(l)^{\frac{1}{6}}} \leq cl^{-\delta}, \tag{80}$$

where c and δ as in Lemma 3.

In S_1 we have to separate the "maximal" term:

$$S_1 \leq \sum_{\substack{m=1 \\ m\neq k+1}}^{ck} (\cdot) + \left|\frac{\Psi_l(x_{k+1})}{\Psi_k(x_{k+1})}\right| \leq \frac{c}{\left(Q^{[-1]}(g(k))\right)^{\frac{1}{4}} \left(Q^{[-1]}(g(l))\right)^{\frac{1}{4}}}$$

$$\times \sum_{\substack{m=1 \\ m\neq k+1}}^{ck} \frac{Q^{[-1]}(g(m))}{g^{\frac{1}{3}}(m)(a_{\Psi_k} - |x_m|)^{\frac{1}{4}}(a_{\Psi_l} - |x_m|)^{\frac{1}{4}}}$$

$$+ \frac{c}{\left(Q^{[-1]}(g(l))\right)^{\frac{1}{4}}} \frac{1}{(a_{\Psi_l} - |x_{k+1}|)^{\frac{1}{4}}} \frac{\left(Q^{[-1]}(g(k))\right)^{\frac{1}{2}}}{g^{\frac{1}{6}}(k)}$$

Because we deal with the case $k << l$ we can estimate

$$|a_{\Psi_l} - |x_{k+1}|| > ca_{\Psi_l}, \tag{81}$$

and so the second term, S_{12}, can be estimated as

$$S_{12} \leq c\frac{1}{\left(Q^{[-1]}(g(l))\right)^{\frac{1}{2}}} \frac{\left(Q^{[-1]}(g(k))\right)^{\frac{1}{2}}}{g^{\frac{1}{6}}(k)}.$$

As in Lemma 3, according to the behavior of the norm of orthogonal polynomials, we have to distinguish two cases: if the second factor is bounded in k, then by (11)

$$S_{12} \le c\frac{1}{\left(Q^{[-1]}(g(l))\right)^{\frac{1}{2}}} \le cl^{-\delta}, \tag{82}$$

where $c < \frac{c_0}{8}$, if n_0 is large enough, and $\delta > \frac{5}{4}$.

If the second factor is increasing, then also by (11)

$$S_{12} \le c\frac{1}{g^{\frac{1}{6}}(l)} \le cl^{-\delta}, \tag{83}$$

where $c < \frac{c_0}{8}$, if n_0 is large enough, and $\delta > \frac{5}{4}$.

Now we have to deal with the first term of S_1.

$$S_{11} \le \frac{c}{\left(Q^{[-1]}(g(k))\right)^{\frac{1}{4}}\left(Q^{[-1]}(g(l))\right)^{\frac{1}{4}}}$$

$$\times \sum_{\substack{m=1\\ m\ne k+1}}^{ck} \frac{Q^{[-1]}(g(m))}{g^{\frac{1}{3}}(m)} \frac{1}{(a_{\Psi_k}-|x_m|)^{\frac{1}{4}}(a_{\Psi_l}-|x_m|)^{\frac{1}{4}}}.$$

As in (78),

$$S_{11} \le \frac{c}{\left(Q^{[-1]}(g(l))\right)^{\frac{1}{2}}} \sum_{\substack{m=1\\ m\ne k+1}}^{ck} \frac{\left(Q^{[-1]}(g(m))\right)^{\frac{1}{2}}}{g^{\frac{1}{3}}(m)} \frac{(a_{\Psi_m})^{\frac{1}{4}}}{(a_{\Psi_k})^{\frac{1}{4}}} \frac{|x_m|^{\frac{1}{4}}}{(|x_{k+1}|-|x_m|)^{\frac{1}{4}}}$$

$$\le \frac{c}{\left(Q^{[-1]}(g(l))\right)^{\frac{1}{2}}} \sum_{\substack{m=1\\ m\ne k+1}}^{ck} m^{-\frac{5}{2}}m^{\frac{1}{4}} \le \frac{c}{\left(Q^{[-1]}(g(l))\right)^{\frac{1}{2}}} \le cl^{-\delta}, \tag{84}$$

where $c < \frac{c_0}{8}$, if n_0 is large enough, and $\delta > \frac{5}{4}$. Here we used (11), and the polynomially growing property of g and Q.

If $k \sim l$, then

$$S_1+S_2+S_3 \le \sum_{\substack{m=1\\ m\ne k+1,l+1}}^{ck} (\cdot) + \left(\left|\frac{\Psi_l(x_{k+1})}{\Psi_k(x_{k+1})}\right| + \left|\frac{\Psi_k(x_{l+1})}{\Psi_l(x_{l+1})}\right|\right) = S_{11}+S_{12},$$

and as in (60), we can estimate by

$$|a_{\Psi_l} - |x_{k+1}|| > \frac{Q^{[-1]}(g(l))}{l}, \tag{85}$$

and so by (11)

$$S_{12} \le c\frac{l^{\frac{1}{4}}}{g^{\frac{1}{6}}(l)} \le cl^{-\delta}, \tag{86}$$

where $c < \frac{c_0}{6}, \delta > \frac{5}{4}$ if n_0 is large enough.

As in the previous calculation

$$S_{11} \leq \frac{c}{\left(Q^{[-1]}(g(l))\right)^{\frac{1}{2}}} \sum_{\substack{m=1 \\ m\neq k+1,l+1}}^{ck} \frac{\left(Q^{[-1]}(g(m))\right)^{\frac{1}{2}}}{g^{\frac{1}{3}}(m)} \frac{|x_m|^{\frac{1}{4}}}{(|x_{k+1}|-|x_m|)^{\frac{1}{4}}} \frac{|x_m|^{\frac{1}{4}}}{(|x_{l+1}|-|x_m|)^{\frac{1}{4}}}$$

$$\leq \frac{c}{\left(Q^{[-1]}(g(l))\right)^{\frac{1}{2}}} \sum_{\substack{m=1 \\ m\neq k+1,l+1}}^{ck} m^{-\frac{5}{2}} m^{\frac{1}{4}} m^{\frac{1}{4}} \leq cl^{-\delta}, \tag{87}$$

where $c < \frac{c_0}{6}$, if n_0 is large enough, and $\delta > \frac{5}{4}$.

When $k \sim l$, the estimations on S_4 are the same as in the previous case, and so the lemma is proved.

Finally applying Lemma 2 with its Corollary to the operator $\hat{A}^T\hat{A}$, and Lemma 4 to $\hat{A}^T$, we get that $\mathrm{Ran}\hat{A}^T = l_2$, and so $\mathrm{Ker}\hat{A}=\{0\}$, which proves the unicity of the solution.

3.2. Finite section method.

As a consequence of invertibility we can apply the so-called "finite section method", which is a very natural (numerical) way to get the solution of the infinite equation $\hat{A}x = \hat{c}$. The process is the following: considering the system $Ax = b$, where A is invertible, but not necessarily Hermitian, we set

$$A_{rn} = P_r A P_n, \quad \text{and} \quad b_{rn} = A^*_{rn} b, \tag{88}$$

where P_r and P_n are projections. That is we can take A_{rn} as it consists of the intersection of the first r rows and the first n columns of A, and b_{rn} as the image of the cut vector b_r. Now we have to try to solve the equation

$$A^*_{rn} A_{rn} x_{rn} = b_{rn}. \tag{89}$$

The convergence of this method is proved by K. Gröchenig, Z. Rzesztonik and T. Strohmer [3]. To state the abovementioned convergence theorem we need some notations. At first we have to note that the original paper works with the index class $\mathbf{Z^d}$, but without any modification we can apply the definitions and results to the index set $\mathbf{N}$.

DEFINITION 7. We say that a matrix A belongs to the Jaffard class $\mathcal{A}_s$, if its elements a_{kl}, $k, l \in \mathbf{N}$ fulfil the following inequality:

$$|a_{k,l}| \leq C(1 + |k-l|)^{-s} \quad \forall k, l \in \mathbf{N}, \tag{90}$$

where C is an absolute constant. The norm in the Jaffard class is $\|A\|_{\mathcal{A}_s} = \sup_{k,l\in\mathbf{N}} |a_{k,l}|(1 + |k-l|)^s$.

NOTATION. Let us denote by $\sigma(A^*A)$ the spectrum of A^*A, and by $\lambda_- = \min \sigma(A^*A)$.

So the (simplified version of the) theorem is the following ([3], Theorem 16):

THEOREM C. *Let $A \in \mathcal{A}_s$ with an $s > 1$, and let $Ax = b$ be given, where $b \in l^2$, and A is invertible on l^2. Consider the finite sections*

$$A_{rn}^* A_{rn} x_{rn} = b_{rn}.$$

Then, for every n there exists an $R(n)$ (depending on λ_- and s) such that $x_{r(n)n}$ converges to x in the norm of l^2, for every choice of $r(n) \geq R(n)$.

Because $\hat{c}$ is in l^2, and $\hat{A}$ is invertible on l^2, for the convergence of the finite section method we have to prove that $\hat{A} \in \mathcal{A}_s$ with an $s > 1$.

LEMMA 6. *Let $\hat{A}$ be as in (30), and let us suppose the assumption of Definition 4. Then there exists an $s > 1$ such that $\hat{A} \in \mathcal{A}_s$.*

PROOF. Because in our matrix $\hat{A}$ the dominant elements are under the principal diagonal, ($|a_{k,k-1}| = 1$), we have to shift the indices, that is we have to prove that there is an absolute constant C, and there is an $s > 1$ such that

$$|a_{k,l-1}| = \left|\frac{\Psi_{l-1}(x_k)}{\Psi_{k-1}(x_k)}\right| \leq C(1 + |k-l|)^{-s}. \tag{91}$$

Using (49) and (52), we get that

$$|a_{k,l-1}| \leq c \frac{\left(Q^{[-1]}(g(k))\right)^{\frac{1}{2}}}{(g(k))^{\frac{1}{6}}} \frac{1}{\left(Q^{[-1]}(g(l))\right)^{\frac{1}{4}}} \frac{1}{|a_l - |x_k||^{\frac{1}{4}}},$$

where c is an absolute constant.

Considering that $x_k \sim a_{\Psi_k}$, we have to distinguish some cases:

a) If $l \sim k$ but $|a_{\Psi_l} - |x_k||$ is not too small, or if $k << l$, then with some $c_1 \neq c_2$

$$D_{lk} = |a_l - |x_k||^{\frac{1}{4}} \geq c \left|c_1 Q^{[-1]}(g(l) - c_2 Q^{[-1]}(g(k)\right|^{\frac{1}{4}} \geq c \left(Q^{[-1]}(g(l)\right)^{\frac{1}{4}}. \tag{92}$$

b) If $l << k$ then

$$D_{lk} \geq c \left(Q^{[-1]}(g(k)\right)^{\frac{1}{4}}. \tag{93}$$

c) If $l \sim k$ and x_k is close to a_l, recalling the estimation on the distance of two consecutive maximum points of Ψ_k-s (see (61)),

$$D_{lk} \geq c \left(\frac{Q^{[-1]}(g(l))}{l}\right)^{\frac{1}{4}}, \tag{94}$$

where we used the polynomially growing property of Q and g.

So in case a), when $l \sim k$ we get that

$$|a_{k,l-1}| \leq c \frac{1}{(g(l))^{\frac{1}{6}}} \leq C(1 + |k-l|)^{-s}, \tag{95}$$

for all $s > 1$. Also in case a), when $k << l$

$$|a_{k,l-1}| \leq c \left(\frac{Q^{[-1]}(g(k))}{Q^{[-1]}(g(l))}\right)^{\frac{1}{2}} \frac{1}{(g(k))^{\frac{1}{6}}}. \tag{96}$$

Here, as in Lemma 3 ((53),(54)), we have to distinguish two cases: in the first case, according to (11)

$$|a_{k,l-1}| \le c\frac{1}{\left(Q^{[-1]}(g(l)\right)^{\frac{1}{2}}} \le cl^{-\frac{5}{4}} \le C(1+|k-l|)^{-s}, \tag{97}$$

with $s = \frac{5}{4}$. In the second case, according to (11) again

$$|a_{k,l-1}| \le c\frac{1}{(g(l))^{\frac{1}{6}}} \le C(1+|k-l|)^{-s}, \tag{98}$$

and $s > \frac{5}{4}$.

In case b), also by (11)

$$|a_{k,l-1}| \le c\left(\frac{Q^{[-1]}(g(k))}{Q^{[-1]}(g(l))}\right)^{\frac{1}{4}}\frac{1}{(g(k))^{\frac{1}{6}}} \le c\frac{\left(Q^{[-1]}(g(k))\right)^{\frac{1}{4}}}{(g(k))^{\frac{1}{6}}} \le C(1+|k-l|)^{-s} \tag{99}$$

with $s = \frac{5}{4}$.

In case c), by (10),(11)

$$|a_{k,l-1}| \le c\frac{l^{\frac{1}{4}}}{(g(l))^{\frac{1}{6}}} \le c\frac{\left(Q^{[-1]}(g(l)\right)^{\frac{1}{4}}}{(g(l))^{\frac{1}{6}}} \le C(1+|k-l|)^{-s}, \tag{100}$$

for all $s > 1$, which proves the lemma.

3.3. Convergence

As it turned out in the introduction, the required form of the elements of the dual space is the following:

$$\varphi_m^* = \frac{\varphi_m - \sum_{k=1}^{\infty} a_{km}\Psi_k}{v^2},$$

which implies that we have to deal with the convergence of the series in the numerator, and we have to give some estimations on the order of the zeros of the numerator.

LEMMA 7. *Let* $\{a_{km}\}_{k=1}^{\infty}$ *be an* l_2*-solution of* (30)*, then*

$$\left|\sum_{k=1}^{n} a_{km}\Psi_k(x)\right| \le cK(x) \quad n \in \mathbb{N}, \tag{101}$$

where

$$K(x) = \begin{cases} 1, \textit{if} \quad |x| \le 1, \\ \frac{Q^{\frac{1}{6}}(x)}{\sqrt{|x|}}, \textit{if} \quad |x| > 1, \end{cases}$$

and the sum $\sum_{k=1}^{\infty} a_{km}\Psi_k(x)$ *is convergent in every* $x \in \mathbb{R}$.

PROOF. At first we will show that the partial sum $\sum_{k=1}^{n} a_{km}\Psi_k(x)$ can be estimated by a function which grows at most polynomially on $\mathbb{R}$. Let us suppose now that $|x|>1$. Using Cauchy Schwarz's inequality, we have to estimate

$$s_n(x) = \sum_{k=1}^{n} \Psi_k^2(x) = \Psi_{j(x)}^2(x) + \sum_{k=1}^{\frac{1}{c}j(x)} \Psi_k^2(x) + \sum_{\substack{\frac{1}{c}j(x)<k<cj(x)\\ k\neq j(x)}} (\cdot) + \sum_{k=cj(x)}^{n} (\cdot)$$

$$= \Psi_{j(x)}^2(x) + S_1 + S_2 + S_3, \tag{102}$$

where $c>1$, and $j(x)$ means that index for which the maximum point of $|\Psi_{j(x)}(x)|$ is the closest to x. (Because the Ψ_k^2-s are even, we can work on the positive part of the real line.) Hence, because in our case the n^{th} orthonormal polynomial p_n attains its maximum around a_n, according to (20, 21) and (48) we obtain that

$$j(x) \sim g^{[-1]}(Q(x)),$$

and so

$$\Psi_{j(x)}^2(x) \leq \|p_{g(j(x))}\|_\infty^2 \sim (g(j(x)))^{\frac{1}{3}}\, a_{g(j(x))}^{-1} \sim \frac{Q^{\frac{1}{3}}(x)}{x} \tag{103}$$

By (48)

$$S_2 \leq c \sum_{\substack{\frac{1}{c}j(x)<k<cj(x)\\ k\neq j(x)}} \frac{1}{\sqrt{a_{\Psi_k}}\sqrt{|a_{\Psi_k}-|x||}}. \tag{104}$$

Taking into consideration the properties of g and Q, we can estimate the difference under the square root as

$$|a_{\Psi_k}-|x|| \geq c|a_{\Psi_k}-a_{\Psi_{j(x)}}| \geq c\left(Q^{[-1]}(g(\cdot))\right)'(j(x))|k-j(x)|$$

$$\geq c\frac{Q^{[-1]}(g(j(x)))}{j(x)}|k-j(x)|, \tag{105}$$

so

$$S_2 \leq c\frac{\sqrt{j(x)}}{Q^{[-1]}(g(j(x)))} \sum_{\substack{\frac{1}{c}j(x)<k<cj(x)\\ k\neq j(x)}} \frac{1}{\sqrt{|k-j(x)|}}$$

$$\leq c\frac{j(x)}{Q^{[-1]}(g(j(x)))} \leq c\frac{g^{[-1]}(Q(x))}{x}. \tag{106}$$

As in Lemma 3 we collected the exponentially small terms in S_1, according to (49),

$$S_1 \leq c\frac{1}{\sqrt{x}} \sum_{k=1}^{\frac{1}{c}j(x)} g(k)^{\frac{1}{6}} \left[\frac{Q^{[-1]}(g(k))}{x}\right]^{c_1 g(k)-\frac{1}{2}}$$

Estimating the sum by an integral, and changing variables, by the properties of g, we obtain that

$$S_1 \le c\frac{1}{\sqrt{x}}\int_1^{\frac{1}{c}j(x)} g(y)^{\frac{1}{6}} \left[\frac{Q^{[-1]}(g(y))}{x}\right]^{g(y)^{\frac{1}{6}}} dy$$

$$\le c\frac{1}{\sqrt{x}}\int_{g(1)^{\frac{1}{6}}}^{cg(j(x))^{\frac{1}{6}}} \frac{z}{(1+c)^z}\left(g^{[-1]}(z^6)\right)' dz \le \frac{K(n_0)}{\sqrt{x}}\int_{g(1)^{\frac{1}{6}}}^{cg(j(x))^{\frac{1}{6}}} \left(g^{[-1]}(z^6)\right)' dz$$

$$\le \frac{K(n_0)}{\sqrt{x}} g^{[-1]}(Q(x)) \le \frac{K(n_0)}{\sqrt{x}} Q(x) \tag{107}$$

In S_3, $x \sim a_{\Psi_{j(x)}}$ is far away from a_{Ψ_k}, so $|a_{\Psi_k} - a_{\Psi_{j(x)}}| \ge c a_{\Psi_k}$, that is

$$S_3 \le c\sum_{k=cj(x)}^{n} \frac{1}{a_{\Psi_k}} \le c\sum_{k=cj(x)}^{n} \frac{1}{Q^{[-1]}(g(k))}. \tag{108}$$

We can estimate this sum by

$$\int_{cj(x)}^{\infty} \frac{1}{Q^{[-1]}(g(y))} dy \le \int_{cQ^{[-1]}(g(j(x))}^{\infty} \frac{1}{z}\frac{g^{[-1]}(Q(z))}{z} dz, \tag{109}$$

where we used the properties of Q and g again. By (10) we have that

$$S_3 \le \sup_{z \ge cQ^{[-1]}(g(j(x))} \frac{g^{[-1]}(Q(z))}{z^{1-\varepsilon}} \int_{cQ^{[-1]}(g(j(x))}^{\infty} \frac{1}{z^{1+\varepsilon}} dz \le c\frac{j(x)}{Q^{[-1]}(g(j(x))}$$

$$\le c\frac{g^{[-1]}(Q(x))}{x}. \tag{110}$$

If $|x| \le 1$, then there is a $k_0 > 1$ such that

$$\Psi^2_{j(x)}(x) + S_1 + S_2 + S_3 \le c\sum_1^{\infty} \frac{1}{a_{\Psi_k}},$$

which can be handled as earlier, that is

$$S_3 \le c\int_1^{\infty} \frac{1}{Q^{[-1]}(g(y))} \le \frac{g^{*[-1]}(Q(n_0))}{n_0} < K,$$

where K is a constant.

Collecting our estimations, if $g^{[-1]}(Q(x))$ is less than $Q^{\frac{1}{3}}(x)$ (see (11)), we obtain that

$$\sum_{k=1}^{n} |a_{km}\Psi_k(x)| \le c\|\{a_{km}\}\|_2 K(x), \tag{111}$$

which gives uniform convergence if $B < 3$, and the second statement of the lemma otherwise.

REMARK. Because the estimation was independent of n, the series in the numerator tends to a function $f(x)$ locally uniformly on $\mathbb{R}$.

Let φ_l be an element of the system (16). Considering that φ_l is a weighted polynomial with an exponential weight, we can immediately get the following

COROLLARY. it There exists a function $g \in L^1(\mathbf{R})$ such that

$$\left|\sum_{k=1}^{n} a_{km}\Psi_k(x)\varphi_l(x)\right| \leq g(x) \quad n \in \mathbb{N}. \tag{112}$$

To state the following lemma we need some notations. Let $S_j := (x_j - \delta(x_j), x_j + \delta(x_j)$ be a ball around x_j such that $x_i \notin S_j$ if $i \neq j$. And let

$$\sigma_n(x) = \sigma_{n,m}(x) = \sum_{k=0}^{n}\left(1 - \frac{l_k}{n+1}\right) a_{km}\Psi_k(x)$$

be the n^{th} Cesàro mean of the Fourier series with respect to $\{p_n(w)\}_{n=0}^{\infty}$ of $S = \sum_{k=1}^{\infty} a_{km}\Psi_k(x)$, where $\{a_{km}\} \in l_2$ is the solution of (30). With these notations we have

LEMMA 8. *Supposing* (10)

$$|\sigma_n(x) - \sigma_\nu(x)| = O\left(\frac{1}{n^\gamma}\right) \quad \textit{if} \quad x \in S_j, \quad n(j) < n < \nu, \quad j = 1, 2 \ldots \tag{113}$$

PROOF. Let x be in S_j. Then

$$|\sigma_n(x) - \sigma_\nu(x)| \leq \left|\sum_{0 \leq l_k \leq n}\left(\frac{1}{n+1} - \frac{1}{\nu+1}\right) l_k a_{km}\Psi_k(x)\right|$$

$$+\left|\sum_{n < l_k \leq \nu}\left(1 - \frac{l_k}{\nu+1}\right) a_{km}\Psi_k(x)\right| =: (*). \tag{114}$$

Let $k(x)$ be that index for which the maximum point of $\Psi_{k(x)}$ is the nearest to x. If x is around $a_{l_{k(x)}}$, then $x \sim Q^{[-1]}(l_{k(x)})$, that is $Q(x) \sim l_{k(x)}$. So if $n > N = N(j)$ (e.g. $cQ(x_j) < n$), then $cl_{k(x)} < n$. Assume now that n is large enough. Then

$$(*) \leq \left|\sum_{\substack{0 \leq l_k \leq n \\ l_k \neq l_{k(x)}}}\left(\frac{1}{n+1} - \frac{1}{\nu+1}\right) l_k a_{km}\Psi_k(x)\right|$$

$$+\left|\left(\frac{1}{n+1} - \frac{1}{\nu+1}\right) l_{k(x)} a_{k(x)m}\Psi_{k(x)}(x)\right|$$

$$+\left|\sum_{n<l_k\le\nu}\left(1-\frac{l_k}{\nu+1}\right)a_{km}\Psi_k(x)\right|=S_1+M+S_2 \tag{115}$$

Let us recall (52). At first we will deal with M:

$$M\le\frac{|a_{k(x)m}|}{n}l_{k(x)}\left(l_{k(x)}\right)^{\frac{1}{6}}\left(a_{l_{k(x)}}\right)^{-\frac{1}{2}}\le c\frac{Q^{\frac{7}{6}}(x_j)}{\sqrt{x_j}}\frac{1}{n}=O\left(\frac{1}{n}\right). \tag{116}$$

We can handle S_2 as S_3 in Lemma 3, that is

$$S_2\le\|\{a_{km}\}_{k=n}^{\infty}\|_2\left(\sum_{n<l_k\le\nu}\left(1-\frac{l_k}{\nu+1}\right)^2\Psi_k^2(x)\right)^{\frac{1}{2}}$$

$$\le c\|\{a_{km}\}_{k=n}^{\infty}\|_2\left(\sum_{n<l_k\le\nu}\frac{1}{a_{\Psi_k}}\right)^{\frac{1}{2}}$$

$$\le c\|\{a_{km}\}_{k=n}^{\infty}\|_2\left(\int_{cg^{[-1]}(n)}^{\infty}\frac{1}{Q^{[-1]}(g(y))}dy\right)^{\frac{1}{2}}=o\left(\sqrt{\frac{g^{[-1]}(n)}{Q^{[-1]}(n)}}\right) \tag{117}$$

We have to decompose S_1 into three parts. Using the Cauchy-Schwarz inequality again we obtain that

$$S_1\le\frac{c}{n}\left(\sum_{0<l_k\le\frac{1}{c}l_{k(x)}}l_k^2\Psi_k^2\right)^{\frac{1}{2}}+\frac{c}{n}\left(\sum_{\frac{1}{c}l_{k(x)}<l_k\le cl_{k(x)}}l_k^2\Psi_k^2\right)^{\frac{1}{2}}$$

$$+\frac{c}{n}\left(\sum_{cl_{k(x)}<l_k\le n}l_k^2\Psi_k^2\right)^{\frac{1}{2}}=S_{11}+S_{12}+S_{13}. \tag{118}$$

Henceforward S_{11} is the collection of the exponentially small terms, that is

$$S_{11}\le\frac{c}{n}\left(\sum_{0<l_k\le\frac{1}{c}l_{k(x)}}\frac{g(k)^{2+\frac{1}{3}}}{Q^{[-1]}(g(k))}\left[\frac{Q^{[-1]}(g(k))}{x}\right]^{c_1g(k)}\right)^{\frac{1}{2}}$$

$$\le\frac{c}{n}\left(\sum_{k=1}^{\infty}\frac{g(k)^{2+\frac{1}{3}}}{Q^{[-1]}(g(k))}\left[\frac{1}{1+c}\right]^{c_1g(k)}\right)^{\frac{1}{2}}=O\left(\frac{1}{n}\right) \tag{119}$$

Applying also the same chain of ideas as in Lemma 3, we obtain that

$$S_{12}\le\frac{c}{n}l_{k(x)}\left(\sum_{\frac{1}{c}k(x)<k\le ck(x)}\Psi_k^2\right)^{\frac{1}{2}}$$

$$\leq \frac{cl_{k(x)}}{n}\sqrt{\frac{g^{[-1]}(Q(x))}{x}} \leq cQ(x_j)\sqrt{\frac{g^{[-1]}(Q(x_j))}{x_j}}\frac{1}{n} = O\left(\frac{1}{n}\right) \tag{120}$$

Similarly

$$S_{13} \leq \frac{c}{n}\left(\sum_{k=ck(x)}^{n} l_k^2 \Psi_k^2\right)^{\frac{1}{2}} \leq \frac{c}{n}\left(\sum_{k=ck(x)}^{n} \frac{l_k^2}{a_{\Psi_k}}\right)^{\frac{1}{2}}$$

$$\sum_{k=ck(x)}^{n} \frac{l_k^2}{a_{\Psi_k}} \leq c\int_{cg^{[-1]}(Q(x))}^{g^{[-1]}(n)} \frac{g^2(y)}{Q^{[-1]}(g(y))}dy$$

$$\leq c\int_{cx}^{Q^{[-1]}(n)} \frac{g^{[-1]}(Q(z))}{z^2}Q^2(z)dz \leq cg^{[-1]}(n)\int_{cx}^{Q^{[-1]}(n)} \frac{Q^2(z)}{z^2}dz$$

Applying [10], 5.4, we can estimate $\frac{Q(z)}{z}$ by $\frac{1}{A}Q^{'}(z)$, where $A > 1$ is in the definition of Freud weights. So

$$\int_{cx}^{Q^{[-1]}(n)} \frac{Q^2(z)}{z^2}dz \leq \frac{1}{2A}\int_{cx}^{Q^{[-1]}(n)} 2Q^{'}(z)Q(z)\frac{1}{z}dz$$

With an integration by parts we get that

$$\int_{cx}^{Q^{[-1]}(n)} \frac{Q^2(z)}{z^2}dz \leq \frac{1}{2A-1}\left(\frac{n^2}{Q^{[-1]}(n)} - c\frac{Q^2(x)}{x}\right) \leq c\frac{n^2}{Q^{[-1]}(n)},$$

if n is large enough. Summarizing the calculations of the previous lines we obtain that

$$S_{13} \leq c\sqrt{\frac{g^{[-1]}(n)}{Q^{[-1]}(n)}}. \tag{121}$$

Hence these estimations yield that if (10) fulfils, then $|\sigma_n(x) - \sigma_\nu(x)| = O\left(\frac{1}{n^\gamma}\right)$.

This lemma shows the order of the roots of $\varphi_m - \sum_{j=1}^{\infty} a_{jm}\Psi_j$ at the x_j-s, that is applying the classical theorem of S. N. Bernstein (see e.g. [17]), and taking into consideration Lemma 7 and its corollary as well, we get the following

COROLLARY. *Let $x \in S_j$, then*

$$\left|\varphi_m(x) - \sum_{j=1}^{\infty} a_{jm}\Psi_j(x)\right| \leq |x - x_j|^\gamma h(x), \tag{122}$$

where $h(x)$ is independent of j, is continuous and it grows polynomially with x.

For the final computations let us prove our last lemma, which follows the same chain of ideas as Lemma 1.1 of J. Szabados [20]:

LEMMA 9. *Let $m_j, \varrho \geq 0, \varepsilon > 0, x_j$ be as in Definition 2, with properties (2),and let*

$$\hat{v}(x) = \prod_{j=1}^{\infty}\left|1 - \frac{x}{x_j}\right|^{m_j} \quad \text{and} \quad \hat{v}_k(x) = \prod_{\substack{1\leq j<\infty \\ j\neq k}}\left|1 - \frac{x}{x_j}\right|^{m_j}.$$

Then

$$\hat{v}(x) \le e^{c|x|^{\varrho+\varepsilon}}, \quad x \in \mathbb{R}, \quad \varepsilon > 0 \tag{123}$$

and

$$\hat{v}(x) \ge e^{-c|x|^{\varrho+\varepsilon}} \quad \textit{for } x \in \mathbb{R} \setminus \cup_{j=1}^{\infty} \left(x_j - \frac{m_j}{|x_j|^{\varrho+\varepsilon}}, x_j + \frac{m_j}{|x_j|^{\varrho+\varepsilon}} \right). \tag{124}$$

Furthermore

$$\hat{v}_k(x) \ge e^{-c|x|^{\varrho+\varepsilon}} \quad \textit{for } x \in \left(x_k - \frac{m_k}{|x_k|^{\varrho+\varepsilon}}, x_k + \frac{m_k}{|x_k|^{\varrho+\varepsilon}} \right), \tag{125}$$

where $c > 0$ depends on $\hat{v}$ and ε, and if $a > b$, then $[a, b] = \emptyset$.

REMARK. If e.g. $x_j = j^{\nu}, \nu > 0$, then $\varrho = \frac{1}{\nu}$, and if $x_j = 2^j$, then $\varrho = 0$.

PROOF. The proof follows the steps of the proof of Lemma 1.1 in [20]. Let

$$N(x) = \sum_{|x_k|<|x|} m_k.$$

According to (4), $N(x) \le c(\varepsilon)|x|^{\varrho+\varepsilon}$, for all $\varepsilon > 0$.

$$\hat{v}(x) \le \prod_{|x_k|<|x|} \left|1 - \frac{x}{x_k}\right|^{m_k} \prod_{\substack{|x_k|\ge|x| \\ x_k x<0}} \left(1 + \left|\frac{x}{x_k}\right|\right)^{m_k} = \hat{v}_1(x)\hat{v}_2(x).$$

As in [20],

$$\hat{v}_1(x) \le \prod_{|x_k|<|x|} \left(2\left|\frac{x}{x_k}\right|\right)^{m_k} \le (2|x|)^{N(x)} \left(\frac{\sum_{|x_k|<|x|} \frac{m_k}{|x_k|^{\varrho+\varepsilon}}}{N(x)} \right)^{\frac{N(x)}{\varrho+\varepsilon}}$$

$$\le \left(\frac{c|x|^{\varrho+\varepsilon}}{N(x)} \right)^{\frac{N(x)}{\varrho+\varepsilon}} \le e^{c|x|^{\varrho+\varepsilon}},$$

and

$$\hat{v}_2(x) \le e^{\sum_{\substack{|x_k|\ge|x| \\ x_k x<0}} m_k \log\left(1+\left|\frac{x}{x_k}\right|\right)} \le e^{|x|^{\varrho+\varepsilon} \sum_{k=1}^{\infty} \frac{m_k}{|x_k|^{\varrho+\varepsilon}}} \le e^{c|x|^{\varrho+\varepsilon}}.$$

For the lower estimation, as in [20], we divide our product into three parts: if $x \ne x_j, j = 1, 2, \ldots$, then

$$\hat{v}(x) = \prod_{|x_j|<|x|} \left|1 - \frac{x}{x_j}\right|^{m_j} \prod_{|x|<|x_j|\le 2|x|} (\cdot) \prod_{|x_j|>2|x|} (\cdot) = P_1 P_2 P_3.$$

As $P_1 \ge \prod_{\frac{x}{x_j}>1}(\cdot), P_2 \ge \prod_{1<\frac{x}{x_j}<2}(\cdot), P_3 \prod_{0<\frac{x}{x_j}<\frac{1}{2}}(\cdot)$ the computations are the same as in [20], so we omit the details.

Also the same computation implies (126).

Now we are in position to prove the theorem.

3.4. Proof of Theorem 1. The properties of g imply that $\varrho < 1$ in the definition of v, so it is obvious from (122) and the definition of Freud weight that there exists μ such that with arbitrary $d > 0$ there is a $v := v_{X,M,\mu,d}$ for which $\varphi_k v \in L^p$. According to (124) and (125) to $c = c(\mu)$ we can choose a $d > 0$ such that $v_{X,M,\mu,d} > ce^{k|x|^{\varrho+\mu}}$ with some $k > 0$ on $\mathbb{R} \setminus \cup_{j=1}^{\infty} \left(x_j - \frac{m_j}{|x_j|^{\varrho+\varepsilon}}, x_j + \frac{m_j}{|x_j|^{\varrho+\varepsilon}}\right)$, and the same fulfils on $v_k = \hat{v}_k(x)e^{d|x|^{\varrho+\mu}}$ on the interval $\left(x_k - \frac{m_k}{|x_k|^{\varrho+\varepsilon}}, x_k + \frac{m_k}{|x_k|^{\varrho+\varepsilon}}\right)$.

Let

$$\varphi_m^* = \frac{1}{v^2}\left(\varphi_m - \sum_{j=1}^{\infty} a_{jm}\Psi_j\right) \quad m = 1, 2, \dots, \tag{126}$$

where $\{a_{jm}\}$ is a solution of (30). We will show that $\{\varphi_m^*\}_{m=1}^{\infty}$ is a system in L_v^q which is biorthonormal with respect to $\{\varphi_m\}_{m=1}^{\infty} \subset L_v^p$.

According to Lemma 7, the series in (101) is convergent in some sense, that is the definition of φ_m^* is clear, and applying the Corollary after Lemma 7, by Lebesgue's theorem we can integrate term by term as follows:

$$\int_{\mathbb{R}} \varphi_m^* \varphi_k v^2 = \int_{\mathbb{R}} \frac{1}{v^2}\left(\varphi_m - \sum_{j=1}^{\infty} a_j m \Psi_j\right)\varphi_k v^2$$

$$= \int_{\mathbb{R}} \varphi_m\varphi_k - \sum_{j=1}^{\infty} a_{jm}\int_{\mathbb{R}} \Psi_j\varphi_k = \delta_{m,k} + 0,$$

where we used the orthonormality of the original system, which was the weighted orthonormal polynomials.

So the only thing we have to prove that φ_m^* is in L_v^q. Let $S_j^* = S_j \cap (x_j - \frac{m_j}{x_j^{\varrho+\varepsilon}}, x_j + \frac{m_j}{x_j^{\varrho+\varepsilon}})$, and let $\delta > 0$ (see 5)) be fixed. Thus

$$\int_{\mathbb{R}} \left|\frac{\varphi_m - \sum_{j=1}^{\infty} a_{jm}\Psi_j}{v}\right|^q = \sum_{j=1}^{\infty}\int_{x\in S_j^*}(\cdot) + \int_{x\in(\mathbb{R}\setminus\ \cup_{j=1}^{\infty} S_j^*)}(\cdot)$$

$$\le c\sum_{j=1}^{\infty}\int_{x\in S_j^*}\left|h(x)x^{m_j}e^{-k|x|^{\varrho+\mu}}\right|^q \left||x-x_j|^{\gamma-m_j}\right|^q +$$

$$\int_{x\in(\mathbb{R}\setminus\cup_{j=1}^{\infty} S_j^*)}\left|k(x)e^{-k|x|^{\varrho+\mu}}\right|^q = (*),$$

where $h(x)$ is as in (122), and according to Lemma 7, $k(x)$ grows polynomially. Hence by Lemma 9, we can estimate $(*)$ on the whole real line with an integral of a function which grows polynomially, times an exponentially small factor, that is

$$\|\varphi_m^*\|_q \le \left(\int_{\mathbb{R}}\left|k_1(x)e^{-c|x|^{\varrho+\mu}}\right|^q\right)^{\frac{1}{q}},$$

where $k_1(x)$ depends only on m and $Q(x)$. So the $q$norm of φ_m^* is bounded if $m_j - \gamma < \frac{1}{q}$, so the dual system is in L_v^q, when p fulfils the inequalities in the theorem.

For completeness we have to prove that if for a $g \in L^q_{v_X,M}$ (where $\frac{1}{p}+\frac{1}{q}=1$) $g(\varphi_k) = \int_{\mathbb{R}} g\varphi_k v^2 = 0, k \in \mathbb{N}$, then $g = 0$. The completeness of the original system implies that g has to be of the form

$$g = \frac{1}{v^2}\sum_{j=1}^{\infty} b_j \Psi_j,$$

and as $g \in L^q_{v_{X,M,d,\mu}}$, $\int_{\mathbb{R}} |gv|^q$ must be finite. By the properties of v, and recalling that $\Psi_j = p_{l_j} w$, the integral on $\mathbb{R} \setminus \cup_j S_j^*$ is finite, so we have to deal with the integral around the roots of v, that is $\sum_{j=1}^{\infty} \int_{S_j^*} \left| \frac{1}{v} \sum_{j=1}^{\infty} b_j \Psi_j \right|^q$ has to be finite. Together with the assumption $p < \inf_{m_j<1} \frac{1}{1-m_j}$, this means that

$$\left(\sum_{j=1}^{\infty} b_j \Psi_j\right)(x_k) = 0, \qquad k = 1, 2, \ldots. \tag{127}$$

So as in (15), we got a homogeneous linear system of equations

$$Ab = 0, \tag{128}$$

where A is the same infinite matrix as in (15). Introducing $\hat{A}$, etc., according to 3.1.2, the homogeneous equation has the only solution in l_2 is $b_j = 0, j = 1, 2\ldots,$ that is $g = 0$.

FINAL REMARKS. (1) If somebody does not take care of the range of the operator A, then, because in our case on the right hand side of the equation there is a fast convergent vector, to get some solution of the equation $Aa_m = c_m$, it is enough to apply Toeplitz's theorem. So it is not necessary to guarantee a not too small element in every row. That is the proof of Lemma 1 ensures a good omission system for arbitrary system of pointss.

(2) The aim of this paper was to show the existence of a "good" point- and a "good" omission system with some assumptions on the functions Q and g. We chose a rather comfortable one. More precisely, our calculations show that besides (10), which is needed for convergence, it is enough to assume for solvability

$$g(x) > x^{\mu}, \quad \mu > \frac{15}{2},$$

$\frac{g^{[-1]}(x)}{(Q^{[-1]}(x))^{1-\varepsilon}}$ is strictly decreasing for a $\varepsilon > 0$, and

$$x^{\delta} \max\left\{ \frac{x^{\frac{1}{4}}}{g^{\frac{1}{6}}(x)}; \frac{1}{(Q^{[-1]}(x))^{\frac{1}{2}}} \right\} \to 0, \quad \text{where} \quad \delta > \frac{5}{4};$$

for unicity:

$$x^{\delta}\max\left\{\frac{1}{g^{\frac{1}{6}}(x)};\frac{1}{(Q^{[-1]}(x))^{\frac{1}{2}}}\right\}\to 0,\qquad \text{where}\qquad \delta>\frac{5}{4};$$

and

$$x^{\nu}\frac{(Q^{[-1]}(g(x)))^{\frac{1}{4}}}{g^{\frac{1}{6}}(x)}\to 0,\qquad \text{where}\qquad \nu>\frac{3}{4};$$

for the convergence of finite section method:

$$x^{\kappa}\max\left\{\frac{(Q^{[-1]}(g(x)))^{\frac{1}{4}}}{g^{\frac{1}{6}}(x)};\frac{1}{(Q^{[-1]}(x))^{\frac{1}{2}}}\right\}\to 0,\qquad \text{where}\qquad \kappa>1$$

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