ASYMPTOTICS FOR RECURRENCE COEFFICIENTS OF X_1-JACOBI EXCEPTIONAL POLYNOMIALS AND CHRISTOFFEL FUNCTION

Á. P. HORVÁTH

Abstract. Computing asymptotics of the recurrence coefficients of X_1-Jacobi polynomials we investigate the limit of the Christoffel function. We also study the relation between the normalized counting measure based on the zeros of the modified average characteristic polynomial and the Christoffel function in limit. The proofs of the corresponding theorems with respect to standard orthogonal polynomials are based on the three-term recurrence relation. The main point is that exceptional orthogonal polynomials possess at least five-term formulae and so Christoffel-Darboux formula also fails. It seems that these difficulties can be handled in combinatorial way.

1. Introduction

Given a (non-trivial) measure μ supported on a set of the complex plane, with finite moments, by Gram-Schmidt orthogonalization a sequence of orthogonal polynomials can be generated, $\{P_n\}_{n=0}^{\infty} \subset L^2(d\mu)$. After normalization the kernel of the projection, $\pi_n : L^2(d\mu) \to \mathcal{P}_n$, where $\mathcal{P}_n$ is the vector subspace of polynomials of degree at most $n-1$, is

$$K_n(x,y) = \sum_{k=0}^{n-1} P_k(x)\overline{P}_k(y).$$

There is an extensive literature of the following two questions investigated in parallel, the asymptotic distribution of zeros of the orthogonal polynomials, $\lim_n \nu_n$, where ν_n is the normalized counting measure based at the zeros, and the limit of the weighted reciprocal of the Christoffel function that is the measure

$$d\mu_n(x) := \frac{1}{n} K_n(x,x) d\mu(x).$$

These questions are in close contact with each other and are related to the recurrence relation of orthogonal polynomials.

For instance on the real line if the recurrence coefficients a_n and b_n have limits a and b, respectively, the polynomials have asymptotic zero distribution with density $\omega_{a,b}(x) = \frac{1}{\pi\sqrt{(b+2a-x)(x-b+2a)}}$ on the interval $[b-2a, b+2a]$, see [?] and the references therein.

On the other hand for a μ supported on the unite circle or on the interval $[-1,1]$ Máté, Nevai and Totik proved that provided the so-called Szegő condition, μ_n tends

2010 *Mathematics Subject Classification.* 33C47,33C45.

Key words and phrases. exceptional orthogonal polynomials, recurrence relation, Chistoffel function.

Supported by the NKFIH-OTKA Grant K128922.

to the equilibrium measures, to the arc- or to the arcsin measure, respectively, cf. [?]. The asymptotics of μ_n is investigated in rather general circumstances, we mention here a result due to Totik on arcs and curves, see [?]. Finally, it can be shown that the zeros of orthogonal polynomials are the eigenvalues of the operator $\pi_n M_x \pi_n$, where π_n is the projection mentioned above and $M_x f = xf$, which ensures that $(\mu_n - \nu_n) \to 0$ in some sense, see [?]. This observation implies a direct relation between the two measures derived by standard orthogonal polynomials and this technique leads to determinantal point processes.

The aim of the paper is to make small contribution to examination of the problem outlined above in case of exceptional orthogonal polynomials. We focus onto the Christoffel function part.

The notion of exceptional orthogonal polynomials is motivated by problems in quantum mechanics. It turned up in the last twenty years; one of the earliest papers is [9], so the topic is fairly new. Owing to the widespread investigation of exceptional systems its literature became rather rich. We mention here just some examples: [7], [?], [10], etc. and see also the references therein.

The study of Christoffel functions of standard orthogonal systems depend on the three-term recurrence formula. Exceptional orthogonal polynomials fulfil $2L+1$-term recurrence relations, where L is at least 2. The other important difference is while in ordinary case xP_n can be expressed by a (three-term) linear combination of orthogonal polynomials, in exceptional case instead of x there is a polynomial of degree at least two. This is why most of the methods developed to ordinary orthogonal polynomials are not applicable to exceptional systems. This investigation based on combinatorial methods inspired by [13] and [14]. Although the family of exceptional Jacobi polynomials is rather rich cf. [3], below we restrict our investigation to exceptional Jacobi polynomials generated by one Darboux transformation applied to Jacobi polynomials cf. [7], and specially X_1 Jacobi polynomials (of codimension 1).

The paper is organized as follows. In the next section we define exceptional orthogonal polynomials and list up some properties which are important for further investigation. In Section 3 we deal with recurrence relation and give the asymptotics of the recurrence coefficients of exceptional Jacobi polynomials derived by one Darboux transformation. In the fourth section we study the asymptotic behavior of the weighted reciprocal of the Christoffel function for codimension 1 exceptional Jacobi case, and in the last section we introduce the modified average characteristic polynomial and study the connection between its zero distribution and μ_n.

2. Exceptional Orthogonal Polynomials

According to Bochner's theorem (cf. [2]) a second order differential operator having polynomial eigenfunctions with full sequence of degrees has to possess polynomial coefficients of degree at most two, and the eigenfunctions of the operator are necessarily the classical orthogonal polynomials (up to an affine transformation). Weakening of the assumptions of Bochner's theorem leads to the notion of exceptional orthogonal polynomials. In this section we summarize some properties of exceptional orthogonal polynomials which are important hereinafter. Finally, we deal with X_1-Jacobi polynomials, but our investigation is based on the general characterization of exceptional systems rather than the known examples. The definition is as follows, cf. [7, Definition 7.4]

By exceptional orthogonal polynomials we denote a co-finite real-valued polynomial sequence $\{p_k\}$, where from the sequence of degrees finite indices $(k_1, \dots, k_m)$ are missing, provided the properties listed below fulfil.
(1) The polynomials are eigenfunctions of a second order linear differential operator with rational coefficients.
(2) There is an interval I and a positive weight function W on I with finite moments and at the endpoints of I $pp_kW \to 0$, where p is the coefficient of the second derivative in the differential operator.
(3) The vector space spanned by the elements of the sequence is dense in the weighted space $L^2(W, I)$.

The second property ensures orthogonality of the exceptional polynomials on I with respect to W. Exceptional orthogonal polynomial systems (XPS) are characterized by their construction cf. [7, Theorem 1.2]: all XPS can be obtained by applying a finite sequence of Darboux transformations to a classical orthogonal polynomial system.

Below we need the construction thus we summarize the 1-step Darboux transformation case in short.

Classical orthogonal polynomials $\{P_n^{[0]}\}_{n=0}^{\infty}$ are eigenfunctions of the second order linear differential operator with polynomial coefficients

$$T[y] = py'' + qy' + ry,$$

and its eigenvalues are denoted by λ_n. T can be decomposed as

$$T = BA + \tilde{\lambda}, \text{ with } A[y] = b(y' - wy), \ B[y] = \hat{b}(y' - \hat{w}y), \tag{1}$$

where b, w are rational functions and

$$\hat{b} = \frac{p}{b}, \quad \hat{w} = -w - \frac{q}{p} + \frac{b'}{b}. \tag{2}$$

Then the exceptional polynomials are the eigenfunctions of $\hat{T}$, that is the partner operator of T, which is

$$\hat{T}[y] = (AB + \tilde{\lambda})[y] = py'' + \hat{q}y' + \hat{r}y, \tag{3}$$

where

$$\hat{q} = q + p' - 2\frac{b'}{b}p, \ \hat{r} = r + q' + wp' - \frac{b'}{b}(q + p') + \left(2\left(\frac{b'}{b}\right)^2 - \frac{b''}{b} + 2w'\right)p, \tag{4}$$

and w fulfils the Riccati equation

$$p(w' + w^2) + qw + r = \tilde{\lambda}, \tag{5}$$

cf. [7, Propositions 3.5 and 3.6].

(1) and (3) ensure that

$$\hat{T}AP_n^{[0]} = \lambda_n AP_n^{[0]}, \tag{6}$$

so exceptional polynomials can be obtained from the classical ones by application of (finite) appropriate first order differential operator(s) to the classical polynomials. This observation motivates the notation below

$$AP_n^{[0]} =: P_n^{[1]}, \tag{7}$$

(and recursively $A_sP_n^{[s-1]} =: P_n^{[s]}$ in case of s Darboux transformations.) The degree of $P_n^{[1]}$ is usually greater than n, so subsequently n is an index not a degree any more; it shows that $P_n^{[1]}$ is generated from $P_n^{[0]}$. Since this notation does not follow where the gaps are in the sequence of degrees, it proved to be useful in handling recurrence formulae, cf. [17].

$\{P_n^{[1]}\}_{n=0}^\infty$ is an orthogonal system on I with respect to the weight

$$W := \frac{pw_0}{b^2}, \tag{8}$$

where w_0 is one of the classical weights. Since at the endpoints of I (if there is any) p may have a zero, b can be zero as well here, but it has no zeros inside I, otherwise W would not have finite moments. Canceling the zeros at the endpoints we introduce the following notation.

Notation.

Let $\frac{p}{b} = \frac{\tilde{p}}{\tilde{b}}$. Hereinafter we assume that $(p, \tilde{b}) = 1$. Let $\deg \tilde{b} = m$.

Remark.

(R1) As $P_n^{[1]}$-s is a polynomials for each n, applying operator A to $P_0^{[0]}$ it can be seen that bw must be a polynomial and to $P_1^{[0]}$ shows that b itself is a polynomial too. Thus we can write w as $w = \frac{g}{b}$, where g is also a polynomial.

(R2) According to (4) $P_n^{[1]}$ satisfies the following differential equation:

$$\begin{aligned} &bpy'' + (b(q+p') - 2b'p)\, y' \\ &+ \left(b(r+q') + gp' - b'(q+p') - pb'' + 2p\frac{b'}{b}(b'-g) + 2pg' \right) y = b\lambda_n y. \end{aligned} \tag{9}$$

Since all the other terms are polynomials $\frac{pb'}{b}(b'-g)P_n^{[1]}$ are also polynomials for all n, thus the coefficient of $P_n^{[1]}$ is a polynomial. Furthermore denoting by

$$L_e[y] := -2b'py' + \left(gp' - b'(q+p') - pb'' + 2p\frac{b'}{b}(b'-g) + 2pg' \right) y, \tag{10}$$

(9) ensures that $L_e P_n^{[1]}$ is divisible by $\tilde{b}$ for all $n \in \mathbb{N}$.

(R3) Observe that if a differential operator D is formulated as $D[f] = Ff' + Gf$ then $D(fg) = fFg' + gDf$.

3. Recurrence Relations

In this section after introducing recurrence relation with respect to exceptional orthogonal polynomials, we investigate the asymptotics of the recurrence coefficients.

Standard orthonormal polynomial systems contain polynomials of all degrees which fulfil the three-term recurrence relation

$$xP_n^{[0]} = a_{n+1}P_{n+1}^{[0]} + b_n P_n^{[0]} + a_n P_{n-1}^{[0]}. \tag{11}$$

Recurrence formulae can be derived for exceptional orthogonal polynomials as well, but these are $2L+1$-term formulae, where $L \geq 2$. As (11) can be rearranged in two ways, for exceptional polynomials there are two different kinds of recurrence relations: with constant- and with variable dependent coefficients. Recalling the Darboux decomposition of the differential operator which generates exceptional polynomials, formulae with variable dependent coefficients can be derived by operators A_i (cf. (7)) (see e.g. [?], [8], [17], [?], [?], [?]) and with constant coefficient ones by operators B_i (see e.g. [?], [?], [11], [?], [5], [?], [4]). The length of the formula, L in the variable dependent coefficient case depends on the number of Darboux transformations, that is

$$P_n^{[L-1]} = \sum_{k=-L}^{L} r_{n,k}^{[L]}(x) P_{n+k}^{[L-1]}.$$

In contrast in the constant coefficient case the length depends on the codimension independently of how many Darboux transformations resulted the codimension of the space of exceptional polynomials. In variable dependent coefficient case it was more convenient to work with monic polynomials, but in constant coefficient case we use the orthonormal form of the polynomials.

Subsequently we deal with exceptional polynomials generated by one-step Darboux transformation, so for sake of simplicity further properties are written in this case. We denote by $P_n^{[i]}$ the orthonormal classical ($i = 0$) or exceptional ($i = 1$) polynomials. Below we summarize the results of Odake, Gómez-Ullate, Kasman, Kuijlaars, and Milson (see [?], [?], [11]). Because we need some steps of the proof too, below the sketch of the proofs are also given in brief and for sake of simplicity in one-step Darboux transform case.

Let us introduce the linear space

$$\mathcal{P}^{[1]} := \left\{ \sum_{n=0}^{N} c_n P_n^{[1]} : c_n \in \mathbb{R}, N \in \mathbb{N} \right\}, \tag{12}$$

and the stabilizer ring

$$\mathcal{S} := \left\{ s \in \mathcal{P} : s P_n^{[1]} \in \mathcal{P}^{[1]} \text{ for all } n \in \mathbb{N} \right\}, \tag{13}$$

where $\mathcal{P}$ denotes the set of polynomials.

Theorem A. *With the notation above let $s \in \mathcal{P}$. If s' is divisible by $\tilde{b}$, then $s \in \mathcal{S}$.*

Proof. By (R2) and by comparison of the codimensions of the two sets in question, it can be seen that a polynomial $s \in \mathcal{P}$ belongs to $\mathcal{P}^{[1]}$ if and only if $L_e s$ is divisible by $\tilde{b}$. According to (R3) $L_e s P_n^{[1]} = -2b' p s' P_n^{[1]} + s L_e P_n^{[1]}$. Thus using (R2) if s' is divisible by $\tilde{b}$ then $L_e s P_n^{[1]}$ is also divisible by $\tilde{b}$ and so it belongs to $\mathcal{P}^{[1]}$. By orthogonality properties it can be expanded as in below

$$s P_n^{[1]} = \sum_{k=-L}^{L} u_{n,k} P_{n+k}^{[1]}. \tag{14}$$

Corollary B. *If*

$$B s P_n^{[1]} = \sum_{k=-L}^{L} a_{n,k} P_{n+k}^{[0]},$$

then

$$u_{n,k} = \frac{a_{n,k}}{\lambda_{n+k} - \tilde{\lambda}}, \;\; k = -L, \dots, L.$$

Proof.

$$BsP_n^{[1]}=\sum_{k=-n}^{L}\frac{a_{n,k}}{\lambda_{n+k}-\tilde\lambda}(\lambda_{n+k}-\tilde\lambda)P_{n+k}^{[0]}=\sum_{k=-n}^{L}\frac{a_{n,k}}{\lambda_{n+k}-\tilde\lambda}BAP_{n+k}^{[0]}$$
$$=\sum_{k=-n}^{L}\frac{a_{n,k}}{\lambda_{n+k}-\tilde\lambda}BP_{n+k}^{[1]}=B\sum_{k=-L}^{L}\frac{a_{n,k}}{\lambda_{n+k}-\tilde\lambda}P_{n+k}^{[1]}.$$

That is

$$0\equiv B\sum_{k=-L}^{L}\left(u_{n,k}-\frac{a_{n,k}}{\lambda_{n+k}-\tilde\lambda}\right)P_{n+k}^{[1]}=\sum_{k=-L}^{L}\left(u_{n,k}-\frac{a_{n,k}}{\lambda_{n+k}-\tilde\lambda}\right)BAP_{n+k}^{[0]}$$
$$=\sum_{k=-L}^{L}\left(u_{n,k}-\frac{a_{n,k}}{\lambda_{n+k}-\tilde\lambda}\right)(\lambda_{n+k}-\tilde\lambda)P_{n+k}^{[0]},$$

which ensures that $u_{n,k}-\frac{a_{n,k}}{\lambda_{n+k}-\tilde\lambda}=0$ for all k.

3.1. **Asymptotics of recurrence coefficients.** The purpose of this section is the determination of the limits of recurrence coefficients i.e.

$$\lim_{n\to\infty}u_{n,j}=:U_{|j|}, \tag{15}$$

cf. (14). Below we examine recurrence coefficients of exceptional Jacobi polynomials generated by one-step Darboux transformation and of codimension m. First we make some general calculations with respect to exceptional polynomials derived by one-step Darboux transform, and we show that recurrence coefficients of exceptional polynomials can be derived from the recurrence coefficients of the classical ones. To this end we return to the expression in Corollary B. Subsequently we use the simplest element of $\mathcal{S}$:

$$Q(x)=\int^x\tilde b,\quad \deg Q=L, \tag{16}$$

where $L=m+1$. Applying (R3), (2) and (1)

$$BQP_n^{[1]}=\hat bQ'P_n^{[1]}+QBP_n^{[1]}$$
$$=\tilde pP_n^{[1]}+(\lambda_n-\tilde\lambda)QP_n^{[0]}=\tilde pb\left(P_n^{[0]}\right)'-\tilde pgP_n^{[0]}+(\lambda_n-\tilde\lambda)QP_n^{[0]}.$$

We need the following notation:

$$\tilde b(x)=\sum_{k=0}^{m}d_kx^k,\quad Q(x)=\sum_{k=1}^{L}\frac{d_{k-1}}{k}x^k\quad\text{and}\quad(\tilde pg)(x)=\sum_{k=0}^{m+1}c_kx^k. \tag{17}$$

In Theorem A the assumption refers to Q' thus we can take Q with an arbitrary constant term. Here the constant is chosen to be zero.

Classical orthonormal polynomials satisfy the following relation (cf. e.g. [**?**]).

$$p\left(P_n^{[0]}\right)'=A_nP_{n+1}^{[0]}+B_nP_n^{[0]}+C_nP_{n-1}^{[0]}, \tag{18}$$

where A_n, B_n and C_n are real numbers and p is the coefficient of the second derivative in the differential equation. Thus

$$BQP_n^{[1]}=\tilde b\left(A_nP_{n+1}^{[0]}+B_nP_n^{[0]}+C_nP_{n-1}^{[0]}\right)-\tilde pgP_n^{[0]}+(\lambda_n-\tilde\lambda)QP_n^{[0]} \tag{19}$$

$$= A_n \sum_{k=0}^{m} d_k \left(x^k P_{n+1}^{[0]} \right) + C_n \sum_{k=0}^{m} d_k \left(x^k P_{n-1}^{[0]} \right)$$
$$+ \left(B_n d_0 - c_0 + \sum_{k=1}^{m} \left(B_n d_k - c_k + \frac{\lambda_n - \tilde{\lambda}}{k} d_{k-1} \right) x^k \right.$$
$$\left. + \left((\lambda_n - \tilde{\lambda}) \frac{d_m}{m+1} - c_{m+1} \right) x^{m+1} \right) P_n^{[0]}.$$

That is knowing the limits of the recurrence coefficients of classical orthonormal polynomials similar limits can be derived to the corresponding exceptional ones.

Wide class of standard orthogonal polynomials supported on $[-1,1]$ have recurrence coefficients with limit $\lim_{n\to\infty} a_n = \frac{1}{2}$; $\lim_{n\to\infty} b_n = 0$, see [**?**, Theorem 4.5.7]. In classical cases, the situation is easier, after a normalization if it is necessary, the limit of the recurrence coefficients; $\lim_{n\to\infty} a_n =: a$; $\lim_{n\to\infty} b_n =: b$ can be got by direct computations. (Since subsequently we use the simplified form $\tilde{b}$ of the coefficient of the derivative in operator A (cf. Darboux transformation), it will no make confusion if we denote by b the limit of the sequence $\{b_n\}$ as it is usual in the literature.)

With this notation (11) can be written as

$$xP_n^{[0]} = (a + \varepsilon_{n,1}) P_{n+1}^{[0]} + (b + \epsilon_{n,0}) P_n^{[0]} + (a + \varepsilon_{n,-1}) P_{n-1}^{[0]}, \tag{20}$$

where $\varepsilon_{n,j}$, $\epsilon_{n,j}$ $(j = -1, 0, 1)$ tend to zero when n tends to infinity. Taking the k-long product of $a + \varepsilon_{l,j}$ and $b + \epsilon_{s,j}$, where $n - k \le l, s \le n + k$, the error term can be estimated by $c(k)\varepsilon_{n,0}$, say, where $c(k)$ is a constant which is independent of n. The symmetry of (11) ensures the same symmetry of the iterated recurrence relation, cf. (25). It has the following form:

$$x^k P_n^{[0]} = \sum_{j=-k}^{k} (s_{k,|j|} + e_{k,n,j}) P_{n+j}^{[0]}, \tag{21}$$

where $|e_{k,n,j}| \le c(k)\varepsilon_{n,0}$, that is it tends to zero when n tends to infinity. Substituting it into (19) we have

$$BQP_n^{[1]} \tag{22}$$
$$= A_n \sum_{k=0}^{m} d_k \sum_{j=-k}^{k} (s_{k,|j|} + e_{k,n+1,j}) P_{n+1+j}^{[0]} + C_n \sum_{k=0}^{m} d_k \sum_{j=-k}^{k} (s_{k,|j|} + e_{k,n-1,j}) P_{n-1+j}^{[0]}$$
$$+ (B_n d_0 - c_0) P_n^{[0]} + \sum_{k=1}^{m} \left(B_n d_k - c_k + \frac{\lambda_n - \tilde{\lambda}}{k} d_{k-1} \right) \sum_{j=-k}^{k} (s_{k,|j|} + e_{k,n,j}) P_{n+j}^{[0]}$$
$$+ \left((\lambda_n - \tilde{\lambda}) \frac{d_m}{m+1} - c_{m+1} \right) \sum_{j=-(m+1)}^{m+1} (s_{m+1,|j|} + e_{m+1,n,j}) P_{n+j}^{[0]}.$$

In view of (22) and Corollary B the coefficients in the expansion of $BQP_n^{[1]}$ are

$$u_{n,\pm(m+1)} = \frac{1}{\lambda_{n\pm(m+1)} - \tilde{\lambda}} \left(\left((\lambda_n - \tilde{\lambda}) \frac{d_m}{m+1} - c_{m+1} \right) (s_{m+1,|m+1|} + e_{m+1,n,m\pm1}) \right.$$
$$\left. + A_n d_m (s_{m,|m|} + e_{m,n+1,m}) \right), \tag{23}$$

and if $0 \leq |j| \leq m$

$$u_{n,\pm j} = \frac{1}{\lambda_{n\pm j} - \tilde{\lambda}} \left(A_n \sum_{k=|\pm j-1|}^{m} d_k (s_{k,|j-1|} + e_{k,n+1,j\pm 1}) \right. \tag{24}$$

$$+ C_n \sum_{k=|\pm j+1|}^{m} d_k (s_{k,|j+1|} + e_{k,n-1,j\pm 1})$$

$$+ \sum_{k=\max\{1,|j|\}}^{m} \left(B_n d_k - c_k + (\lambda_n - \tilde{\lambda}) \frac{d_{k-1}}{k} \right) (s_{k,|j|} + e_{k,n,j})$$

$$\left. + \left((\lambda_n - \tilde{\lambda}) \frac{d_m}{m+1} - c_{m+1} \right) (s_{m+1,|j|} + e_{m+1,n,j}) \right).$$

To finish this general part we compute $s_{k,j}$.

Lemma 1. *In* (21) *the iterated coefficients in limit are*

$$s_{k,j} = \sum_{i=0}^{\left[\frac{k-|j|}{2}\right]} \binom{k}{|j|+2i} \binom{|j|+2i}{i} a^{|j|+2i} b^{k-|j|-2i}. \tag{25}$$

In particular, if $a = \frac{1}{2}$ *and* $b = 0$*,*

$$s_{2p,j} = \begin{cases} \binom{2p}{p-l} \frac{1}{2^{2p}}, & |j| = 2l, \ 0 \leq l \leq p \\ 0, & |j| = 2l+1, \ 0 \leq l \leq p-1; \end{cases} \tag{26}$$

$$s_{2p+1,j} = \begin{cases} \binom{2p+1}{p-l} \frac{1}{2^{2p+1}}, & |j| = 2l+1, \ 0 \leq l \leq p \\ 0, & |j| = 2l, \ 0 \leq l \leq p. \end{cases}$$

Remark. Certainly similar results can be found in the literature, see e.g. [1] with certain Freud weights. We introduce here the method we use mainly in the next section to get the result in the required form.

Proof. Recalling that $s_{k,j}$ is the main part of the coefficient of $P_{n+j}^{[0]}$ in the expansion of $x^k P_n^{[0]}(x)$ we can compute it on an oriented weighted graph $G = (V, E)$, where the vertices $V = \mathbb{N}^2$ and the edges

$$E := \{e : (n,m) \to (n+1, m+k), n, m \in \mathbb{N}, \ -1 \leq k \leq 1\}.$$

The weight on an edge is given by

$$w(e) = \begin{cases} a, & \text{if } k = 1 \text{ or } k = -1 \\ b, & \text{if } k = 0. \end{cases}$$

We start from zero level, say and we have to reach level j (or $-j$) in k steps. As we can step forward (on the same level) with weight b, up and down with weight a, on the k-long path we can take $j + 2i$ steps up or down which contains $j + i$ steps upwards and i downwards, and the remainder ones are taken forward. (For $-j$ we get the same.) That is

$$s_{k,j} = \sum_{\gamma:(0,n)\to(k,n+j)} \prod_{e\in\gamma} w(e) = \sum_{i=0}^{\left[\frac{k-|j|}{2}\right]} \binom{k}{|j|+2i} \binom{|j|+2i}{i} a^{|j|+2i} b^{k-|j|-2i},$$

which is just the statement. Simple substitution implies (26).

In the rest of this section we compute the asymptotic recurrence coefficients $U_{|j|}$.

3.2. X_m **Jacobi polynomials.** X_m stands for exceptional Jacobi polynomials derived by 1-step Darboux transform from the classical ones, and of codimension m.

Let the n^{th} Jacobi polynomial defined by Rodrigues' formula:

$$(1-x)^\alpha(1+x)^\beta P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right)^{(n)},$$

where $\alpha, \beta > -1$.

$$p_n^{\alpha,\beta} = \frac{P_n^{\alpha,\beta}}{\sqrt{\sigma_n}},$$

the orthonormal Jacobi polynomials which fulfil the following differential equation (cf. [**?**, (4.2.1)])

$$(1-x^2)y'' + (\beta-\alpha-(\alpha+\beta+2)x)y' + n(n+\alpha+\beta+1)y = 0. \tag{27}$$

With

$$P_n^{[0]} = p_n^{(\alpha,\beta)}, \tag{28}$$

and according to (7) and (1) one can derive $P_n^{[1]}$ as

$$P_n^{[1]} = AP_n^{[0]} = b(P_n^{[0]})' - bwP_n^{[0]}. \tag{29}$$

(Examples of X_m Jacobi polynomials, that is of appropriate Darboux transformations, can be found e.g. in [12].)

Recalling (14), (15) and (17) we are in position to state the main result of this section.

Proposition 1. *With the notation of* (15) *for* X_m *Jacobi polynomials*

$$U_{|j|} = \begin{cases} \sum_{p=\max\{l,1\}}^{\left[\frac{m+1}{2}\right]} \frac{d_{2p-1}}{2p}\binom{2p}{p-l}\frac{1}{2^{2p}}, & \text{if} \quad |j| = 2l \\ \sum_{p=l}^{\left[\frac{m}{2}\right]} \frac{d_{2p}}{2p+1}\binom{2p+1}{p-l}\frac{1}{2^{2p+1}}, & \text{if} \quad |j| = 2l+1. \end{cases} \tag{30}$$

Proof. According to the general calculations first we need the coefficients A_n, B_n and C_n, cf. (23) and (24). Taking into consideration that $\lim_{n\to\infty}\frac{\sigma_{n-1}}{\sigma_n} = 1$, etc. (for σ_n see [**?**, (4.3.4)]), according to [**?**, (4.5.5)]

$$\lim_{n\to\infty}\frac{A_n}{n} = \frac{1}{2}, \quad \lim_{n\to\infty} B_n = \frac{\alpha-\beta}{2} \quad \lim_{n\to\infty}\frac{C_n}{n} = -\frac{1}{2}. \tag{31}$$

$A_n := A_n(\alpha,\beta)$, etc. - for sake of simplicity we omit α and β. Again after normalization in view of [**?**, (4.5.1)]

$$a_n \to \frac{1}{2}, \quad b_n \to 0. \tag{32}$$

To compute the limit coefficients first we take into consideration that $\tilde\lambda$ does not depend on n, cf. (1). Thus by (27) $|\lambda_{n\pm j} - \tilde\lambda| \sim n^2$. ($X \sim Y$ means that $\frac{X}{Y}$ is between two positive constants.) Taking into consideration (23) and (24), for all $0 \le |j| \le m+1$ we have

$$U_{|j|} = 0 + \lim_{n\to\infty}\frac{\lambda_n-\tilde\lambda}{\lambda_{n+j}-\tilde\lambda}\sum_{k=\max\{1,|j|\}}^{m+1}\frac{d_{k-1}}{k}(s_{k,|j|}+e_{k,n,j}) = \sum_{k=\max\{1,|j|\}}^{m+1}\frac{d_{k-1}}{k}s_{k,|j|}.$$

Taking into account that $s_{k,|j|} = 0$ if $k - j$ is odd, by (26) we arrive to (30).

4. Exceptional Jacobi Polynomials of Codimension 1

For standard orthogonal polynomials it is proved in rather general circumstances that $\frac{1}{n}K_n(x,x)w(x)dx$ tends to the equilibrium measure of the (essential) support of μ in some sense, where w is the Radon-Nikodym derivative of μ and $K_n(x,x)$ the Christoffel-Darboux kernel, see e.g. [?] and the references therein. All the proofs of these types of theorems use the three-term recurrence relation of orthogonal polynomials, or in case of polynomial ensembles, with higher term recurrence relations the proof uses the fact that $xP_n \in \operatorname{span}\{P_{n-L}, \dots, P_{n+L}\}$, cf. [14]. As it is pointed out in section 3, in our case at least one of the conditions above fails. In this section we show in the simplest, one codimensional case, that is with five-term recurrence formula and with $\deg s = 2$, that the statement above still fulfils.

With the notation above let us choose

$$\tilde{b}(x) = d_1 x + d_0, \text{ that is } \quad Q(x) = \frac{d_1}{2}x^2 + d_0 x.$$

Here we choose the constant term of Q to be 0 again. To define $\tilde{b}$ monic polynomials are also appropriate. Since in the known examples $\tilde{b}$ is not monic (cf. e.g. [12]) we take it in the form given above and it does not cause any further difficulty. Let

$$K_n(x,y) := \sum_{k=0}^{n-1} P_k^{[1]}(x) P_k^{[1]}(y),$$

where $P_k^{[1]}(x)$ are the k^{th} exceptional Jacobi polynomials orthonormal with respect to W on $[-1,1]$, where, according to (28) and (8),
$W(x) := W(\alpha,\beta)(x) = (1-x^2)\frac{(1-x)^\alpha(1+x)^\beta}{b^2(x)}$. We introduce the following notation:

$$d\mu_N(x) = \frac{1}{N}K_N(x,x)W(x)dx, \tag{33}$$

$$\langle f,g\rangle := \int_{-1}^{1} f(x)g(x)W(x).$$

The equilibrium measure of $[-1,1]$, μ_e is defined by

$$d\mu_e(x) = \omega(x)dx = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx.$$

Let us recall the Wallis' formula:

$$\int_{-1}^{1} x^l d\omega(x) = \begin{cases} \binom{2k}{k}\frac{1}{2^{2k}}, & l = 2k, \\ 0, & l = 2k+1. \end{cases}$$

Our main result is as follows.

Theorem 1. *If μ_N is defined as in* (33) *by exceptional Jacobi polynomials of codimension 1, on* $[-1,1]$

$$\lim_{N\to\infty} \mu_N = \mu_e$$

in weak-star sense.

To prove Theorem 1 we need the following Lemma.

Lemma 2. *For all* $k = 0, 1, \dots$

$$\lim_{n\to\infty} \langle Q^k P_n^{[1]}, P_n^{[1]} \rangle = \int_{-1}^{1} Q^k(x)\omega(x)dx. \tag{34}$$

Proof. First we compute the effect of ω to Q^k, $k = 0, 1, \dots$. Binomial theorem and Wallis' formula give

$$\int_{-1}^{1} Q^k(x)\omega(x)dx \tag{35}$$

$$= \sum_{j=0}^{k} \binom{k}{j} d_0^j d_1^{k-j} \frac{1}{2^{k-j}} \int_{-1}^{1} x^{2k-j}(x)d\omega(x) = \sum_{i=0}^{\left[\frac{k}{2}\right]} \binom{k}{2i} \binom{2(k-i)}{k-i} \frac{d_0^{2i} d_1^{k-2i}}{2^{3k-4i}}.$$

On the other hand, according to Theorem A for all (fixed) k there exist coefficients $c_{n,j}^{(k)}$ such that

$$Q^k P_n^{[1]} = \sum_{j=-2k}^{2k} c_{n,j}^{(k)} P_{n+j}^{[1]}, \tag{36}$$

thus

$$\lim_{n\to\infty} \langle Q^k P_n^{[1]}, P_n^{[1]} \rangle = \lim_{n\to\infty} c_{n,0}^{(k)} =: c^{(k)},$$

where the limit exists with regard to Proposition 1. We can proceed as previously, that is recalling the oriented weighted graph $G = (V, E)$ with $V = \mathbb{N}^2$, we define edges as

$$E := \{e : (n, m) \to (n+1, m+k), n, m \in \mathbb{N},\ -2 \le k \le 2\},$$

and weights on the edges as

$$w(e) = \begin{cases} U_0, & \text{if } k = 0 \\ U_1, & \text{if } k = 1 \text{ or } k = -1 \\ U_2, & \text{if } k = 2 \text{ or } k = -2, \end{cases}$$

where, according to Proposition 1

$$U_0 = \frac{d_1}{4}, \quad U_1 = \frac{d_0}{2}, \quad U_2 = \frac{d_1}{8}. \tag{37}$$

Thus $c^{(k)}$ can be expressed as the collection of all possible weighted k-long paths from $P_n^{[1]}$ to $P_n^{[1]}$,

$$c^{(k)} = \sum_{\gamma(0,n)\to(k,n)} \prod_{e\in\gamma} w(e).$$

That is in each step we can move forward, denoted by (0) (with weight U_0); one level up or down denoted by (± 1) (with weight U_1) and two levels up or down denoted by (± 2) (with weight U_2). Since finally we have to arrive to the starting level it is clear that we need even number of (± 1). Let the number of (± 1) be $2i$. Then we can choose $s + m$ pieces of (2) (that is two levels big steps upwards) and m (-2). The rest of the steps are zeros (i.e. forward). That is

$$c^{(k)} = \sum_{i=0}^{\left[\frac{k}{2}\right]} \binom{k}{2i} \tag{38}$$

$$\times \sum_{s=0}^{\min\{i,k-2i\}} \binom{2i}{s+i} \sum_{m=0}^{\left[\frac{k-2i-s}{2}\right]} \binom{k-2i}{s+2m} \binom{s+2m}{m} \frac{d_0^{2i} d_1^{k-2i}}{2^{2k-2i+2m+\max\{0,s-1\}}}.$$

Indeed, we can choose the $2i$ (± 1)-s in $\binom{k}{2i}$ ways, and to neutralize the s pieces of (2) we need $2s$ (-1) and we have further $\frac{2i-2s}{2}$ (-1), that is the number of the choices of (-1)-s is $\binom{2i}{s+i}$. Then we choose from the remainder $k-2i$ elements the $s+2m$ elements with absolute value 2, and from these the m pieces of (-2). If $s>0$, we can do the same with s (-2)-s, etc. and it gives a factor 2 in these cases. Now taking into consideration the weights we arrive to the formula above.

Notice that (38) can be rearranged as

$$c^{(k)} = \sum_{i=0}^{\left[\frac{k}{2}\right]} \binom{k}{2i} \frac{d_0^{2i} d_1^{k-2i}}{2^{2k-2i}} S_{k,i},$$

where

$$S_{k,i} = \sum_{s=0}^{\min\{i,k-2i\}} \binom{2i}{s+i} \sum_{m=0}^{\left[\frac{k-2i-s}{2}\right]} \binom{k-2i}{s+2m} \binom{s+2m}{m} \frac{1}{2^{2m+\max\{0,s-1\}}}.$$

In order to prove that $c^{(k)} = \int_{-1}^{1} Q^k(x) d\omega(x)$, it is enough to show that for all $0 \le i \le \left[\frac{k}{2}\right]$,

$$S_{k,i} = \binom{2(k-i)}{k-i} \frac{1}{2^{k-2i}}. \tag{39}$$

First we observe that if $i=0$ and k is arbitrary, then

$$S_{k,0} = \sum_{m=0}^{\left[\frac{k}{2}\right]} \binom{k}{2m} \binom{2m}{m} \frac{1}{2^{2m}}$$

$$= \int_{-1}^{1} (1+x)^k d\omega(x) = \frac{2^{k+1}}{\pi} \int_0^{\frac{\pi}{2}} \cos^{2k} \xi d\xi = \binom{2k}{k} \frac{1}{2^k}. \tag{40}$$

If k is even and $i = \frac{k}{2}$, then on both sides of (39) stands $\binom{k}{\frac{k}{2}}$.

To show (39) we use induction from (k,i) to $(k+1,i+1)$. Notice that replacing $(k+1,i+1)$ instead of (k,i) the right-hand side of (39) doubles up. We show the same for the left-hand side of (39).

We can consider $S_{k,i}$ as all the weighted paths from zero level to zero level in k steps with fixed $2i$ (± 1)-s and with the following new weights: (± 2) have weight $\frac{1}{2}$ and (0), (± 1) have weight 1. In $S_{k+1,i+1}$ we have one more step and two more (± 1)-s. Thus from a paths of $S_{k,i}$ we can get a path of $S_{k+1,i+1}$ if we replace a (2) by two (-1)-s (or a (-2) by two (1)-s), or a (0) by (1) and (-1) and there are no other ways. Observe that all these steps double up the weighted sum, because if we replace a (2) by (-1)-s we replace $\frac{1}{2}$ by 1 and when we replace a (0), we can change the places of (1) and (-1), so we get two different paths. Proceeding this way and then omitting the paths we have got more than one times, we get all the paths of $S_{k+1,i+1}$. Conversely, taking two paths from $S_{k+1,i+1}$ such that one can be derived from the other by a change of a pair of (± 1) and replacing them by a (0), or two (1)-s by a (-2), etc., omitting the repetitions again we get all the paths of $S_{k,i}$, thus $S_{k+1,i+1} = 2S_{k,i}$ and together with (40) it gives the statement.

Proof. (of Theorem 1) By arithmetic mean law Lemma 2 ensures for all $k = 0, 1, \dots$

$$\lim_{N\to\infty} \int_{-1}^{1} Q^k(x) d\mu_N(x) = \int_{-1}^{1} Q^k(x) d\omega(x).$$

The linear space

$$\mathcal{A} := \operatorname{span}\left\{Q^k : k \in \mathbb{N}\right\}$$

is also an algebra with unit element. Recalling that W has finite moments, and $\tilde{b}^2$ is in the denominator of W, the zero of $\tilde{b}$ is out of the interval of orthogonality, that is $\tilde{b} = Q'$ has constant sign here and so Q is monotone on $[-1, 1]$ which ensures that $\mathcal{A}$ separates the points of $[-1, 1]$. So by Stone-Weierstrass theorem $\mathcal{A}$ is dense in $C[-1, 1]$. As μ_N for all N and ω are probability measures, the theorem follows from density.

Remark. The main point of this theorem and of Proposition 1 is that on the left-hand side of (36) stands $Q^k P_n$ instead of $x^k P_n$; cf. (14) as well.
The first comment on this, that as the process depends only on Q, that is independent of the number of Darboux transformations which resulted the codimension of the exceptional family, the investigation is restricted to one Darboux transformation case only for sake of simplicity of the notation. Of course, one codimension can be generated just by one Darboux transformation.
The conjecture in connection with Theorem 1 is certanly that $\mu_n \xrightarrow{w^*} \mu_e$ in X_m-Jacobi case too. The complexity of the method above grows with the degree of Q.

5. "Average Characteristic Polynomial"

In this section we look at μ_N (see (33)) from another perspective. It shows up in connection with determinantal point processes. A point process on a Polish space (X) is a probability measure on the space of the boundedly finite counting measures on X; for more details see e.g. [18]. A point process is determinantal if its correlation functon has a determinantal form, cf. (41), below. At this point this notion is related to random matrix theory, see e.g. [**?**].

Without going into this theory deep, we sketch how it relates to average characteristic polynomial, χ_N.
Let $\{P_n\}_{n=0}^{\infty}$ be a standard or an exceptional orthonormal system of polynomials on a real interval I with respect to an appropriate measure μ or weight function W (which is the Radon-Nikodym derivative of μ), that is orthonormal in $L^2(\mu)$ and $\deg P_n \geq n$. Let

$$\mathcal{P}_N := \operatorname{span}\{P_n : n = 0, \dots, N-1\}, \quad \mathcal{P} = \bigcup_{N\geq 1} \mathcal{P}_N$$

and as above,

$$K_N(x, y) := \sum_{k=0}^{N-1} P_k(x) P_k(y).$$

$x_1, \dots, x_N$ are random variables, the joint probability distribution on $\mathbb{R}^N$ is

$$\varrho_{N,n}(x_1, \dots, x_N) = c(n, N) \det |K_n(x_i, x_j)|_{i,j=1}^{N} \prod_{i=1}^{N} W(x_i), \tag{41}$$

where $c(n,N)$ is a normalization factor. The expectation $\mathbb{E}$ refers to ϱ. The average characteristic polynomial is

$$\chi_N(z) := \mathbb{E}\left(\prod_{i=1}^{N}(z-x_i)\right).$$

In standard case, as it is well-known, orthogonal polynomials can be expressed as

$$P_n(x) = c_n \int_{I^n} \prod_{k=0}^{n-1}(x-x_k) \prod_{1\le i<j\le n-1}(x_i-x_j)^2 \prod_{k=0}^{n-1} d\mu(x_k),$$

see [**?**, (2.2.10)]. The factor $\prod_{1\le i<j\le n-1}(x_i-x_j)^2$ is the square of a Vandermonde determinant, V. Expressing $V^2 = \det|K_n(x_i,x_j)|_{i,j=1}^n$, we arrive to χ_N. This correspondence ensures results about limit zero distribution of standard orthogonal polynomials. The known examples show that normalized counting measures based on the zeros of exceptional polynomials tend to the equilibrium measure too. For instance, for some exceptional Laguerre and Jacobi polynomials it is pointed out that the regular zeros tends to the zeros of the corresponding classical polynomials, see e.g. [12]. For some exceptional Hermite polynomials it is proved directly that, after certain normalization, the zero counting measure based on their regular zeros tends to μ_e, see e.g. [**?**]. For special examples there are some more results about the behavior of the zeros, see e.g. [15], [16], [**?**], but a general description is still missing. The connection between χ_N and $P_N^{[1]}$ is a little bit more complicated than above. To get relationship between μ_N and χ_N similar to the standard case, we have to modify the average characteristic polynomial - we denote the modified version with χ_N again.

We define a polynomial s from the corresponding stabilizer ring and $u_{n,j}$ such that

$$sP_n \in \mathcal{P} \quad \forall\ n\in\mathbb{N} \quad \text{and} \quad sP_n = \sum_{j=-L}^{L} u_{n,j}P_{n+j} \tag{42}$$

with some fixed L depending only on s. We also define two mappings acting on $L^2(\mu)$

$$\pi_N : f(x) \mapsto \int_I K_N(x,t)f(t)d\mu(t) \in \mathcal{P}_N, \qquad M : f(x) \mapsto s(x)f(x).$$

Note that in standard case $\deg P_n = n$, $s(x) = x$ as in [13] and [14].

We need the following property of the joint distribution ϱ_N:
Let f be real valued, bounded and measurable function with bounded support contained by I and assume that

$$\sum_{N=0}^{\infty} \frac{\|f\|_\infty^N}{N!} \int_{I^N} \varrho_N(x_1,\dots,x_N)dx_1\dots dx_N < \infty. \tag{43}$$

Then

$$\mathbb{E}\left(\sum_{i_1\ne\dots\ne i_k} f(x_{i_1})\dots f(x_{i_k})\right) \tag{44}$$

$$= c(n,k) \int_{I^k} f(x_1) \dots f(x_k) \det |K_n(x_i, x_j)|_{i,j=1}^k \prod_{i=1}^k d\mu(x_i),$$

see e.g. [18, Proposition 2.2].

The empirical distribution is

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}.$$

In particular, if $k = 1$, $n = N$, then (44) reads

$$\mathbb{E}\left(\int f \hat{\mu}_N \right) = \int_I f(x) \frac{1}{N} K_N(x,x) W(x) dx = \int_I f d\mu_N. \tag{45}$$

We also need

$$\hat{\mu}_N^s = \frac{1}{N} \sum_{i=1}^N \delta_{s(x_i)},$$

and the modified average characteristic polynomial

$$\chi_N(z) := \chi_N(z)^s(z) = \mathbb{E}\left(\prod_{i=1}^N (z - s(x_i)) \right).$$

Denote by z_i the zeros of $\chi_N(z)$ and define

$$\nu_N = \frac{1}{N} \sum_{i=1}^N \delta_{z_i}.$$

After introducing notation above we are in position to state the next theorem.

Theorem 2. *Let I be bounded. If there is a B such that $|u_{n,j}| \le B$ for all n, j, then for all $l \ge 0$*

$$\lim_{N \to \infty} \left| \mathbb{E}\left(\int x^l d\hat{\mu}_N^s(x) \right) - \int x^l d\nu_N(x) \right| = 0.$$

Let Q be defined as in (16). We define y_i (not necessarily unique) so that $z_i = Q(y_i)$, $i = 1, \dots, N$. Let $\tilde{\nu}_N = \frac{1}{N} \sum_{i=1}^N \delta_{y_i}$, that is $\int Q^l(y) d\tilde{\nu}_N(y) = \int z^l d\nu_N(z)$. Proposition 1 and Theorem 2 immediately ensure the next corollary.

Corollary. In the X_m Jacobi case for all $l \ge 0$

$$\lim_{N \to \infty} \left| \int_{-1}^1 Q^l d\mu_N - \int Q^l d\tilde{\nu}_N \right| = 0.$$

The proof of Theorem 2 is based on the next lemma.

Lemma 3.

$$\mathbb{E}\left(\int x^l d\hat{\mu}_N^s(x) \right) = \frac{1}{N} \mathrm{Tr}(\pi_N M^l \pi_N), \tag{46}$$

$$\int x^l d\nu_N(x) = \frac{1}{N} \mathrm{Tr}\left((\pi_N M \pi_N)^l \right). \tag{47}$$

Proof. To apply (44) first we have to check the conditions assumed above in our case. We take $f = s|_I$, which has the required properties. Orthogonality together with Dyson's theorem (see [**?**, Theorem 5.14]) ensures the convergence in (43).
(46): In view of (45)

$$\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N} s^l(x_i)\right) = \int_I s^l(x)\frac{1}{N}K_N(x,x)W(x)dx$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\int_I s^l(x)P_k^2(x)W(x)dx = \frac{1}{N}\sum_{k=0}^{N-1}\langle \pi_N M^l \pi_N P_k, P_k\rangle = \frac{1}{N}\mathrm{Tr}(\pi_N M^l \pi_N).$$

(47): Applying (44)

$$\mathbb{E}\left(\prod_{i=1}^{N}(z - s(x_i))\right) = z^N + \sum_{k=1}^{N}\frac{(-1)^k z^{N-k}}{k!}\mathbb{E}\left(\sum_{i_1\neq\ldots\neq i_k} s(x_{i_1})\ldots s(x_{i_k})\right)$$

$$= z^N + \sum_{k=1}^{N}\frac{(-1)^k z^{N-k}}{k!}\int_{I^k} s(x_1)\ldots s(x_k)\det|K_N(x_i,x_j)|_{i,j=1}^k \prod_{i=1}^{k} W(x_i)dx_i.$$

Considering the Fredholm's expansion of integral operator $\pi_N M \pi_N$ acting on $\mathcal{P}_N$ with kernel $s(y)K_N(x,y)$ we have

$$\det(z - \pi_N M\pi_N) = z^N + \sum_{k=1}^{N}\frac{(-1)^k z^{N-k}}{k!}\int_{I^k}\det|s(x_j)K_N(x_i,x_j)|_{i,j=1}^k \prod_{i=1}^{k} W(x_i)dx_i,$$

cf. [6], from which follows the next equality

$$\det(z - \pi_N M\pi_N) = \chi_N(z) := \mathbb{E}\left(\prod_{i=1}^{N}(z - s(x_i))\right).$$

That is the zeros of $\chi_N(z)$ are the eigenvalues of $\pi_N M \pi_N$, so $\frac{1}{N}\mathrm{Tr}\left((\pi_N M\pi_N)^l\right) = \frac{1}{N}\sum_{i=1}^{N} z_i^l = \int x^l d\nu_N(x)$.

Let

$$D_N := \{(m,n) \in \mathbb{N}^2 : n \geq N\}.$$

The proof of the next lemma is independent of the form of s. The weight of a path, $w(\gamma) = \prod_{e\in\gamma} w(e)$.

Lemma 4. [13, Lemma 2.5, Lemma 2.6] *For all* $l \in \mathbb{N}$

$$\mathrm{Tr}(\pi_N M^l \pi_N) = \sum_{k=0}^{N-1}\sum_{\gamma:(0,k)\to(l,k)} w(\gamma), \tag{48}$$

$$\mathrm{Tr}\left((\pi_N M\pi_N)^l\right) = \sum_{k=0}^{N-1}\sum_{\substack{\gamma:(0,k)\to(l,k)\\ \gamma\cap D_N=\emptyset}} w(\gamma). \tag{49}$$

To make the discussion self-contained we insert the proof of Theorem 2 in brief, cf. [14].

Proof. (of Theorem 2) In view of Lemmas 3 and 4 we estimate

$$\frac{1}{N}\sum_{k=0}^{N-1}\sum_{\substack{\gamma:(0,k)\to(l,k)\\ \gamma\cap D_N\neq\emptyset}} w(\gamma). \tag{50}$$

According to (42), if $k < N - lL$ then $\gamma \cap D_N = \emptyset$. By the assumption on $u_{k,j}$ the weight of an l-long path is at most B^l and the vertices of admissible paths are contained by the set $\{(n,m) : 0 \leq n \leq l,\ N-l \leq m \leq N+l\}$ which has at most $(2l)^l$ elements. That is

$$\frac{1}{N}\sum_{k=0}^{N-1}\sum_{\substack{\gamma:(0,k)\to(l,k)\\ \gamma\cap D_N\neq\emptyset}} w(\gamma) \leq \frac{(2lB)^l}{N}.$$

Remark. The location of the zeros of the modified average characteristic polynomial is not clear yet, it needs further study.

Acknowledgments

The author would like to express her gratitude to the anonymous referee for his/her thorough reading of the manuscript and for the helpful suggestions which were all taken into consideration in the present form of the paper, and also for the ideas and inspiration for further investigation.

References

[1] S. S. Bonan, D. S. Clark, Estimates of the Hermite and the Freud polynomials, *J. Approx. Theory*, **63** (1990), 210-224.

[2] S. Bochner, Über Sturm-Liouvillesche Polynomsysteme, *Math. Z.* **29** (1929), 730-736.

[3] N. Bonneux, Exceptional Jacobi polynomials, *J. Approx. Theory*, **239** (2019), 72-112.

[4] P. A. Clarkson, D. Gómez-Ullate, Y. Grandati, R. Milson, Rational solutions of higher order Painlevé systems I., (2018) arXiv:1811.09274.

[5] A. J. Durán, Higher order recurrence relation for exceptional Charlier, Meixner, Hermite and Laguerre orthogonal polynomials, *Integral Transforms Spec. Funct.*, **26** (2015), 357-376.

[6] I. Gohberg, S. Goldberg and N. Krupnik, *Traces and Determinants of Linear Operators* Birkhäuser, Basel, 2000.

[7] M. Á. García-Ferrero, D. Gómez-Ullate, R. Milson, A Bochner type classification theorem for exceptional orthogonal polynomials, *J. Math. Anal. Appl.*, **472** (1), (2019) 584-626.

[8] D. Gómez-Ullate, Y. Grandati and R. Milson, Rational extensions of the quantum Harmonic oscillator and exceptional Hermite polynomials, *J. Phys. A*, **47** (2014), 015203.

[9] D. Gómez-Ullate, N. Kamran, R. Milson, An extended class of orthogonal polynomials defined by a Sturm-Liouville problem, *J. Math. Anal. Appl.*, **359** (2009), 352-367.

[10] D. Gómez-Ullate, N. Kamran, R. Milson, On Orthogonal Polynomials Spanning a Non-standard Flag, *Algebraic Aspects of Darboux Transformations, Quantum Integrable Systems and Supersymmetric Quantum Mechanics*, Contemp. Math., **563** (2012), 51-72.

[11] D. Gómez-Ullate, A. Kasman, A. B. J Kuijlaars, and R. Milson, Recurrence relations for exceptional Hermite polynomials, *J. Approx. Theory*, **204** (2016), 1-16.

[12] D. Gómez-Ullate, F. Marcellán, R. Milson, Asymptotic and interlacing properties of zeros of exceptional Jacobi and Laguerre polynomials, *J. Math. Anal. Appl.* **399** (2013) 480-495.

[13] A. Hardy, Average characteristic polynomials of determinantal point processes, *Ann. Inst. Henri Poincaré - Probabilités et Statistiques*, **51** (1) (2015), 283-303.

[14] A. Hardy, Polynomial Ensembles and Recurrence Coefficients, *Constr. Approx.* **48** (2018), 137-162.

[15] Á. P. Horváth, The Electrostatic Properties of Zeros of Exceptional Laguerre and Jacobi Polynomials and Stable Interpolation, *J. Approx. Theory*, **194** (2015), 87-107.

[16] Á. P. Horváth, The Energy function with respect to the zeros of exceptional Hermite polynomials, *Acta Math. Sci.*, **37** (5) (2017), 1483-1496.

[17] Á. P. Horváth, Recurrence relation and multi-indexed polynomials of the second kind, *Acta Math. Hungar.*, **157** (2) (2019), 537-550.
[18] K. Johansson, Random matrices and determinantal processes, In *Mathematical Statistical Physics: Lecture Notes of the Les Houches Summer School 2005* Bovier et al. (Eds.). Elsevier, Amsterdam, (2006) MR2581882 1-55.
[19] A. B. J. Kuijlaars, W. Van Assche, The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients, *J. Approx. Theory*, **99** (1999), 167-197.
[20] A. B. J. Kuijlaars, R. Milson, Zeros of exceptional Hermite polynomials, *J. Approx. Theory*, **200** (2015), 28-39.
[21] Y.C. Lun, Behavior of zeros of X_1-Jacobi and X_1-Laguerre exceptional polynomials, (2018) arXiv: 1807.00034.
[22] M. L. Mehta, Random Matrices, 2nd edition, Academic Press, New York, 1991.
[23] A. Máté, P. Nevai and V. Totik, Szegő's extremum problem on the unit circle, *Ann. of Math.*, **134**, (2) (1991), 433-453.
[24] H. Miki, S. Tsujimoto, A new recurrence formula for generic exceptional orthogonal polynomials, *J. Math. Phys.*, **56** (2015), 033502.
[25] P. Nevai, Géza Freud, Orthogonal Polynomials and Christoffel Functions. A Case Study, *J. Approx. Theory* **48**, (1986), 3-167.
[26] S. Odake, Recurrence Relations of the Multi-Indexed Orthogonal Polynomials, *J. Math. Phys.*, **54** (2013), 083506.
[27] S. Odake, Recurrence Relations of the Multi-Indexed Orthogonal Polynomials : II, *J. Math. Phys.* **56** (2015), 053506.
[28] S. Odake, Recurrence Relations of the Multi-Indexed Orthogonal Polynomials : III, *J. Math. Phys.* **57** (2016), 023514.
[29] S. Odake, Recurrence Relations of the Multi-Indexed Orthogonal Polynomials IV: closure relations and creation/annihilation operators, *J. Math. Phys.* **57** (2016), 113503.
[30] S. Odake, R. Sasaki, Infinitely Many Shape Invariant Potentials and New Orthogonal Polynomials, *Phys. Lett. B* **679** (2009), 414.
[31] S. Odake, R. Sasaki, Multi-indexed Wilson and Askey-Wilson polynomials, *J. Phys. A* **46** (2013), 045204.
[32] C. Quesne, Exceptional Orthogonal Polynomials, Exactly Solvable Potentials and Supersymmetry, *J. Phys. A: Math. Theor.* **41** (2008) 392001.
[33] R. Sasaki, S. Tsujimoto, A. Zhedanov, Exceptional Laguerre and Jacobi polynomials and the corresponding potentials through Darboux-Crum transformations, *J. Phys. A: Math. Gen.* **43** (2010), 315204.
[34] B. Simon, The Christoffel-Darboux kernel, Proceedings of Symposia in Pure Mathematics (Perspectives in Partial Differential Equations, Harmonic Analysis and Applications) Ed. D. Mitrea, E, Mitrea *79* (2008) 295-335.
[35] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ. **23.** Amer. Math. Soc. Providence RI, 1975 (fourth edition).
[36] V. Totik, Asymptotics of Christoffel functions on arcs and curves, *Adv. in Math.*, **252** (2014), 114-149.

Department of Analysis,
Budapest University of Technology and Economics
g.horvath.agota@renyi.hu