RECURRENCE RELATION AND MULTI-INDEXED POLYNOMIALS OF THE SECOND KIND

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Abstract. Exceptional orthogonal polynomials fulfil recurrence relations with constant, and with variable dependent coefficients. Considering the second type relations we can define multi-indexed polynomials of the second kind. In some cases they are also exceptional orthogonal polynomials. The other types of multi-indexed polynomials of the second kind are investigated in case of 2-step Darboux transform.

1. Introduction

Exceptional orthogonal polynomials are complete systems of polynomials with respect to a positive measure. They are different from the generalized orthogonal polynomials, for instance Freud or generalized Jacobi polynomials, since they are the polynomial eigenfunctions of a Sturm-Liouville problem as the classical (Hermite, Laguerre, Jacobi) families are cf. [13], [5]. They are also differ from the classical orthogonal polynomials since there are a finite number of degrees for which the second order differential operator in question has no polynomial eigenfunction. That is an exceptional orthogonal polynomial family has finite codimension in the space of polynomials.

Exceptional orthogonal polynomials were introduced recently by D. Gomez-Ullate, N. Kamran and R. Milson, cf. e.g. [7], [10] and the references therein. These families of polynomials play a fundamental role for instance in the construction of bound-state solutions to exactly solvable potentials in quantum mechanics. In the last few years have seen a great deal of activity in this area both by mathematicians and physicists, cf. e.g. [23], [22], [12], [20], [21] [6], [7], [8], [10]. The relationship between exceptional orthogonal polynomials and the Darboux transform is observed by C. Quesne [22]. After constructing higher codimension spaces of exceptional orthogonal polynomials cf. e.g. [20], [14], multi-indexed orthogonal polynomials are constructed by s-step Darboux transform [9], [21].

Classical and generalized orthogonal polynomials satisfy three-term recurrence relations. In monic form the recurrence formula is

$$P_n^{[0]}(x) = (x - d_n)P_{n-1}^{[0]}(x) - c_n P_{n-2}^{[0]}(x) = u_n P_{n-1}^{[0]} - c_n P_{n-2}^{[0]}, \tag{1}$$

where $P_n^{[0]}$ is the nth monic orthogonal polynomial. Moreover according to Favard's theorem [1, Theorem 4.4], arbitrary sequences of complex numbers $\{d_n\}$ and $\{c_n\}$

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define a family of orthogonal polynomials by the corresponding three-term recurrence formula. Exceptional orthogonal polynomials also fulfil some recurrence relations, but higher-term ones. Since formula (1) can be rearranged to

$$xP_n^{[0]}(x) = c_{n+1}P_{n-1}^{[0]}(x) + d_{n+1}P_n^{[0]}(x) + P_{n+1}^{[0]}(x),$$

two types of recurrence formulae are examined with respect to exceptional polynomials; formulae with constant coefficients and formulae with variable dependent coefficients. The constant coefficient cases are investigated for instance in [17], [18], [11], [19], [3], [15]. In this case the coefficients can be polynomials or rational functions of n (see e.g. [17, Example 3] or [11, (40)] for the rational function cases). Note that by some convenient rescalling of exceptional polynomials (cf. [2, Proposition 3.6]) can turn these coefficients into polynomials of n, but the genral closed form expression for them is still not known. Formulas with variable dependent coefficients are investigated for instance in [23], [16], [6].

Below we make some observations with respect to the symmetry properties of recurrence relations of the second type. Denoting by $P_n^{[s]}$ the polynomial which is derived from the classical monic orthogonal polynomial $P_n^{[0]}$ by applying an s-step Darboux transformation to the original differential operator, we have the recurrence relation (cf. [16])

$$\sum_{k=-(s+1)}^{s+1} a_{n,k}^{[s]}(x)P_{n+k}^{[s]} = 0.$$

Our main observation is that the left-hand side of this equality can be divided into groups (in s way) such that the polynomials which are the sums of the elements of the groups, fulfil the recurrence relations of $P_n^{[k]}$, $1 \le k \le s$. Some properties of these new families of polynomials - multi-indexed polynomials of the second kind - are investigated.

2. Preliminaries

$$T_0[y] := py'' + qy' + ry, \quad T_0[P_n^{[0]}] = \lambda_n P_n^{[0]} \tag{2}$$

the Sturm-Liouville differential operator and equation of the classical orthogonal polynomials (cf. e.g. [24]). The eigenfunctions of this operator are orthogonal on the interval in question ((a, b)) with respect to the weight $W_0 = \frac{1}{p}\exp\left(\int^x \frac{q}{p}\right)$, that is

$$\int_a^b P_n^{[0]}P_m^{[0]}W_0 = c(n)\delta_{m,n}.$$

A second order linear differential operator, $T[y] = py'' + qy' + ry$ with rational coefficients has a rational factorization and a partner operator $\hat{T}$ as it follows (cf. [5, Propositions 3.5 and 3.6]). Let Φ be a quasi-rational (i.e. $\frac{\Phi'}{\Phi}$ is rational) eigenfunction of T with eigenvalue λ and let b be an arbitrary non-zero rational function. Let $w = \frac{\Phi'}{\Phi}$, $\hat{b} = \frac{p}{b}$, $\hat{w} = -w - \frac{q}{p} + \frac{b'}{b}$.

$$T = BA + \tilde{\lambda}, \text{ with } A[y] = b(y' - wy), \; B[y] = \hat{b}(y' - \hat{w}y),$$

and

$$\hat{T}[y] = (AB + \tilde{\lambda})[y] = py'' + \hat{q}y' + \hat{r}y,$$

where

$$\hat{q} = q + p' - 2\frac{b'}{b}p,\ \ \hat{r} = r + q' + wp' - \frac{b'}{b}(q+p') + \left(2\left(\frac{b'}{b}\right)^2 - \frac{b''}{b} + 2w'\right)p.$$

So let

$$T_0 = B_1 A_1 + \tilde{\lambda}_1.$$

$$T_1 = \hat{T}_0 = A_1 B_1 + \tilde{\lambda}_1, \quad T_1 A_1 = A_1 T_0, \quad T_1 A_1 P_n^{[0]} = \lambda_n A_1 P_n^{[0]}.$$

Continuing this procedure s times we obtain

$$T_s A_s A_{s-1} \dots A_1 = A_s A_{s-1} \dots A_1 T_0, \quad T_s A_s A_{s-1} \dots A_1 P_n^{[0]} = \lambda_n A_s A_{s-1} \dots A_1 P_n^{[0]}.$$

$$A_k[y] = b_k(y' - w_k y).$$

Let us denote by

$$A_1 P_n^{[0]} =: P_n^{[1]}, \dots A_s P_n^{[s-1]} =: P_n^{[s]}. \tag{3}$$

$P_n^{[k]}$-s are the exceptional orthogonal polynomials. The degree of $P_n^{[k]}$ is $n_k > n$ if $k > 0$. $P_n^{[k]}$-s are orthogonal on (a,b) with respect to the weight $W_k := \frac{p^k W_0}{(b_1 \dots b_k)^2}$ and the set has finite codimension in the set of polynomials. Furtheremore $\{P_n^{[k]}\}_{n=0}^{\infty}$ is a complete system in the weighted Hilbert space in question, cf. [5] and the references therein.

A simple computation shows that

$$P_n^{[1]} = A_1 P_n^{[0]} = A_1(u_n P_{n-1}^{[0]}) - A_1 c_n P_{n-2}^{[0]} = u_n P_{n-1}^{[1]} - c_n P_{n-2}^{[1]} - b_1 P_{n-1}^{[0]}. \tag{4}$$

Denoting by $(e:n)$ the equation above and by $(e:n+1)$ and $(e:n-1)$ the similar equations with respect to $P_{n+1}^{[1]}$ and $P_{n-1}^{[1]}$ respectively, we have

$$(e:n+1) - u_n(e:n) + c_n(e:n-1)$$

$$= P_{n+1}^{[1]} - (u_n + u_{n+1})P_n^{[1]} + (c_n + c_{n+1} + u_n^2)P_{n-1}^{[1]} - c_n(u_n + u_{n-1})P_{n-2}^{[1]} + c_n c_{n-1} P_{n-3}^{[1]}$$

$$= \left(P_{n+1}^{[1]} - u_{n+1}P_n^{[1]} + c_{n+1}P_{n-1}^{[1]}\right) - u_n\left(P_n^{[1]} - u_n P_{n-1}^{[1]} + c_n P_{n-2}^{[1]}\right)$$

$$+ c_n\left(P_{n-1}^{[1]} - u_{n-1}P_{n-2}^{[1]} + c_{n-1}P_{n-3}^{[1]}\right)$$

$$= Q_{n+1}^{[1]} - u_n Q_n^{[1]} + c_n Q_{n-1}^{[1]} = 0,$$

which is exactly the five term recurrence formula for one-step exceptional orthogonal polynomials, cf. [16, (3.13)]. This formula can be expressed as a three term relation with respect to the new polynomials $Q_n^{[1]}$. These new polynomials can be derived immediately from the classical orthogonal polynomials as it follows. Taking into consideration that $(P_n^{[0]})' = P_{n-1}^{[0]} + u_n(P_{n-1}^{[0]})' - c_n(P_{n-2}^{[0]})'$, for $n \geq 1$ we have

$$Q_n^{[1]} = A_1 P_n^{[0]} - u_n A_1 P_{n-1}^{[0]} + c_n A_1 P_{n-2}^{[0]}$$

$$= b_1\left((P_n^{[0]})' - u_n(P_{n-1}^{[0]})' + c_n(P_{n-2}^{[0]})'\right) - b_1 w_1\left(P_n^{[0]} - u_n P_{n-1}^{[0]} + c_n P_{n-2}^{[0]}\right)$$

$$= b_1 P_{n-1}^{[0]}.$$

That is $Q_1^{[1]} = b_1$, the degree of $Q_n^{[1]}$ is $n-1+$ the degree of b_1 and $Q_n^{[1]}$-s are orthogonal with respect to $\frac{W_0}{b_1^2}$ and $\{Q_n^{[1]}\}_{n=1}^{\infty}$ is a complete system in $L^2_{\frac{W_0}{b_1^2}}$.

To prove the same for multi-indexed exceptional polynomials we rewrite the recurrence formula of S. Odake into monic form. Expressing (1) as

$$\sum_{k=-1}^{1} v_{n,k}^{[0]} P_{n+k}^{[0]} = 0, \tag{5}$$

we have

Proposition 1. [16, (3.10), (3.11), (3.13)]

$$\sum_{k=-2s-1}^{1} v_{n,k}^{[s]} P_{n+k}^{[s]} = 0, \tag{6}$$

where

$$v_{n,k}^{[s]} = v_{n,k}^{[s-1]} - u_{n+1-s} v_{n-1,k+1}^{[s-1]} + c_{n+1-s} v_{n-2,k+2}^{[s-1]} \tag{7}$$

and

$$(v_{n,k}^{[s]})' = -(s+1) v_{n-1,k+1}^{[s-1]}. \tag{8}$$

To make the presentation complete, we give this proof here. The first step of the induction (with respect to s) is given above, that is with $s = 0$ (6), with $s = 1$ (6), (7), (8) are satisfied for all $n \geq 0$ with the remark that if $l < 0$ $P_l^{[s]} = 0$ $\forall s \geq 0$, and if $k > 1$ or $k < -2s-1$ then $v_{n,k}^{[s]} = 0$ $\forall s, n \geq 0$. Supposing that (6), (7), (8) are satisfied for $s-1$ and repeating the previous arguments we apply A_s to the recurrence formula (6) (with $s-1$). Taking into consideration again that $A_k(vP) = vA_kP + b_k v'P$ and (8)

$$0 = A_s\left(\sum_{k=-2(s-1)-1}^{1} v_{n,k}^{[s-1]} P_{n+k}^{[s-1]}\right)$$

$$= \sum_{k=-2(s-1)-1}^{1} v_{n,k}^{[s-1]} P_{n+k}^{[s]} - sb_s \sum_{l=-2(s-2)-1}^{1} v_{n-1,l}^{[s-2]} P_{n-1+l}^{[s-1]}. \tag{9}$$

Denoting by $(e1:n+1)$ the equation above and by $(e1:n)$ and $(e1:n-1)$ the similar equations with respect to $P_n^{[s-1]}$ and $P_{n-1}^{[s-1]}$, we have

$$0 = (e1:n+1) - u_{n+1-s}(e1:n) + c_{n+1-s}(e1:n-1) \tag{10}$$

$$= \sum_{k=-2(s-1)-1}^{1} \left(v_{n,k}^{[s-1]} P_{n+k}^{[s]} - u_{n+1-s} v_{n-1,k}^{[s-1]} P_{n-1+k}^{[s]} + c_{n+1-s} v_{n-2,k}^{[s-1]} P_{n-2+k}^{[s]}\right)$$

$$-sb_s \sum_{l=-2(s-2)-1}^{1} \left(v_{n-1,l}^{[s-2]} P_{n-1+l}^{[s-1]} - u_{n+1-s} v_{n-2,l}^{[s-2]} P_{n-2+l}^{[s-1]} + c_{n+1-s} v_{n-3,l}^{[s-2]} P_{n-3+l}^{[s-1]}\right)$$

$$= \sum_{k=-2(s-1)-1}^{1} (v_{n,k}^{[s-1]} - u_{n+1-s} v_{n-1,k+1}^{[s-1]} + c_{n+1-s} v_{n-2,k+2}^{[s-1]}) P_{n+k}^{[s]}$$

$$-sb_s \sum_{k=-2(s-2)-1}^{1} \left(v_{n-1,k}^{[s-2]} - u_{n+1-s} v_{n-2,k+1}^{[s-2]} + c_{n+1-s} v_{n-3,k+2}^{[s-2]}\right) P_{n-1+k}^{[s-1]}$$

$$= \sum_{k=-2s-1}^{1} v_{n,k}^{[s]} P_{n+k}^{[s]}.$$

We used that by the assumption the second term is zero, so (6) and (7) are proved and it ensures (8).

3. Multi-indexed polynomials of the second kind

Theorem 1. (6) *can be expressed in the following equivalent form:*

$$Q_{n+1}^{[s]} - u_{n+1-s} Q_n^{[s]} + c_{n+1-s} Q_{n-1}^{[s]} = 0, \tag{11}$$

where

$$Q_n^{[s]} = \sum_{k=-2(s-1)-1}^{1} v_{n-1,k}^{[s-1]} P_{n-1+k}^{[s]}. \tag{12}$$

Moreover for $n \geq s$

$$Q_n^{[s]} = s! \left(\prod_{k=1}^{s} b_k \right) P_{n-s}^{[0]}. \tag{13}$$

Denoting by $S = \prod_{k=1}^{s} b_k$ we have the following

Corollary. $\{Q_n^{[s]}\}_{n=s}^{\infty}$ is an exceptional (co-finite, real-valued Sturm-Liouville) orthogonal polynomial system (cf. [5, Definition 7.1]), that is

$$\int_a^b Q_k^{[s]} Q_n^{[s]} \frac{W_0}{S^2} = 0 \quad k \neq n, \tag{14}$$

$Q_n^{[s]}$ fulfils the differential equation

$$py'' + \left(-2p\frac{S'}{S} + q\right) y' + \left(-p\frac{S''}{S} + 2p\left(\frac{S'}{S}\right)^2 - q\frac{S'}{S} + r\right) y = \lambda_{n-s} y, \tag{15}$$

and $\{Q_n^{[s]}\}_{n=s}^{\infty}$ is a closed system in $L^2_{\frac{W_0}{S^2}}$.

Proof. With the notation above we can express (9) as

$$0 = Q_{n+1}^{[s]} - s b_s Q_n^{[s-1]}. \tag{16}$$

Then the equation (10) is equivalent with (11). (16) immediately ensures (13) and then (14). Since $b_1 \ldots b_s \neq 0$ on (a, b), the last statement can be derived immediately from (2).

Remark. Generally if a sequence of polynomials fulfills the recurrence formula $p_n(x) = (x - D_n)p_{n-1}(x) - C_n p_{n-2}(x)$ with $\deg p_0 = n_0 > 0$ $(p_{-1} \equiv 0)$, then for all $n > 0$ p_n is divisible by p_0 that is $p_n = p_0 q_n$ and $\deg p_n = n + n_0 = N$. q_n-s are polynomials of degree n and they fulfil the same recurrence formula as p_n-s. So if p_0 is "nice", to find a measure μ, $\{p_n\}$ to be orthogonal with respect to μ is equivalent to find a measure $\tilde{\mu}$, $\{q_n\}$ to be orthogonal with respect to $\tilde{\mu}$.

Indeed by standard arguments (cf. e.g. [1]), we define a "moment functional" as it follows. For a sequence of (complex) numbers let us define a (complex valued) linear functional on the space of polynomials $Pp_0 := \{\pi p_0 : \pi \text{ is a polynomial}\}$ $L(x^k p_0) = \mu_k$, $L(a\pi_1 p_0 + b\pi_2 p_0) = aL(\pi_1 p_0) + bL(\pi_2 p_0)$. For the sequences $\{C_k\}$ and $\{D_k\}$ there is a unique "moment functional" L such that $L(p_0^2) = C_1$, $L(p_m p_n) = 0$

if $m \neq n$: let $L(p_0^2) = C_1 = \mu_0$, and we define "moments" μ_k recursively such that $L(p_n p_0) = 0$ if $n > 0$: $0 = L(p_1 p_0)$ if $\mu_1 = L(xp_0^2) = D_1\mu_0$, etc., $\mu_k = L(x^k p_0^2)$ is given by $C_1, \dots, C_k, D_1, \dots, D_k, \mu_0, \dots, \mu_{k-1}$ such that $L(p_k p_0) = 0$. Thus, since $xp_n = D_{n+1}p_n + C_{n+1}p_{n-1} + p_{n+1}$, $L(xp_n p_0) = 0$ if $L(p_{n-1}p_0) = L(q_{n-1}p_0^2) = 0$ that is if $n > 1$, and recursively $L(x^k p_n p_0) = 0$ if $n > k \geq 0$. Thus if $m \neq n$ $0 = L(q_m p_n p_0) = L(p_m p_n)$. Moreover $L(p_n^2) = L(q_n p_n p_0) = L(x^n p_n p_0) = C_{n+1}L(x^{n-1}p_{n-1}p_0) = \dots = \prod_{k=1}^{n+1} C_k$. Obviously if the sequences $\{C_k\}$ and $\{D_k\}$ define a measure $d\tilde{\mu}(x) = w_0(x)dx$ on (a, b) where $w_0 > 0$ (a, b) for which q_n-s are orthogonal and p_0 is not zero on (a, b), then $d\mu = \frac{d\tilde{\mu}}{p_0^2} = d\mu = \frac{w_0(x)dx}{p_0^2(x)}$.

Theorem 2. (6) *can be expressed in the following equivalent form:*

$$\sum_{k=-2(s-l)-1}^{1} v_{n-l,k}^{[s-l]} Q_{n+k}^{[l,s]} = 0, \tag{17}$$

where

$$Q_n^{[0,s]} = P_n^{[s]},\ Q_n^{[s,s]} = Q_n^{[s]},\ Q_n^{[l,s]} = \sum_{k=-1}^{1} v_{n-l,k}^{[0]} Q_{n-1+k}^{[l-1,s]},\ \ 1 < l < s,\ \ n \geq l. \tag{18}$$

Proof. For $l = 0$ (17) coincides with (6). To prove (17) for l, by (18) we have to show that

$$0 = \sum_{k=-2(s-l)-1}^{1} \sum_{j=-1}^{1} v_{n-l,k}^{[s-l]} v_{n+k-l,j}^{[0]} Q_{n+k-1+j}^{[l-1,s]}$$

$$= \sum_{p=-2(s-l+1)-1}^{1} \left(\sum_{\substack{-1 \leq j \leq 1 \\ k+j-1=p}} v_{n-l,k}^{[s-l]} v_{n+k-l,j}^{[0]} \right) Q_{n+p}^{[l-1,s]}.$$

Supposing that (17) is valid for $l - 1$ it is enough to show that

$$v_{n+1,p}^{[s+1]} = v_{n,p}^{[s]} - u_{n+p+2}v_{n,p+1}^{[s]} + c_{n+p+3}v_{n,p+2}^{[s]}. \tag{19}$$

Comparing (19) with (7) one can check that for $s = 0$ (19) is valid, that is $v_{n+1,-3}^{[1]} = c_n c_{n+1}$, $v_{n+1,-2}^{[1]} = -c_{n+1}(u_n + u_{n+1})$, $v_{n+1,-1}^{[1]} = c_{n+1} + c_{n+2} + u_{n+1}^2$, $v_{n+1,0}^{[1]} = -(u_{n+1} + u_{n+2})$, $v_{n+1,1}^{[1]} = 1$. Now by induction again, we can show that

$$v_{n,p}^{[s]} - u_{n+p+2}v_{n,p+1}^{[s]} + c_{n+p+3}v_{n,p+2}^{[s]} = v_{n+1,p}^{[s]} - u_{n+1-s}v_{n,p+1}^{[s]} + c_{n+1-s}v_{n-1,p+2}^{[s]}. \tag{20}$$

Applying (7) to the left-hand side of (20), and by the induction assumption (19) (with $s - 1$) to the right-hand side of (20) one can verify the equality.

Remark. Comparing (17) and (6) we can observe that $Q_n^{[l,s]}$ and $P_{n-l}^{[s-l]}$ fulfil the same recurrence formula. As a second solution of the recurrence relation, $Q_n^{[l,s]}$-s can be called multi-indexed polynomials of the second kind. Considering the case of $Q_n^{[s,s]}$ the question is immediately arisen whether $Q_n^{[l,s]}$ is different from $P_{n-l}^{[s-l]}$ only in a factor which is independent of n, as in the case $l = s$. The situation is different as we see below.

4. Example: $s = 2$.

Recalling the Crum-Darboux decomposition above and (13) $Q_n^{[2,2]} = Q_n^{[2]} = 2b_1b_2P_{n-2}^{[0]}$, $n \geq 2$ and it fulfils the three term recurrence relation (1). According to (18) $Q_n^{[0,2]} = P_n^{[2]}$, $n \geq 0$ and it fulfils the seven term recurrence relation (6). $Q_n^{[2,2]}$ and $Q_n^{[0,2]}$ are both exceptional orthogonal polynomials. By (17) $Q_{n+1}^{[1,2]}$ fulfils the same five term recurrence relation as $P_n^{[1]}$, $n \geq 0$. The difference is in the initial values. Since

$$P_n^{[1]} = A_1P_n^{[0]}, \quad A_1[y] = b_1(y' - w_1y),$$

$(P_{-1}^{[1]} = 0)$, $P_0^{[1]} = -b_1w_1$, $P_1^{[1]} = b_1(1 - w_1u_1)$, $P_2^{[1]} = b_1(u_1 + u_2 - w_1(u_1u_2 - c_2))$, and by (18) and (17):
$(Q_0^{[1,2]} = 0)$, $Q_1^{[1,2]} = b_1b_2\left(\frac{b_1'}{b_1} - w_1 - w_2\right)$, $Q_2^{[1,2]} = b_1b_2\left(2 + u_1\left(\frac{b_1'}{b_1} - w_1 - w_2\right)\right)$, $Q_3^{[1,2]} = b_1b_2\left(2(u_1+u_2) + (u_1u_2 - c_2)\left(\frac{b_1'}{b_1} - w_1 - w_2\right)\right)$. That is we have

Proposition 2.

$$Q_{n+1}^{[1,2]} = RP_n^{[0]}, \quad R[y] = 2b_1b_2(y' - My), \quad M = \frac{1}{2}\left(w_1 + w_2 - \frac{b_1'}{b_1}\right). \tag{21}$$

Proof. By (17)

$$Q_{n+1}^{[1,2]} = -\sum_{k=-3}^{0} v_{n-1,k}^{[1]}Q_{n+k}^{[1,2]} = -\sum_{k=-3}^{0} v_{n-1,k}^{[1]}RP_{n+k-1}^{[0]}$$

$$= -2b_1b_2\sum_{k=-3}^{0} v_{n-1,k}^{[1]}\left(\left(P_{n+k-1}^{[0]}\right)' - MP_{n+k-1}^{[0]}\right)$$

$$= -2b_1b_2\left(\left(\sum_{k=-3}^{0} v_{n-1,k}^{[1]}P_{n+k-1}^{[0]}\right)' - \sum_{k=-3}^{0}\left(v_{n-1,k}^{[1]}\right)'P_{n+k-1}^{[0]}\right)$$

$$+2b_1b_2M\sum_{k=-3}^{0} v_{n-1,k}^{[1]}P_{n+k-1}^{[0]}.$$

By (8) and (1) the second term in the last bracket is zero:
$\sum_{k=-3}^{0}\left(v_{n-1,k}^{[1]}\right)'P_{n+k-1}^{[0]} = -2\sum_{l=-1}^{1} v_{n-2,l}^{[0]}P_{n-2+l}^{[0]} = 0$. By (7)

$$-\sum_{k=-3}^{0} v_{n-1,k}^{[1]}P_{n+k-1}^{[0]} = -\sum_{k=-1}^{0} v_{n-1,k}^{[0]}P_{n+k-1}^{[0]} + u_{n-1}\sum_{l=-1}^{1} v_{n-2,l}^{[0]}P_{n-2+l}^{[0]}$$

$$+c_{n-1}\sum_{l=-1}^{1} v_{n-3,l}^{[0]}P_{n-3+l}^{[0]} = P_n^{[0]} + 0 + 0,$$

that is

$$Q_{n+1}^{[1,2]} = 2b_1b_2\left(\left(P_n^{[0]}\right)' - MP_n^{[0]}\right),$$

which is (21).

To find an operator B with which $T_0 = BR + \lambda$, M has to fulfil the Riccati equation $p(M' + M^2) + qM + r = \lambda$ (cf. e.g. [5, Proposition 3.5]). Taking into consideration that

$$p(w_1' + w_1^2) + qw_1 + r = \tilde{\lambda}_1,$$

and

$$p(w_2' + w_2^2) + \hat{q}w_2 + \hat{r} = p(w_2' + w_2^2) + \left(q + p' - 2p\frac{b_1'}{b_1}\right) w_2$$

$$+r + q' + w_1 p' - \frac{b_1'}{b_1}(q + p') + p\left(2\left(\frac{b_1'}{b_1}\right)^2 - \frac{b_1''}{b_1} + 2w_1'\right) = \tilde{\lambda}_2,$$

we have

$$p(M' + M^2) + qM + r = \frac{1}{2}(\tilde{\lambda}_1 + \tilde{\lambda}_2 - q') + \frac{1}{2}f, \tag{22}$$

where

$$f = -\frac{p}{2}\left(w_1 - w_2 + \frac{b_1'}{b_1}\right)^2 - (pw_1)' + p'\frac{b_1'}{b_1} - pw_1'. \tag{23}$$

That is if $B[y] = \frac{p}{2b_1b_2}\left(y' + \left(M + \frac{q}{p} - \frac{(b_1b_2)1}{b_1b_2}\right)y\right)$,

$$T_0[y] = BR[y] + \frac{1}{2}(\tilde{\lambda}_1 + \tilde{\lambda}_2 - q')y + \frac{1}{2}fy = \lambda_n y,$$

and $Q_{n+1}^{[1,2]} = RP_n^{[0]}$ satisfies the inhomogeneous differential equation

$$(RB + \frac{1}{2}(\tilde{\lambda}_1 + \tilde{\lambda}_2 - q') + \frac{1}{2}f - \lambda_n)y = -b_1b_2f'P_n^{[0]}. \tag{24}$$

Since

$$RB[y] = py'' + \left(p' + q - 2p\frac{(b_1b_2)'}{b_1b_2}\right)y'$$

$$+\left((pM)' - pM^2 + q' - qM - (q+p)\frac{(b_1b_2)'}{b_1b_2} + p\frac{(b_1b_2)''}{b_1b_2}\right)y = py'' + Hy' + Ky,$$

multiplying (24) by h_n we have

$$(RB+\frac{1}{2}f+\frac{1}{2}(\tilde{\lambda}_1+\tilde{\lambda}_2-q'))(h_ny)-(ph_n''y-2ph_n'y'-Hh_n'y) = \lambda_n(h_ny)-b_1b_2f'h_nP_n^{[0]}.$$

That is we have the following

Proposition 3. *Let h_n be the solution of the following linear differential equation of the second order*

$$pQ_{n+1}^{[1,2]}y'' + \left(2p\left(Q_{n+1}^{[1,2]}\right)' + HQ_{n+1}^{[1,2]}\right)y' - b_1b_2f'P_n^{[0]}y = 0. \tag{25}$$

Then

$$\mathcal{R}_n := h_nQ_{n+1}^{[1,2]}$$

is the eigenfunction of the differential operator $T^{[1,2]}$ with eigenvalue λ_n, where

$$T^{[1,2]} = RB + \frac{1}{2}f + \frac{1}{2}(\tilde{\lambda}_1 + \tilde{\lambda}_2 - q'). \tag{26}$$

According to [5, Proposition 2.2] the eigenfunctions $\mathcal{R}_n$ are orthogonal with respect to the weight $w = \frac{pW_0}{(b_1b_2)^2}$, provided that $pw\mathcal{R}_n'\mathcal{R}_m$ tends to zero at the endpoint of the interval. Unfortunately h_n-s are not necessarily polynomials.

4.1. **An example with Laguerre weight.** In this subsection we give a numerical example of the procedure above, and verify that eigenfunctions $\mathcal{R}_n$ form an orthogonal system.

Based on the results of [9] we investigate $\mathcal{R}_n$ with respect to exceptional Laguerre polynomials are given by a special two-step Darboux transform. To this end we have to examine the order of h_n at infinity. First we rewrite the differential equation (25). Recalling (21) $Q_{n+1}^{[1,2]} = 2b_1b_2\left(\left(P_n^{[0]}\right)' - MP_n^{[0]}\right)$ and $b_i \neq 0$, dividing the equation by $2b_1b_2$ and taking into consideration the differential equation of $P_n^{[0]}$, we have

$$p\left(\left(P_n^{[0]}\right)' - MP_n^{[0]}\right)y''$$
$$+\left((-3q-2pM+p')\left(P_n^{[0]}\right)' + (r-(pM)'+Mq-\lambda_n)P_n^{[0]}\right)y' - \frac{1}{2}f'P_n^{[0]}y = 0.$$

Now introducing the notation

$$\Theta_n := h_n\sqrt{\varrho_n},$$

where

$$\varrho_n = \exp\left(\int^x \frac{(-3q-2pM+p')\left(P_n^{[0]}\right)' + (r-(pM)'+Mq-\lambda_n)P_n^{[0]}}{p\left(\left(P_n^{[0]}\right)' - MP_n^{[0]}\right)}\right),$$

(25) is transformed to

$$u'' = \left(\frac{\frac{1}{2}f'P_n^{[0]}}{p\left(\left(P_n^{[0]}\right)' - MP_n^{[0]}\right)} + \frac{\sqrt{\varrho_n}''}{\sqrt{\varrho_n}}\right)u, \tag{27}$$

which is satisfied by Θ_n.

Let $T_0[y] = xy'' + (\alpha+1-x)y'$ the Laguerre differential operator on $(0,\infty)$. Recall that $T_0 = B_1A_1 + \tilde{\lambda}_1$, $T_1 = \hat{T}_0 = A_1B_1 + \tilde{\lambda}_1 = B_2A_2 + \tilde{\lambda}_2$. By the notations of [9] with $m_1 = 1, m_2 = 2$ here $A_i[y] = b_i(y' - w_iy)$, where

$$b_1(x) = xL_1^{(-\alpha)}(x);\ \ w_1(x) = -\frac{1}{L_1^{(-\alpha)}(x)} - \frac{\alpha}{x};$$

$$b_2(x) = \frac{x\eta(x)}{L_1^{(-\alpha)}(x)};\ \ w_2 = \frac{L_1^{(-\alpha)}(x)}{\eta(x)} - \frac{\alpha-1}{x},$$

where

$$L_1^{(-\alpha)}(x) = -x-\alpha+1,\quad \eta(x) = -\frac{1}{2}x^2 - (\alpha-1)x - \frac{1}{2}(\alpha^2-3\alpha+2).$$

Thus

$$M(x) = \frac{1}{2}\frac{L_1^{(-\alpha)}(x)}{\eta(x)} - \frac{\alpha}{x} = -\frac{\alpha-1}{2x} + O\left(\frac{1}{x^2}\right).$$

With $P_n^{[0]} = (-1)^n n! L_n^{(\alpha)}$ (cf. [24])

$$p\left(\left(P_n^{[0]}\right)' - MP_n^{[0]}\right) = \left(n + \frac{\alpha-1}{2}\right)x^n + O\left(x^{n-1}\right),$$

$$(-3q-2pM+p')\left(P_n^{[0]}\right)'+(r-(pM)'+Mq-\lambda_n)P_n^{[0]}$$

$$=3nx^n+O\left(x^{n-1}\right)+\left(n+\frac{\alpha-1}{2}\right)x^n+O\left(x^{n-1}\right)=\left(4n+\frac{\alpha-1}{2}\right)x^n+O\left(x^{n-1}\right),$$

That is at infinity

$$\sqrt{\varrho_n}(x)\sim xe^{2x}.$$

Now we estimate the coefficient of u in (27). Since

$$w_1-w_2+\frac{b_1'}{b_1}=-\frac{1}{x}-\frac{L_1^{(-\alpha)}(x)}{\eta(x)}-\frac{2x+\alpha}{L_1^{(-\alpha)}(x)}=2+O\left(\frac{1}{x}\right);\ \left(w_1-w_2+\frac{b_1'}{b_1}\right)'=O\left(\frac{1}{x^2}\right),$$

$$-2w_1'-p(w_1''+w_1')+\frac{b_1'}{b_1}=O\left(\frac{1}{x}\right),$$

$$f'=-1+O\left(\frac{1}{x}\right).$$

Thus by the previous computations the coefficient of u in (27) is $4-\frac{1}{2n+\alpha^1}+O\left(\frac{1}{x}\right)$, that is according to [4, Theorem 2] the order of the solutions of (27) at infinity is at most $p(x)e^{2x}$, where p is a polynomial. This ensures that h_n grows at most polynomially at infinity, that is $\mathcal{R}_n$-s satisfy the conditions at the endpoints of the interval, so we have an orthogonal system.

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