RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 68 (2002) pp. 491-511

CHARACTERIZATION OF FOURIER SERIES WITH (C, 1) MEANS[1]

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Introduction

Let $(a,b) \subset \mathbf{R}$ is a finite or an infinite interval on the real line, and let $w(x)$ be a non-negative function whose support is in (a,b), and is measurable in Lebesgue's sense, and for which $\int_a^b w(x)dx > 0$, moreover w^2 has finite moments, that is for every $k \in \mathbf{N}$ $\int_a^b x^k w^2(x)dx < \infty$. In the followings the set of these kind of functions will be denoted by W, and our weight functions will be $w \in W$. Let $p_{w^2,n}(x)$ be the n^{th} orthonormal polynomial with respect to the weight $w^2(x)$, that is

$$\int_a^b p_{w^2,n}(x)p_{w^2,m}(x)w^2(x)dx = \delta_{n,m} \tag{1}$$

After these preliminaries we can construct the Fourier series of a measurable function $f(x)$ if the corresponding c_k-s exist:

$$f(x) \sim \sum_{k=0}^{\infty} c_k p_{w^2,k}(x), \quad \text{where} \quad c_k = c_k(f,w) = \int_a^b f(x)p_{w^2,k}(x)w^2(x)dx. \tag{2}$$

With the above notations we have

$$S_k(f,x) = \sum_{i=0}^{k} c_k p_{w,k}(x), \tag{3}$$

is the k^{th} partial sum of the Fourier series of f with respect to the orthogonal polynomials for w,

$$\sigma_n(f,x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(f,x), \tag{4}$$

[1]This paper is in final form and no version of it will be submitted for publication elsewhere.

[2]Research was supported by OTKA No. T32872 and T32374

is the first Cesàro mean of f.

$\gamma_k > 0$ is the highest coefficient of the k^{th} orthonormal polynomial.

Definition 1 (a) *The Christoffel function is*

$$\lambda_n(w^2, x) = \inf_{p\in\Pi_n} \int_a^b \frac{p^2(t)w^2(t)dt}{p^2(x)}, \quad x \in (a,b). \tag{5}$$

(b) *The weighted norm of the reciprocal of the Christoffel function is denoted by*

$$\Lambda_n = \|\lambda_n^{-1}(w^2, x)w^2(x)\|_\infty, \tag{6}$$

where the infinity norm is on (a, b).

As it is wellknown (e.g. [4]),

$$\Lambda_n(x) = w^2(x) \sum_{k=0}^{n} p_{w^2,k}^2(x), \tag{7}$$

and

$$\Lambda_n = \|\Lambda_n(x)\|_\infty . \tag{8}$$

Now we can formulate our problem: if we have a formal sum in some weighted space, how can we recognize that it is the Fourier series of a certain function of that space. For the investigation of this question we need a lemma on Cèsaro sums in the weighted cases, which was proven at first by G. Freud in 1973, for the weights $w_k(x) = e^{-\frac{1}{2}x^{2k}}$. His result is the following:

Theorem 1 (3, L. 6, Th. 3) *We have*

$$\frac{1}{n} w_k(x) \sum_{l=0}^{n-1} |S_l(w_k^2, f, x)| \le C\|w_k f\|_\infty,$$

and so we have for every $1 \le p \le \infty$

$$\|\sigma_n(w_k^2, f)w_k\|_p \le C(p)\|f w_k\|_p.$$

This lemma appears in several papers for different weights. For generalized Jacobi weights the proof was given by Nguyen Xuan Ky and F. Schipp in 1986 [16, L. 3], and for Erdős weights it was given by D. S. Lubinsky and T. Z. Mthembu in 1992 [13 Th. 2.1]. The most recent result of D. Mache and D. S. Lubinsky [13] gives a general proof of Freud's lemma for exponential weights, and they gave an improvement of it in exponential cases which works better than the oroginal form close to the endpoints of the main interval of approximation. But it is not clear how does it work in Jacobi case. Introducing some new notations and a function (different from [13]), we are able to give a common proof for the results in question, furthermore for any weights fulfilling the natural assumptions in the introduction.

Results

Let

$$\tau(x) : (a, b) \longrightarrow (a, b)$$

an increasing bijection wich is piecewise differentiable. Let us denote

$$f_\tau(x) = f(x)(\tau'(x))^{-\frac{1}{2}} \tag{9}$$

for an arbitrary function f. Furthermore let us suppose that, applying the above notation, $w_\tau \in W$ is a weight function too, and $(\tau')^{-1}$ is bounded on (a, b). Let

$$\Lambda_{\tau,n} = \|\lambda_n^{-1}(w^2, x)w_\tau^2(x)\|_\infty. \tag{10}$$

We define the weighted L_p space as follows:

$$f \in L_{w,p}, \text{ if } fw \in L_p, \quad \|f\|_{w,p} = \|fw\|_p, \quad 1 \le p \le \infty, \tag{11}$$

and let us denote by

$$\Gamma_n = \sup_{k \le n} \frac{\gamma_k}{\gamma_{k+1}}. \tag{12}$$

Then we have

Lemma 1 *Let $(a, b) \subset \mathbf{R}$, and $w \in W$. If with a τ given above, there is a constant K such that*

$$\frac{\Gamma_n \Lambda_{\tau,n}}{n} < K, \tag{13}$$

then we have

$$\|\sigma_n(f_\tau)\|_{w_\tau,\infty} \le C\|f\|_{w_\tau,\infty}, \text{ for } f \in L_{w_\tau,\infty}, \tag{14}$$

and

$$\|\sigma_n(f_\tau)\|_{w,1} \le C\|f\|_{w,1}, \text{ for } f \in L_{w,1}, \tag{15}$$

and

$$\|\sigma_n(f_\tau)\|_{w_\tau,p} \le C(p)\|f\|_{w,p}, \text{ for } f \in L_{w,p} \; 1 < p < \infty. \tag{16}$$

Remark. The last inequality holds for $p = 1$ and $p = \infty$ too, but it is weaker than the statement.

Proof. At first we will prove the statement for $p = \infty$. Let x be in (a, b) arbitrary but fixed, and $\{\delta_n\}$ an arbitrary positive sequence. Let

$$F_n(t) = \begin{cases} \frac{f(t)}{x-t} & \text{if } |x-t| > \delta_n, t \in (a, b) \\ 0 & \text{if } |x-t| \le \delta_n, t \in (a, b). \end{cases}$$

Now by the help of this auxiliary function and by the Christoffel-Darboux formula let us estimate the weighted norm of the k^{th} partial sum ($k < n$). Denoting by

$$K_k(x,y) := \sum_{l=0}^{k} p_{w^2,l}(x) p_{w^2,l}(y), \tag{17}$$

we have

$$\|S_k(f_\tau, x)\|_{w_\tau,\infty} = \left\| \sum_{l=0}^{k} \int_a^b p_{w^2,l}(x) p_{w^2,l}(y) f_\tau(y) w^2(y) dy \right\|_{w_\tau,\infty} \leq$$

$$\left\| \int_{[x-\delta_n, x+\delta_n]\cap(a,b)} K_k(x,y) f_\tau(y) w^2(y) dy \right\|_{w_\tau,\infty} +$$

$$\left\| \frac{\gamma_k}{\gamma_{k+1}} \left(p_{w^2,k+1}(x) c_k(F_{\tau,n}) - p_{w^2,k}(x) c_{k+1}(F_{\tau,n}) \right) \right\|_{w_\tau,\infty} = I_k + II_k.$$

Let us suppose that f is zero outside (a,b). Using Schwarz's inequality we get that

$$\left| \left(\int_{[x-\delta_n, x+\delta_n]\cap(a,b)} K_k(x,y) f_\tau(y) w^2(y) dy \right) w_\tau(x) \right| \leq$$

$$\|f_\tau w\|_\infty \int_{[x-\delta_n, x+\delta_n]\cap(a,b)} |K_k(x,y) w(y)|\, dy w_\tau(x) \leq$$

$$\|f\|_{w_\tau,\infty} \left(\int_{[x-\delta_n, x+\delta_n]\cap(a,b)} 1^2 \right)^{\frac{1}{2}} \left(\int_{[x-\delta_n, x+\delta_n]\cap(a,b)} K_k^2(x,y) w^2(y) dy \right)^{\frac{1}{2}} w_\tau(x) \leq$$

$$\|f\|_{w_\tau,\infty} \sqrt{2\delta_n} \left(\int_a^b K_k^2(x,y) w^2(y) dy \right)^{\frac{1}{2}} w_\tau(x).$$

By the orthonormality

$$\int_a^b K_k^2(x,y) w^2(y) dy = \sum_{l=0}^{k} p_{w^2,l}^2(x),$$

that is

$$\|f\|_{w_\tau,\infty} \sqrt{2\delta_n} \left(\int_a^b K_k^2(x,y) w^2(y) dy \right)^{\frac{1}{2}} w_\tau(x) \leq \|f\|_{w_\tau,\infty} \sqrt{2\delta_n} \left(\sum_{l=0}^{n-1} p_{w^2,l}^2(x) w_\tau^2(x) \right)^{\frac{1}{2}} =$$

$$\|f\|_{w_\tau,\infty} \sqrt{2\delta_n} \left(\lambda_n^{-1}(w^2, x) w_\tau^2(x) \right)^{\frac{1}{2}}.$$

Thus we have that

$$\left| \left(\int_{[x-\delta_n, x+\delta_n]\cap(a,b)} K_k(x,y) f_\tau(y) w^2(y) dy \right) w_\tau(x) \right| \leq \|f\|_{w_\tau,\infty} \sqrt{2\delta_n} \left(\lambda_n^{-1}(w^2, x) w_\tau^2(x) \right)^{\frac{1}{2}},$$

and so taking supremum we get

$$I_k \le C\|f\|_{w_\tau,\infty}\sqrt{\delta_n\Lambda_{\tau,n}}. \tag{18}$$

Bessel's inequality implies that

$$\frac{1}{n+1}\sum_{k=0}^{n} II_k \le \frac{C}{n+1}\left(\sup_{k\le n}\frac{\gamma_{k-1}}{\gamma_k}\right)\sqrt{\Lambda_{\tau,n}}\left(\sum_{k=0}^{n} c_k^2(F_{\tau,n})\right)^{\frac{1}{2}} \le$$

$$\frac{C}{n+1}\left(\sup_{k\le n}\frac{\gamma_{k-1}}{\gamma_k}\right)\sqrt{\Lambda_{\tau,n}}\left(\int_a^b F_{\tau,n}^2(y)w^2(y)dy\right)^{\frac{1}{2}}$$

$$\le \frac{C}{n}\left(\sup_{k\le n}\frac{\gamma_{k-1}}{\gamma_k}\right)\sqrt{\frac{\Lambda_{\tau,n}}{\delta_n}}\|f(x)\|_{w_\tau,\infty} \tag{19}$$

Now if we choose

$$\delta_n = \frac{1}{\Lambda_n},$$

by the definition of σ_n, the statement of the lemma is a consequence of (1.18) and (1.19) in the case $p=\infty$. To prove it for $p<\infty$, we use the well-known device : at first it has to be proved by duality method for $p=1$: using Fubini's theorem :

$$\|\sigma_n(f_\tau,x)\|_{w,1} = \sup_{g,\|g\|_{w,\infty}\le 1}\int_a^b g(x)\sigma_n(f_\tau,x)w^2(x)dx =$$

$$\sup_{g,\|g\|_{w,\infty}\le 1}\frac{1}{n+1}\sum_{k=0}^{n}\int_a^b\int_a^b g(x)f_\tau(y)\sum_{l=0}^{k}p_{w^2,l}(x)p_{w^2,l}(y)w^2(x)w^2(y)dydx =$$

$$\sup_{g,\|g\|_{w,\infty}\le 1}\int_a^b f_\tau(y)\sigma_n(g,y)w^2(y)dy \le C\|f(x)\|_{w,1},$$

because denoting by $g=h_\tau$ we can apply the previous part of the lemma with the remark that $\|h_\tau\|_{w,\infty}=\|h\|_{w_\tau,\infty}$. By the boundedness of $(\tau')^{-\frac{1}{2}}$ we can observe that

$$\|\sigma_n(f_\tau)w_\tau\|_\infty \le C\|f\|_{w_\tau,\infty} \le C\|fw\|_\infty = C\|f_\tau w\|_{\sqrt{\tau'},\infty},$$

and

$$\|\sigma_n(f_\tau)w_\tau\|_1 \le C\|\sigma_n(f_\tau)\|_{w,1} \le C\|fw\|_1 = C\|f_\tau w\|_{\sqrt{\tau'},1}.$$

So for the operator

$$T:\quad g\longrightarrow \sigma_n\left(\frac{g}{w}\right)w_\tau,$$

$$T: L_{\sqrt{\tau'},\infty}\longrightarrow L_\infty$$

and

$$T: L_{\sqrt{\tau'},1}\longrightarrow L_1$$

we can apply the Riesz-Thorin interpolation theorem which proves (1.16).

Definition 2 $(a,b) \subset \mathbf{R}, w \in W$ *is a weight function on* (a,b).

$$C_w(a,b) := \{f \in C(a,b) | \lim_{x \longrightarrow a+}(fw)(x) = \lim_{x \longrightarrow b-}(fw)(x) = 0\}$$

Definition 3 $1 \leq p \leq \infty$:

$$E_n^w(f)_p := \inf_{p_n \in \Pi_n} \|f(x) - p_n(x)\|_{w,p}.$$

Remark.

(1) If $w \leq 1$ and (a,b) is finite then Weierstrass theorem ensures that

$$\lim_{n \longrightarrow \infty} E_n^w(f)_p = 0$$

for all f continuous in (a,b).

(2) If (a,b) is finite then it is also clear that if $f \in C_w(a,b)$ then $\lim_{n \longrightarrow \infty} E_n^w(f)_p = 0$, namely it is enough to show for $p = \infty$. Let us choose to an arbitrary $\varepsilon > 0$ a δ such that if $x \in (a, a+\delta) \cup (b-\delta, b)$, then $|(fw)(x)| < \varepsilon$. Let now

$$f_\varepsilon(x) = \begin{cases} f(x) & \text{if } x \in [a+\delta, b-\delta], \\ l_1(x) & \text{if } x \in (a, a+\delta) \\ l_2(x) & \text{if } x \in (b-\delta, b) \end{cases}$$

where $l_i(x)$ are linear functions such that f_ε is continuous. Because of this continuity there is a polynomial $p_{n(\varepsilon)}(x)$ such that

$$|p_{n(\varepsilon)}(x) - f_\varepsilon(x)| < \varepsilon, \quad x \in (a,b).$$

That is

$$\|f(x) - p_{n(\varepsilon)}(x)\|_{w,\infty} \leq \|f - f_\varepsilon(x)\|_{w,\infty} + \|f_\varepsilon(x) - p_{n(\varepsilon)}(x)\|_{w,\infty} \leq (2+K)\varepsilon,$$

where $K = \|w(x)\|_\infty$ on (a,b).

(3) If (a,b) is infinite, we have the following

Theorem 2 (10,Th. 1.1) *Let* $w = e^{-Q}$ *where* $Q : \mathbf{R} \longrightarrow [0,\infty)$ *is even, and* $Q(e^x)$ *is convex on* $(0,\infty)$. *Then the polynonials are dense in the corresponding weighted space* $(C_w(-\infty,\infty))$, *if and only if*

$$\int_0^\infty \frac{Q(x)}{1+x^2}dx = \infty.$$

(4) As a corollary of the previous theorem we get the following

Lemma 2 *Suppose besides the assumptions of Lemma 1 that $w(x)$ is continuous and positive on* R. *In this case the density of the polynomials in C_w entails the density in $L_{w,1}$.*

Proof. Let $f \in L_{w,1}$ arbitrary. Because compactly supported continuous functions are dense in L_1, for every $\varepsilon > 0$ there is a $g \in C_0(\mathbf{R})$ such that

$$\|(fw)(x) - g(x)\|_1 < \varepsilon.$$

Furthermore $\frac{g}{w} \in C_0(\mathbf{R}) \subset C_{\sqrt{w}}$, thus there exists a polynomial p such that

$$\|\frac{g}{w} - p\|_{\sqrt{w},\infty} < \varepsilon.$$

These imply that

$$\|f - p\|_{w,1} \le \|fw - g\|_1 + \|g - pw\|_1 \le$$

$$\varepsilon + \int_{\mathbf{R}} \left| \frac{g(x)}{\sqrt{w(x)}} - p(x)\sqrt{w(x)} \right| \sqrt{w(x)}dx \le \varepsilon(1 + \|\sqrt{w}\|_1).$$

After these preliminaries we can formulate the main result of the paper. After the general statement we will investigate the fields of application, namely how does it work on special weights. For example in the trigonometric case the following theorem has a much simpler form; every statement has the form "if and only if". The summary of the trigonometric case can be found in K. Hoffman's book, published in 1962 [5]. Some details for the Hermite case are proved by I. Joó in 1988 [7], and the complete theorem in the Hermite case is given in [6] in 1990. We will see in the section of applications that not only in Hermite case, but for any Freud weight Theorem 4 has as "nice" form as in the trigonometric case.

Let τ be an increasing, piecewise differentiable function, $(\tau')^{-1}$ is bounded again, w is continuous and positive, and let us list now some properties of a weight function:

(A) the assumptions of Lemma 1 are valid ,

(B) polynomials are dense in $C_w(a, b)$,

(C) polynomials are dense in $L_{w\sqrt{\tau'},1}(a, b)$,

(D) $\|p_{w^2,n}\|_{w_\tau,\infty} \le K$, where $K \in \mathbf{R}$ is independent of n,

(E) $\|p_{w^2,n}\|_{w_\tau,1} \le K$, where $K \in \mathbf{R}$ is independent of n.

Using the notation

$$\int_c^d f(x)d\mu_\tau(x) := \int_c^d f_\tau(x)d\mu(x),$$

we have

Theorem 3 *Let $(a,b) \subset \mathbf{R}$ be a finite or infinite interval, $w \in W$ a bounded weight function on it. With the above notations:*

(1)

(a) *If* (A) *holds and $f \in L_{w,p}(a,b)$ $1 < p < \infty$, then the Cesàro means of f_τ : $\sigma_n(f_\tau, x)$ are uniformly bounded in $L_{w_\tau,p}(a,b)$.*

(b) *If the Cesàro means of a formal sum $\sum_{k=0}^{\infty} c_k p_{w^2,k}$ are uniformly bounded in $L_{w,p}(a,b), 1 < p < \infty$, then this formal sum is the Fourier series of a function $f \in L_{w,p}(a,b)$.*

(2)

(a) *If* (A), (C) *hold, and $f \in L_{w,1}(a,b)$, then $\sigma_n(f) \longrightarrow f$ with respect to the norm of $L_{w,1}$.*

(b) *If* (D) *holds and the Cesàro means $\sigma_n(\cdot, x)$ of the formal sum $\sum_{k=0}^{\infty} c_k p_{w^2,k}$ are convergent in $L_{w\sqrt{\tau'},1}(a,b)$, then this formal sum is the Fourier series of a function $f \in L_{w\sqrt{\tau'},1}(a,b)$.*

(3)

(a) *If* (A), (B) *hold, and $f \in C_w(a,b)$, then*

$$\|f - \sigma_n(f)\|_{w_\tau,\infty} \longrightarrow 0, \quad n \longrightarrow \infty.$$

(b) *If* (E) *holds and the Cesàro means $\sigma_n(\cdot, x)$ of the formal sum $\sum_{k=0}^{\infty} c_k p_{w^2,k}$ are convergent in $L_{w\sqrt{\tau'},\infty}(a,b)$, then this formal sum is the Fourier series of a function $f \in L_{w\sqrt{\tau'},\infty}(a,b)$.*

(4)

(a) *If* (A) *holds and μ is a signed measure with $\int_a^b w(x)|d\mu(x)| < \infty$, then the Cesàro means $\sigma_n(d\mu_\tau, x)$ are uniformly bounded in $L_{w,1}(a,b)$.*

(b) *If the Cesàro means $\sigma_n(\cdot, x)$ of a formal sum $\sum_{k=0}^{\infty} c_k p_{w^2,k}$ are uniformly bounded in $L_{w,1}(a,b)$, then this sum is the Fourier series of a signed measure μ, with $\int_a^b w(x)|d\mu(x)| < \infty$.*

(5)

(a) *If* (A) *holds and μ is a signed measure with $0 < \int_a^b w(x)d\mu(x) < \infty$, then the Cesàro means $\sigma_n(d\mu_\tau, x)$ are uniformly bounded in $L_{w,1}(a,b)$, and there exists a subsequence n_k for which*

$$\int_a^b \sigma_{n_k}(d\mu_\tau, x)w(x)dx > 0.$$

(b) *If the Cesàro means $\sigma_n(\cdot, x)$ of a formal sum $\sum_{k=0}^{\infty} c_k p_{w^2,k}$ are uniformly bounded in $L_{w,1}(a,b)$, and there exists a subsequence n_k for which*

$$\int_a^b \sigma_{n_k}(\cdot, x)w(x)dx > 0,$$

then this sum is the Fourier series of a signed measure μ, with $0 < \int_a^b w(x)d\mu(x) < \infty$.

Proof.

(1)

The implication (a) is contained in Lemma 1.

For proving the converse direction we have to consider $L_{w,p}(a,b)$ as a reflexive Banach space. A bounded sequence in it has a weakly convergent subsequence σ_{n_k}, that is there is an $f \in L_{w,p}(a,b)$ such that $\sigma_{n_k} \longrightarrow f$ in weak sense, and so specially for $p_{w^2,l}(x) \in L_{w,q}(a,b)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $l \in \mathbf{N}$ we have

$$\int_a^b \sigma_{n_k}(\cdot,x)p_{w^2,l}(x)w^2(x)dx \longrightarrow \int_a^b f(x)p_{w^2,l}(x)w^2(x)dx \quad (k \longrightarrow \infty)$$

On the other hand

$$\int_a^b \sigma_{n_k}(\cdot,x)p_{w^2,l}(x)w^2(x)dx = \left(1 - \frac{l}{n_k+1}\right)c_l \longrightarrow c_l \quad (k \longrightarrow \infty)$$

This means that the c_k-s in σ_n are the Fourier coefficients of f.

(2)

For the proof of convergence we have to estimate $\|f(x) - \sigma_n(f,x)\|_{w,1}$ where $f \in L_{w,1}(a,b)$. Let ε be arbitrary and $p = p(\varepsilon, x)$ be a polynomial such that

$$\|f(x) - p(\varepsilon,x)\|_{w\sqrt{\tau'},1} \leq \varepsilon.$$

The existence of such a p is guaranteed by assumption (C). With this polynomial

$$\|f(x) - \sigma_n(f,x)\|_{w,1} \leq \left\|(\tau')^{-\frac{1}{2}}\right\|_\infty \|f(x) - p(\varepsilon,x)\|_{w\sqrt{\tau'},1} + \|p(\varepsilon,x) - \sigma_n(p,x)\|_{w,1} +$$

$$\|\sigma_n(f-p,x)\|_{w,1} \leq C\varepsilon + \frac{1}{n+1}\|S_0(p,x) + \cdots + S_l(p,x)\|_{w,1} +$$

$$\left(1 - \frac{n-l}{n+1}\right)\|S_l(p,x)\|_{w,1} + C\varepsilon,$$

where for the estimation of the third term we used Lemma 1, and in the calculation of the second term $l = l(\varepsilon)$ is the degree of $p(\varepsilon,x)$, and so if $n > N = N(\varepsilon)$, then

$$\|f(x) - \sigma_n(f,x)\|_{w,1} \leq C\varepsilon.$$

Proof of direction (b): Supposing that for some $f \in L_{w\sqrt{\tau'},1}$

$$\sigma_n(\cdot,x) \longrightarrow f(x)$$

in the norm of $L_{w\sqrt{\tau'},1}$, we can estimate the distance of the l^{th} Fourier coefficients of $f(x)$ and $\sigma_n(\cdot,x)$, namely if $n \geq l$, then

$$|c_l(f) - c_l(\sigma_n)| = \left| \int_a^b (f(x) - \sigma_n(\cdot,x)) p_{w^2,l}(x) w^2(x) dx \right| \leq$$

$$\|p_{w^2,l}(x)\|_{w_\tau,\infty} \|f(x) - \sigma_n(\cdot,x)\|_{w\sqrt{\tau'},1},$$

which tends to 0 by the assumptions, that is as in part (1) we can see that $f(x)$ has the expansion $\sum_{k=0}^{\infty} c_k p_{w^2,k}(x)$.

(3)

Using $L_{w_\tau,\infty}$ instead of $L_{w,1}$, the proof of this case is similar to case (2): Let $p = p(\varepsilon, x)$ be a polynomial such that

$$\|f(x) - p(\varepsilon,x)\|_{w,\infty} \leq \varepsilon.$$

With this polynomial guaranteed by (B):

$$\|f(x) - \sigma_n(f,x)\|_{w_\tau,\infty} \leq \|f(x) - p(\varepsilon,x)\|_{w_\tau,\infty} +$$

$$\|p(\varepsilon,x) - \sigma_n(p,x)\|_{w_\tau,\infty} + \|\sigma_n(f-p,x)\|_{w_\tau,\infty} \leq$$

$$C\varepsilon + \frac{1}{n+1}\|S_0(p,x) + \cdots + S_l(p,x)\|_{w_\tau,\infty} + \left(1 - \frac{n-l}{n+1}\right) \|S_l(p,x)\|_{w_\tau,\infty} + C\varepsilon.$$

Here l is the degree of $p(\varepsilon,x)$, and we applied Lemma 1 to $h := (f-p)\sqrt{\tau'}$. Part (b) is the same as case (2), thus we omit the details.

(4)

At first we will prove the boundedness of the Cesàro means as in Lemma 1:

$$\|\sigma_n(d\mu_\tau, x)\|_{w,1} = \sup_{g, \|g\|_{w,\infty} \leq 1} \left| \int_a^b \sigma_n(d\mu_\tau, x) g(x) w^2(x) dx \right| =$$

$$\sup_{g, \|g\|_{w,\infty} \leq 1} \left| \int_a^b \int_a^b w^2(t) w^2(x) g(x) \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) p_{w^2,k}(t) p_{w^2,k}(x) dx d\mu_\tau(t) \right| =$$

$$\sup_{g, \|g\|_{w,\infty} \leq 1} \left| \int_a^b (\tau'(t))^{-\frac{1}{2}} \sigma_n(g,t) w^2(t) d\mu(t) \right| \leq$$

$$\sup_{g, \|g\|_{w,\infty} \leq 1} \|\sigma_n(g,x)\|_{w_\tau,\infty} \int_a^b w(t) |d\mu(t)| \leq C.$$

We used here Fubini's theorem and assumption (A) with $g = f_\tau$.

For the proof of the opposite direction let us introduce the following measures:

$$d\mu_n(x) = w(x) \sigma_n(\cdot, x) dx.$$

The total variation of these measures is uniformly bounded, that is

$$\int_a^b |d\mu_n(x)| = \int_a^b w(x)|\sigma_n(\cdot,x)|dx = \|\sigma_n(\cdot,x)\|_{w,1} \leq K$$

Let now $\{[A_k, B_k]\}$ be a sequence of intervals in (a,b) such that

$$\bigcup_{k=1}^{\infty}[A_k, B_k] = (a,b)$$

Let us denote the dual space of continuous functions on $[A_k, B_k]$ (which are the Borel measures on $[A_k, B_k]$) by

$$B_k := (C[A_k, B_k])^*$$

$$\nu \in B_k, \quad \|\nu\| = \int_{A_k}^{B_k} |d\nu|.$$

According to the Banach-Alaoglu theorem the closed balls in B_k are w^*-compact. Consequently for a given k one can choose a w^*-convergent subsequence $(\mu_{n_{i,k}})$ wich also has a subsequence which is w^*-convergent on $[A_{k+1}, B_{k+1}]$, etc. Using the diagonal method of Cantor, we can extract a subsequence (denoted again by μ_{n_i}) such that

$$\mu_{n_i} \xrightarrow{w^*} \hat{\mu} \;\text{ on } [A_k, B_k] \text{ for every } k \in \mathbf{N}.$$

Let

$$w(x)d\mu(x) := d\hat{\mu}(x).$$

First of all

$$\int_a^b w(x)|d\mu(x)| < \infty,$$

because for every k

$$\int_{A_k}^{B_k} w(x)|d\mu(x)| = \int_{A_k}^{B_k} |d\hat{\mu}(x)| \leq \sup_i \int_{A_k}^{B_k} |d\mu_{n_i}(x)| \leq \sup_n \|\sigma_n(\cdot,x)\|_{w,1} \leq K < \infty.$$

Secondly it has to be shown that the Fourier coefficients of the above mentioned measure μ are equal to the c_k-s.

$$\left|c_k - \int_a^b p_{w^2,k}(x)w^2(x)d\mu(x)\right| \leq \left|\int_{(a,b)\setminus[A_N,B_N]} p_{w^2,k}(x)w^2(x)d\mu(x)\right| +$$

$$\left|c_k - \int_{A_N}^{B_N} p_{w^2,k}(x)w^2(x)d\mu(x)\right| = I + II$$

$$II = \left|c_k - \lim_{i\to\infty}\int_{A_N}^{B_N} p_{w^2,k}(x)w(x)d\mu_{n_i}(x)\right| =$$

$$\left| c_k - \lim_{i\longrightarrow\infty} \int_{A_N}^{B_N} p_{w^2,k}(x) w^2(x) \sigma_{n_i}(x) dx \right| =$$

$$\left| c_k - \lim_{i\longrightarrow\infty} \sum_{j=0}^{n_i} \left(1 - \frac{j}{n_i+1}\right) c_j \left(\int_a^b p_{w^2,j}(x) p_{w^2,k}(x) w^2(x) dx - \right.\right.$$

$$\left.\left. \int_{x\in(a,b)\setminus[A_N,B_N]} p_{w^2,j}(x) p_{w^2,k}(x) w^2(x) dx \right) \right| \leq$$

$$\frac{k|c_k|}{n_i+1} + \|\sigma_{n_i}(\cdot,x)\|_{w,1} \sup_{x\in(a,b)\setminus[A_N,B_N]} |p_{w^2,k}(x) w(x)|.$$

When i and N tend to infinity, then the right-hand side tends to zero. On the other hand,

$$I \leq \sup_{x\in(a,b)\setminus[A_N,B_N]} |p_{w^2,k}(x) w(x)| \int_a^b w(x) |d\mu(x)|.$$

Because of finite moments and boundedness of the second factor this expression also tends to zero when N tends to infinity, which completes the proof.

(5)

The proof of this part is the same as that of the previous part and from the construction we can easily see that μ is positive in the sense stated in the theorem if and only if the μ_{n_i}-s are also positive in that sense.

Remark. Introducing a new notation for $p = \infty$ and $p = 1$,

$$\|f\|^*_{w_\tau,p} := \|f(\tau^{-1}(y))\|_{w_\tau(\tau^{-1}(y))[\tau(a),\tau(b)],p}$$

we get (2) and (3) parts of Theorem 4 in a more symmetric form:

(2.b) If

$$\|p_{w^2,n}\|^*_{w_\tau,p} \leq K,$$

and σ_n is convergent in $L^*_{w_\tau,1}$, then the formal sum $\sum_{k=0}^\infty c_k p_{w^2,k}(x)$ is the Fourier series of a function $f \in L^*_{w_\tau,1}$.

(3.b) is similar. These symmetric forms follow from the change of variables $\tau(x) = y$ which works by the assumptions on τ.

Applications

Let $(a,b) = (-a,a)$ be symmetric. Before we detail the applications of second theorem, we need a new notion again, namely let us denote by $a_n = a_n(w)$ the so-called Mhaskar-Rahmanov-Saff number associated with w. This number shows

"Where does the sup norm of a weighted polynomial live?" [16]. Applying it to symmetric intervals it means that

$$\begin{cases} \|p\|_{w,\infty} = \max_{|x|\leq a_n} |p(x)|w(x) & \text{for all } p \in \Pi_n \\ \|p\|_{w,\infty} > |p(x)|w(x) & \text{for all } |x| > a_n \end{cases} \tag{20}$$

More precisely if $w(x) = e^{-Q(x)}, x \in (-a, a)$ and Q is even and convex on $(-a, a)$ then a_u is the positive root of the equation

$$u = \frac{2}{\pi}\int_0^1 a_u Q'(a_u t)\frac{t}{\sqrt{1-t^2}}dt, \quad u > 0. \tag{21}$$

In the applications for a symmetric weight function $w(x) = e^{-Q(x)}$, we can define

$$T(x) = 1 + x\frac{Q''(x)}{Q'(x)}, \quad x \in (0, a). \tag{22}$$

If T is positive and unbounded then let

$$\varphi(x) = \int_0^x \sqrt{T(t)}dt,$$

and

$$\tau(x) := \tau_w(x) = \begin{cases} \varphi(x) & \text{if } 0 < x < a \\ 0 & \text{if } x = 0 \\ -\varphi(-x) & \text{if } -a < x < 0 \end{cases} \tag{23}$$

This τ is an increasing function on (a, b), and $W_\tau(x) = w(x)(T(x))^{-\frac{1}{4}}$ is a weight function too if $\frac{1}{T}$ is bounded.

Freud Weights

Definition 4 *Let $(a, b) = \mathbf{R}, w(x) = e^{-Q(x)}$, where $Q : \mathbf{R} \longrightarrow \mathbf{R}$ is even and continuous in $\mathbf{R}$, Q'' is continuous in $(0, \infty)$, and $Q' > 0$ in $(0, \infty)$. Furthermore for some $A, B > 1$, let*

$$A \leq \frac{(xQ'(x))'}{Q'(x)} \leq B, \quad x \in (0, \infty). \tag{24}$$

Fulfilling the above assumptions, w is called a Freud weight.

We have to note that (1.21) means $1 < A \leq T(x) \leq B$, and so

$$\tau(x) = x$$

is a good choice. Because $\tau'(x) = 1$, we get for Freud weights the most beautiful form of Lemma 1 and Theorem 4, namely it means that

$$\|\sigma_n(f)\|_{w,p} \leq C(p)\|f\|_{w,p} \quad 1 \leq p \leq \infty,$$

and every part of Theorem 2 becomes "if and only if" statements. The validity of properties (A)-(E) implies the following theorem:

Theorem 4 (8) *Let w be a Freud weight.*

(a) [Th 1.1. b] *Then for all $x \in \mathbf{R}$, and $n \geq 1$,*

$$\lambda_n(w^2, x) \geq C \frac{a_n}{n} w^2(x) \left(\max \left\{ n^{-\frac{2}{3}}, 1 - \frac{|x|}{a_n} \right\} \right)^{-\frac{1}{2}}. \tag{25}$$

(b) [Th. 12.3.] *Moreover,*

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n, \quad n \geq 1 \tag{26}$$

(c) [Cor 1.4.] *and*

$$\|p_{w^2,n}\|_{w,\infty} \sim a_n^{-\frac{1}{2}} n^{\frac{1}{6}} \tag{27}$$

With the help of the previous theorem we can see that the assuptions of Lemma 1 hold for Freud weights, namely

Theorem 5 implies that

$$\Lambda_n \leq C \frac{n}{a_n},$$

and so we can choose

$$\delta_n := \frac{a_n}{n}.$$

in the proof of Lemma 1.

To verify property (B) we have Theorem 3. As it is well-known, the typical examples for Freud weights are

$$w_\gamma(x) = e^{-|x|^\gamma}.$$

In this case the assumption in Definition 4 is the following: $A = \gamma = B > 1$, which guarantees that the integral in question is divergent. Lemma 2 shows that property (C) is valid in this case if so is (B).

From (Def. 4) and from (Th 5. c) it follows that

$$\|p_{w^2,n}\|_{w,\infty} \leq C n^{\frac{1}{6} - \frac{1}{2B}},$$

which shows that (D) is valid only if $B \leq 3$. Assumption (E) is not valid even in the Hermite case. This was used for the proof of the direction (b) of (3) in the fourth theorem. It follows that a Jackson theorem gives us an estimation on the speed of convergence. For this we need the definition of the weighted modulus of smoothness:

Definition 5

$$\omega_{r,p}(f, w, t) = \sup_{0<h\leq t} \|\Delta_h^r(f, x, \mathbf{R})\|_{w,p,[-\sigma_h,\sigma_h]} + \inf_{p\in\Pi_{n-1}} \|f - p\|_{w,p,|x|\geq\sigma_t,} \tag{28}$$

where

$$\sigma(t) := \inf\left\{a_n : \frac{a_n}{n} \le t\right\}, \quad t > 0,$$

$$\Delta_h^r(f, x, I) := \sum_{i=0}^{r} \binom{r}{i}(-1)^i f\left(x + \frac{rh}{2} - ih\right).$$

In the last equation we assumed that all arguments of f lie in I, and the corresponding term is 0 otherwise.

Theorem 5 (1, Th. 1.2.) *Let w be a Freud weight, $r \ge 1$ and $0 < p \le \infty$. Let $f \in L_{w,p}(\mathrm{R})$ if $p < \infty$, and $f \in C_w(\mathrm{R})$, if $p = \infty$. Then there exist C_1 and C_2 depending only on w, r, such that*

$$E_n^w(f)_p \le C_1\omega_{r,p}\left(f, w, C_2\frac{a_n}{n}\right), \quad n \ge r - 1. \tag{29}$$

Also in [1] Corollary 1.6. yields that

$$\omega_{r,p}(f, w, t) = O(t^\alpha) \iff E_n^w(f)_p = O\left(\left(\frac{a_n}{n}\right)^\alpha\right), \quad 0 < \alpha < r. \tag{30}$$

Comparing the above relation and the estimation of the difference of a function f and it's Cesàro mean we get that if the $(r-1)^{th}$ derivative of f is in Lip_β $(\beta = \alpha - r + 1)$ and f is in $L_{w,p}$ or in C_w, then

$$\|\sigma_{n-1}(f, x) - f(x)\|_{w,p} \le C\|f(x)\|_{w,p}\left(\frac{a_n}{n}\right)^\alpha, \quad 1 \le p \le \infty.$$

Generalized Jacobi Weights

This section deals with the application of the (second) theorem to the generalization of $w(x) = (1-x)^\alpha(1+x)^\beta$, $x \in (-1, 1)$, $\alpha, \beta > -1$. A kind of generalization can be found in [2, Def.8.1.1.]. Further generalization allows some zeros or singularities inside the interval. We will follow this later variation [15].

Definition 6 *$w \in GJ$ is a generalized Jacobi weight if it has the form*

$$w(x) = H(x)\left(\sqrt{1-x^2}\right)^{-1} w_0\left(\sqrt{1-x}\right) w_{m+1}\left(\sqrt{1+x}\right)\prod_{r=1}^{m} w_r\left(|x - t_r|\right), \tag{31}$$

where $t_r \in (-1, 1), r = 1 \ldots m$,

$$w_r(\delta) = \prod_{s=1}^{l_r}\left(\omega_{r,s}(\delta)\right)^{\alpha_{r,s}}, \tag{32}$$

$m, l_r \in \mathbf{N}, \alpha_{r,s} \in \mathbf{R}, \omega_{r,s}(\delta)$ are concave moduli of continuity ($s = 1, \dots, l_r, r = 0, \dots, m+1$), and the function H satisfies

$$H(x) > 0, \quad H \textit{ and } \frac{1}{H} \in L_\infty,$$

Furthermore

$$\int_0^\delta w_r(\tau) d\tau = O(\delta w_r(\delta)), \quad \delta \longrightarrow +0 \ (r = 0, \dots, m+1),$$

and

$$\omega(H(\cos\theta), \delta)_\infty \delta^{-1} \in L_{1,[0,1]} \textit{ or } \omega(H(\cos\theta), \delta)_2 = O(\sqrt{\delta}), \ \delta \longrightarrow +0.$$

Here $\omega(f, \delta)_p = \sup_{|\tau| \le \delta} \|f(\cdot + \tau) - f(\cdot)\|_p$ is the usual modulus of continuity.

After this rather complicated definition we need a perturbed version of it, because in the case of not even weight function with inner singularities calculating with the perturbed weight function plays the role of the M-R-S number.

Definition 7 *Let*

$$w(\varrho, x) = \left(\sqrt{1-x^2} + \varrho^{-1}\right)^{-1} w_0\left(\sqrt{1-x} + \varrho^{-1}\right)$$
$$\times w_{m+1}\left(\sqrt{1+x} + \varrho^{-1}\right) \prod_{r=1}^{m} w_r\left(|x - t_r| + \varrho^{-1}\right). \tag{33}$$

Remark.

One can easily see that on finite intervals

$$\frac{\gamma_{k-1}}{\gamma_k} = O(1). \tag{34}$$

Namely, let us write

$$p_{w^2,k}(x) = \frac{\gamma_k}{\gamma_{k-1}} x p_{w^2,k-1}(x) + q_{k-1}(x), \text{ where } q_{k-1} \in \Pi_{k-1}.$$

Because of the orthogonality we have

$$1 = \int_a^b \left(\frac{\gamma_k}{\gamma_{k-1}} x p_{w^2,k-1}(x) + q_{k-1}(x)\right) p_{w^2,k}(x) w^2(x) dx$$

and using Cauchy-Schwarz's inequality, we get that

$$\frac{\gamma_{k-1}}{\gamma_k} = \int_a^b x p_{w^2,k-1}(x) p_{w^2,k}(x) w^2(x) dx \le \max\{|a|, |b|\}.$$

(We have to note here that on infinite intervals the same chain of ideas gives that $\frac{\gamma_{k-1}}{\gamma_k} = O(a_k)$.)

In the case of generalized Jacobi weights we will use as $\tau(x)$ the τ_w of ultraspherical weights without constants, which means that if $Q(x) = \ln \frac{1}{(1-x^2)^\gamma}$, then $\tau_w(x) = -2\left(\arccos x - \frac{\pi}{2}\right)$, and we will use

$$\tau(x) = -\arccos x.$$

Theorem 6 (15) *If $w \in GJ$, then*

(a) [14, Th.3.1.]

$$\lambda_n(w,x) \sim \frac{w(n,x)\varphi(n,x)}{n} \tag{35}$$

(b) [14, Th. 3.2 and 3.3]

$$\|p_{w^2,n}(x)\|_{w\sqrt{\varphi},\infty} \sim 1, \tag{36}$$

where $\varphi(x) = \sqrt{1-x^2}$.

Applying part (a) of the above theorem we get that

$$\frac{\Lambda_{\tau,n}}{n} = \left\| \frac{w^2(x)\varphi(x)}{w^2(n,x)\varphi(n,x)} \right\|_\infty .$$

This yields that it is enough to investigate a factor

$$\left(\frac{\omega_{r,s}(|x-t_r|)}{\omega_{r,s}\left(|x-t_r| + \frac{1}{n}\right)} \right)^{\alpha_{r,s}} .$$

If $\alpha_{r,s} \geq 0$, the above expression is less than 1, because a modulus of continuity is nondecreasing. This is the case of w_0 and w_{m+1} too. If $\alpha_{r,s} < 0$, then we have to use a consequence of concavity : $\frac{\delta}{\omega(\delta)}$ is nondecreasing, and hence

$$\left(\frac{\omega_{r,s}\left(|x-t_r| + \frac{1}{n}\right)|x-t_r|}{\omega_{r,s}(|x-t_r|)|x-t_r|} \right)^{-\alpha_{r,s}} \leq \left(\frac{|x-t_r| + \frac{1}{n}}{|x-t_r|} \right)^{-\alpha_{r,s}} \leq 2^{-\alpha_{r,s}}.$$

Summarizing

$$\frac{\Lambda_{\tau,n}}{n} \leq K = K(m,r),$$

wich makes sure that assumption (A) (before Theorem 4) holds. We have to note that this case shows the importance of τ, i.e. however $\frac{\Lambda_{\tau,n}}{n}$ is bounded,

$$\frac{\Lambda_n(x)}{n} \sim \varphi(x)^{-1},$$

which is not bounded. Properties (B) and (C) are obviously valid on finite interval and Theorem 7 (b) means that (D) and (E) are valid too.

Erdős Weights

Definition 8 *Let $w := e^{-Q}$, where $Q : \mathrm{R} \longrightarrow \mathrm{R}$ is even and continuous in* R*; assume that Q''' exists on $(0,\infty)$, and Q' is positive on $(0'\infty)$. Let*

$$T(x) = 1 + x\frac{Q''(x)}{Q'(x)}, \quad x \in (0,\infty)$$

be increasing on $(0,\infty)$ with

$$\lim_{x\longrightarrow 0+} T(x) > 1, \tag{37}$$

$$\lim_{x\longrightarrow \infty} T(x) = \infty, \tag{38}$$

and for each $\varepsilon > 0$,

$$T(x) = O(Q'(x)^{\varepsilon}), \quad x \longrightarrow \infty. \tag{39}$$

Assume further that

$$\frac{Q''(x)}{Q'(x)} \sim \frac{Q'(x)}{Q(x)} \quad (\, x \text{ large}). \tag{40}$$

and for some $C > 0$,

$$\frac{|Q'''(x)|}{Q'(x)} \leq C\left(\frac{Q'(x)}{Q(x)}\right)^2 \quad (x \text{ large})\ . \tag{41}$$

In the definition given in [13, Def. 1.1] there are some technical assumptions, but we will use this form for simplicity. Typical examples for Erdős weights are

$$w_{k,\alpha}(x) = \exp(-\exp_k(|x|^{\alpha}), \ \ \alpha > 0, k \geq 1,$$

where $\exp_k = \exp(\exp(\exp(\cdots)))$ (k times). If $\tau(x)$ is given as in (1.20), we have

Theorem 7 (13, 3.13;9, 10.33)

$$\Lambda_{\tau,n} \leq C\frac{n}{a_{n}}, \tag{42}$$

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n, \tag{43}$$

where a_n is the M-R-S number. This shows that choosing $\delta_n = \frac{a_n}{n}$ we get Lemma 1 for Erdős weights, which is the same for $p = 1$ and for $p = \infty$ as Theorem 2.1 in [14]. Theorem 3 and Lemma 2 contain the information on property (B) and (C).

Exponential Weights on $(-1, 1)$.

Definition 9 (9) *$w \in \hat{W}$ if $w := e^{-Q}$, where $Q : (-1,1) \longrightarrow \mathbf{R}$ is even and is twice continuously differentiable in $(-1,1)$. Assume moreover that*

$$Q^{'} \geq 0, \;\; Q^{''} \geq 0 \;\; in \; (0.1), \tag{44}$$

$$\lim_{t \longrightarrow 1-} Q(t) = \infty, \tag{45}$$

The function

$$T(t) := 1 + t\frac{Q^{''}(t)}{Q^{'}(t)}, \quad t \in [0,1) \tag{46}$$

is increasing in $[0,1)$ with

$$T(0+) > 1 \tag{47}$$

and

$$T(t) \sim \frac{Q^{'}(t)}{Q(t)}, \; t \text{ close enough to } 1. \tag{48}$$

Examples for $w \in \hat{W}$:

$$w_{0,\alpha}(x) = \exp(-(1-x^2)^{-\alpha}), \;\; \alpha > 0,$$

$$w_{k,\alpha}(x) = \exp(-\exp_k(1-x^2)^{-\alpha}), \;\; \alpha > 0, k \geq 1.$$

We have to note that for Jacobi weights (1.46) doesn't hold, so they aren't in $\hat{W}$.

Similarly to the GJW section the quotients of the leading coefficients are bounded thus to verify property (A) it is enough to show that $\frac{\Lambda_{r,n}}{n}$ is bounded.

Theorem 8 (9) [Cor 1.3.] *There exists n_1 such that for $n \geq n_1$,*

$$\sup_{x \in (-1,1)} \lambda_n^{-1}(w^2, x) w^2(x) \sim nT(a_n)^{\frac{1}{2}}. \tag{49}$$

[Th 1.2] *Uniformly for $n > n_1$*

$$\lambda_n^{-1}(w^2, x) w^2(x) \sqrt{1 - \frac{|x|}{a_n}} \sim n, \quad |x| < a_n \left(1 - \frac{C}{T(a_n)}\right) \tag{50}$$

[L 3.2] *Given fixed $r > 1, u \in [1, \infty)$:*

$$\frac{a_{ru}}{a_u} - 1 \sim \frac{1}{T(a_u)}. \tag{51}$$

On the same chain of ideas given in [10, Th 3.3] we get the following

Theorem 9 *We have*

$$\Lambda_{r,n} \leq n. \tag{52}$$

This shows that Lemma 1 is valid for exponential weights on $(-1, 1)$ too.

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