Müntz-type theorems on the half-line with weights*

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Abstract

We consider the linear span S of the functions t^{a_k} (with some $a_k > 0$) in weighted L^2 spaces, with rather general weights. We give one necessary and one sufficient condition for S to be dense. Some comparisons are also made between the new results and those that can be deduced from older ones in the literature.

1 Introduction

The first "if and only if" solution of the problem of S. N. Bernstein [5] was given by Ch. H. Müntz [22]:

Theorem A
Let $0 = \lambda_0 < \lambda_1 <, \dots$ be an increasing sequence of real numbers. The linear subspace span$\{t^{\lambda_k} : k = 0, 1, \dots\}$ is dense in $C([0, 1])$, if and only if $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty$.

This classical result was first proved in $L_2[0, 1]$ and then extended to $C[0, 1]$, as stated above. Also, it was stated only for increasing sequences λ_k. Subsequently, this theorem has had several different proofs and generalizations, and there are several surveys in this topic (see for instance the papers of J. Almira and A. Pinkus [1], [24]).

On $C[0, 1]$ and $L_p(0, 1)$, "full Müntz theorems", i.e. theorems with rather general exponents, were later proved by eg. P. Borwein, T. Erdélyi, W. B. Johnson and V. Operstein ([8], [14], [13], [23]). Versions of Müntz's theorem on

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compact subsets of positive measure [9], [10], and on countable compact sets [2] were also proved. Further results can be found for instance in the monographs of P. Borwein, T. Erdélyi [11], and B. N. Khabibullin [17].

In this paper we are interested in Müntz-type theorems on $(0, \infty)$. Several papers were written in the '40s on the completeness of the set $\{t^{\lambda_k} e^{-t}\}$ in $L_2(0, \infty)$ (see eg. [15], [6], [7]). In particular, we will use some ideas of W. Fuchs. His theorem is the following:

Theorem B

Let a_k be positive numbers, such that $a_{k+1} - a_k \geq d > 0$ $(k = 1, 2, \ldots)$, and let $\log \Psi(r) = 2 \sum_{a_k < r} \frac{1}{a_k}$, if $r > a_1$, and $\log \Psi(r) = \frac{2}{a_1}$ if $r \leq a_1$. Then $\{e^{-t} t^{a_k}\}$ is complete in $L_2(0, \infty)$, if and only if

$$\int_1^\infty \frac{\Psi(r)}{r^2} dr = \infty.$$

This weighted Müntz problem (with weight function $w(t) = e^{-t}$) has several generalizations with different weights. The basic paper in this respect was written by P. Malliavin [21]. His results were completed by closure theorems of J. M. Anderson and K. G. Binmore [3]. A. F. Leont'ev [19] and G. V. Badalyan [4] proved similar theorems with more general weights. In 1980, by the Hahn-Banach theorem technique, R. A. Zalik [29] proved a Müntz type theorem on the half-line with weights $|w| \leq c \exp(-|\log t|^a)$ $(a > 0)$. In 1996 Kroó and Szabados [18] also had a related result on $(0, \infty)$.

Closely related to our topic (by a $\log t$ substitution) are the results on the whole real line for exponential systems. Some nice generalizations of the above mentioned results were given by for instance by B. V. Vinnitskii, A. V. Shapovalovskii [26], by G. T. Deng [12], and by E. Zikkos [30].

In Theorem 1 and Theorem 2 below we will prove Müntz-type theorems on the half-line with more general weights, namely the previously investigated weights fulfil some log-convexity property, which is not necessary in our aspect (see the examples below).

2 Definitions, Results

We will work in weighted Banach spaces on $(0, \infty)$, so let us begin with the definitions of them. Let w be a weight function. (The rather general definition of w is given below.) Now let us introduce the following notations:

$$L_w^p := L_w^p(0, \infty) = \{f : \|fw\|_{p,(0,\infty)} < \infty\} \quad 1 \leq p < \infty,$$

where $\|fw\|_{p,(0,\infty)} = \left(\int_0^\infty |f(x)w(x)|^p dx\right)^{\frac{1}{p}}$, is the usual p-norm on the half-line, and let us denote the norm on this space by

$$\|f\|_{p,w} = \|fw\|_{p,(0,\infty)}.$$

Let

$$C_w := C_{w,(0,\infty)} = \{f \in C(0,\infty) : \lim_{\substack{t\to 0+\\ t\to\infty}} f(t)w(t) = 0\}$$

with the norm

$$\|f\|_{\infty,w} = \|fw\|_{\infty,(0,\infty)}.$$

Principally the L^2_w case will be examined in this note.

At first the weight function will be defined. Some specific examples are given subsequently. Stating Theorem 1 we need the following type of weights:

Definition 1 *We say that a weight function $w(t) = \nu(t)\mu(t)$ is admissible on $[0,\infty)$, if $\nu(t)$ and $\mu(t)$ are nonnegative and continuous on $[0,\infty)$, positive on $(0,\infty)$, w^2 has finite moments, and the followings are valid:*

$$\lim_{t\to 0+} \mu(t) \in (0,\infty), \tag{1}$$

and there is an $a \geq 1$ such that

$$\int_0^1 \left(\frac{t^{a-1}}{\nu(t)}\right)^2 dt < \infty. \tag{2}$$

Furthermore let us assume that there is a function γ on $[0,\infty)$, such that

$$\gamma(t) = \sum_{k=0}^{\infty} c_k t^{\gamma_k},$$

where $c_k > 0$ for all k, and $0 = \gamma_0 < \gamma_1 < \gamma_2 < \ldots$, $\lim_{k\to\infty}\gamma_k = \infty$. Let us assume that $\forall\ t > 1$

$$\frac{1}{w^2(t)} \leq \gamma(t) \tag{3}$$

and there is a $C > 1$, such that

$$\int_0^\infty \gamma\left(\frac{t}{C}\right) w^2(t)dt < \infty. \tag{4}$$

Here and in the followings C, C_i and c, c_i are absolute constants, and the value of them will not be the same at each occurrence.

Remark:
(1) The factor $\mu(t)$ is responsible for the behavior of the weight at infinity, and the factor $\nu(t)$ at zero, furthermore (2) ensures that the weight tends to zero at zero at most polynomially.
(2) If (2) is valid with an a_0, then one can choose any $a \geq a_0$ instead of a_0.
(3) If $\nu(t) \equiv 1$ (as in Theorem B) then one can choose $a = 1$.

Examples :
$w(t) = t^\beta e^{-Dt^\alpha}$, where $\nu(t) = t^\beta$, $\beta \geq 0$ and $\alpha > 0$ is admissible, that is it has finite moments, and choosing $\gamma(t) = e^{3Dt^\alpha}$ (3) and (4) are valid. The

original case of Fuchs (Theorem B) is $\beta = 0$ and $D = \alpha = 1$. When $\beta \geq 0$, $D = \frac{1}{2}$ and $\alpha = 1$, then w^2 is a Laguerre weight. When $\beta = 0$, $D > 0$ and $\alpha > 1$, then w is a Freud weight.

Let $w(t) = (4 + \sin t)t^\beta \prod_{k=1}^{n} e^{-D_k t^{\alpha_k}}$, where $\nu(t) = t^\beta$ again, and let us assume, that $\beta \geq 0$, and $0 \leq \alpha_1 < \alpha_2 \ldots < \alpha_n$, and $D_n > 0$. Then w is admissible, and $e^{Dt^{\alpha_n}}$ is a suitable choice for $\gamma(t)$, if D is large enough. In particular, it is easy to check that if $w(t) = t(4 + \sin t)e^{-t}$ then the second derivative of $-\log(w(e^t))$ takes some negative values on (A, ∞) for any $A > 0$. This property ensures that the results of [30] are not applicable to admissible weights.

Furthermore let us define another property of a weight function. The classical weight functions, and also our examples above, fulfil this "normality" condition, as we can see later.

Definition 2 *Let us call a weight function w^2 with finite moments "normal", if the largest zero of the n^{th} orthogonal polynomial $(x_{1,n})$ with respect to w^2, can be estimated as:*

$$x_{1,n} \leq e^{cn},$$

where $c = c(w)$ is a positive constant independent of n.

Examples :

In the cases of Laguerre and Freud weights $x_{1,n} \leq cn^\lambda$, where $\lambda = \lambda(w)$ is a positive constant depending on the weight function, moreover the same estimation is valid for a more general classes of weights on the real axis ([20] p. 313. Th. 11.1). As an application of the result of A. Markov ([25] p. 115. Th. 6.12.2), we can get a similar estimation on the examples above; for instance $w(x) = x^\gamma e^{-x^\alpha}$ with $\alpha \geq 1$, there is a $\beta > 0$ integer, such that with $W(x) = x^\beta e^{-x}$, the quotient $\frac{W}{w}$ is increasing on $(0, \infty)$; if $w(x) = x(4 + \sin x)e^{-x}$, then the corresponding W can be $W(x) = x^2 e^{-\frac{x}{2}}$, that is $\frac{W}{w} = \frac{xe^{\frac{x}{2}}}{4+\sin x}$ is increasing on $(0, \infty)$. So by Markov's theorem $x_{1,n}(w) \leq x_{1,n}(W) \leq cn$ (see [25], p. 127. Th. 6.31.2).

Remark:

Müntz-type theorems will be proved in L^2_w-spaces. As it was mentioned in the introduction, there are several results with respect to this. Examples above show that all the classical, and previously used weights are included in the classes of weights given above. Our main example is $w(t) = (4 + \sin t)t^\beta \prod_{k=1}^{n} e^{-D_k t^{\alpha_k}}$, which is admissible and normal, and for which the convexity properties required before, are false.

In connection with the weight function, let us introduce another notation, which will be very useful in formulating our results.

Definition 3 *Let w be a positive continuous weight function with w^2 having finite moments. Then define $K(x)$ corresponding to $w(x)$ as*

$$K(x) = \int_0^\infty t^{2x} w^2(t) dt, \quad x > -\frac{1}{2}, \tag{5}$$

and

$$\varphi(x) = \left(\int_0^\infty t^{2x} w^2(t) dt \right)^{\frac{1}{2x}} = (K(x))^{\frac{1}{2x}}, \quad x > 0. \tag{6}$$

Remark:
If $w(0) \neq 0$, $K(x)$ tends to infinity as x tends to $-\frac{1}{2}$. So in this case $\frac{1}{K(x)}$ will be defined at $-\frac{1}{2}$ as 0.

Our aim is formulating a theorem similar to Theorem B, thus we need some further definitions with respect to the exponent system.

Definition 4 *Let us suppose that the real numbers* $\{a_k\}$ *fulfil*

$$a_{k+1} - a_k \geq d > 0 \quad k = 0, 1, \ldots, \quad \textit{with} \quad a_0 = 0. \tag{7}$$

Now let

$$S = span\{t^{a_k} : k = 1, 2, \ldots\}. \tag{8}$$

Definition 5 *With the notations of the previous definition, let us define (as in [15] and Theorem B above)*

$$m(r) = \begin{cases} \frac{1}{a_1}, & \textit{if } 0 \leq r \leq a_1 \\ \sum_{a_k < r} \frac{1}{a_k}, & \textit{if } r > a_1 \end{cases} \tag{9}$$

and let

$$\Psi(r) = e^{2m(r)}. \tag{10}$$

We are now in a position to state the main results of this note. Proofs will follow in the next Section.

Theorem 1 *Let* w *be an admissible and normal weight function on* $[0, \infty)$*. If there exists a monotone increasing, nonnegative function* f *on* $[0, \infty)$*, such that for all* $0 < x \leq r$

$$x \log \frac{\Psi(r)}{\varphi(x)} \leq f(r), \tag{11}$$

and

$$\int_1^\infty \frac{f(r)}{r^2} dr < \infty, \tag{12}$$

then S *is not dense in* $L^2_w(0, \infty)$*.*

This result is then complemented by the following positive result.

Theorem 2 *Let w be positive, continuous, normal weight function on $(0,\infty)$ with finite moments.*

If there exists a monotone increasing, nonnegative function h on $[0,\infty)$, for which there is a constant $D>0$ such that

$$h(2r) < Dh(r) \tag{13}$$

on $[0,\infty)$, and there are $\alpha, C, c > 0$, such that for all $0 < x \le r$

$$0 < h(r) \le C^{\frac{1}{x}} \frac{cx}{\varphi^\alpha(x)} \Psi^\alpha(r), \tag{14}$$

and

$$\int_1^\infty \frac{h(r)}{r^2} dr = \infty, \tag{15}$$

then S is dense in $L^2_w(0,\infty)$.

Remark:

Let

$$B_\alpha(r) = \inf_{x\in(0,r)} C^{\frac{1}{x}} \frac{x}{\varphi^\alpha(x)}.$$

By this function, depending only on r, one can formulate Theorem 2 as

"If there exists an increasing function on $[0,\infty)$ with $0 \le h(r) \le cB_\alpha(r)\Psi^\alpha(r)$, for which (13) and (15) are valid, then S is dense in $L^2_w(0,\infty)$."

Theorem 2 can be stated also in $L^p_w(0,\infty)$, with $1 \le p < \infty$, and in $C_{w,(0,\infty)}$ with the same proof. That is, let us define

Definition 6

$$\varphi_p(x) = \left(\int_0^\infty t^{px} w^p(t)dt\right)^{\frac{1}{px}}, \quad x > 0, \quad 1 \le p < \infty$$

and

$$\varphi_c(x) = \left(\sup_{t>0} t^x w(t)\right)^{\frac{1}{x}}, \quad x > 0.$$

Now we formulate the following theorem:

Theorem 3 *Let w be positive, continuous and normal on $(0,\infty)$, and let us assume that $t^x w(t) \in L^p_{(0,\infty)}$ in the L^p_w-case, and that for all $a > 0$ $\lim_{\substack{t\to 0+\\ t\to\infty}} t^a w(t) = 0$ in the C_w-case. If there is a monotone increasing function h on $(0,\infty)$ with the properties (13) and (15), and for which there are $\alpha, C, c > 0$, such that for all $0 < x \le r$*

$$0 < h(r) \le C^{\frac{1}{x}} \frac{cx}{\varphi^\alpha_{p/c}(x)} \Psi^\alpha(r),$$

then S is dense in $L^p_w(0,\infty)$/in $C_{w,(0,\infty)}$.

Comparing the conditions of Theorem 1 and Theorem 2, we conclude the following:

Corollary:

If w is admissible and normal on $[0,\infty)$, and if there is a nonnegative increasing function h on $[0,\infty)$ for which (11), (13) and (14) are valid with some positive constants α, C, c, then S is dense in $L^2_w(0,\infty)$ if and only if

$$\int_1^\infty \frac{h(r)}{r^2}dr = \infty.$$

Remark:

Let us investigate now the connection of our result with the result of Fuchs. By the substitution $t = Du^\alpha$ (without any further restrictions on the exponents a_k for $\alpha \geq 1$, and with the restriction $a_k \neq \frac{1}{2}(\frac{1}{\alpha} - 1)$ for $0 < \alpha < 1$), after some obvious estimations one can deduce from the result of Fuchs (Theorem B), that $\{t^{a_k}e^{-Dt^\alpha}\}$ is dense if and only if $\int_1^\infty \frac{\Psi^\alpha(r)}{r^2}dr = \infty$. We will show that we get the same from Theorems 1 and 2, when $w(t) = \sqrt{m}e^{-Dt^\alpha}$. (Without loss of generality we can deal with cw instead of w so we can assume that $K(0) < 1$.) At first let us observe, that according to the remark after Theorem 2, if $B_\alpha(r) > B > 0$ $(\forall r \geq 1)$, then $h(r) = B\Psi^\alpha(r)$ can be a good choice. Therefore we will compute, that when $w(t) = \sqrt{m}e^{-Dt^\alpha}$, then $\inf_{x\in(0,r)} \frac{x}{\varphi^\alpha(x)} > B > 0$. Indeed with this weight

$$K(x) = m\int_0^\infty t^{2x}e^{-2Dt^\alpha}dt = \frac{m}{\alpha(2D)^{\frac{2x+1}{\alpha}}}\Gamma\left(\frac{2x+1}{\alpha}\right).$$

By Stirling's formula (see eg. [16], Vol. 2. p. 42.)

$$\frac{x}{\varphi^\alpha(x)} = \frac{x}{\left(\frac{m\sqrt{2\pi}\left(\frac{2x+1}{\alpha}\right)^{\frac{2x+1}{\alpha}-\frac{1}{2}}e^{-\frac{2x+1}{\alpha}}e^{J\left(\frac{2x+1}{\alpha}\right)}}{\alpha(2D)^{\frac{2x+1}{\alpha}}}\right)^{\frac{\alpha}{2x}}} = (*),$$

where J is the Binet function (see the literature cited above). For $t > 0$ we have $0 < J(t) < \frac{1}{12t}$. At first we estimate the $(*)$ expression from below.

$$(*) \geq \frac{2De\alpha x}{2x+1}\left(\frac{2De\alpha}{2x+1}\left(\frac{\alpha(2x+1)}{2\pi}\right)^{\frac{\alpha}{2}}\frac{1}{m^\alpha e^{\frac{\alpha^2}{12(2x+1)}}}\right)^{\frac{1}{2x}} = b(D,\alpha,x)$$

It can be seen, that $b(D,\alpha,x)$ tends to $De\alpha$ when x tends to infinity. Since $K(0) < 1$, $\lim_{x\to 0+}\frac{x}{\varphi^\alpha(x)} = \infty$. The behavior at the endpoints establishes the boundedness of $\frac{x}{\varphi^\alpha(x)}$ from below on $(0,\infty)$.

(In the case of Fuchs, when $\alpha = 1$, $D = \frac{1}{2}$, let $m < 1$, and so $K(0) \leq 1$.)

So let us choose $h(r) = cf(r) = c_1\Psi^\alpha(r), r \geq 0$.

After the previous chain of ideas we need to check the assumptions of Theorem 1. Recalling that $\Psi(r) \geq 1$, around zero it is obvious that

$$x \log \Psi(r) - \log \sqrt{K(x)} \leq c\Psi^{\alpha}(r)$$

with some constant c. On the interval $\left(\frac{1}{2}, \infty\right)$, say we give an estimation on the expression $(*)$ from above.

$$(*) \leq \frac{2De\alpha x}{2x+1} \left(\frac{2De\alpha}{2x+1} \left(\frac{\alpha(2x+1)}{2\pi} \right)^{\frac{\alpha}{2}} \frac{1}{m^{\alpha}} \right)^{\frac{1}{2x}},$$

that is $(*)$ tends to $De\alpha$ at infinity again, so it is bounded by a constant from above on $\left(\frac{1}{2}, \infty\right)$. That is $\varphi(x) \geq cx^{\frac{1}{\alpha}}$ here, and so

$$\log \frac{\Psi(r)}{\varphi(x)} \leq \frac{1}{\alpha} \log \frac{\Psi^{\alpha}(r)}{cx} \leq c_1 \frac{\Psi^{\alpha}(r)}{x},$$

therefore (11) is valid with $f(r) = c\Psi^{\alpha}(r)$ on $(0, \infty)$.

Finally the choice $h(r) = cf(r) = c_1 \Psi^{\alpha}(r)$ is exactly the original result of Fuchs.

After getting back the old results, we show an example to the new one.

Example :

It can be easily seen that if $w^2(t) = t(4+\sin t)e^{-t}$, then w is also admissible and normal, and $-\log\left(w\left(e^t\right)\right)$ is not convex. We will show that with this weight function $h(r) = cf(r) = c_1\Psi(r)$ is a good choice again, that is by the Corollary of Theorems 1. and 2. one can get an "if and only if" result:

If $w(t) = \sqrt{t(4+\sin t)e^{-t}}$, *then* S *is dense in* $L^2_w(0,\infty)$ *if and only if*

$$\int_1^{\infty} \frac{\Psi(r)}{r^2} dr = \infty.$$

As in the classical case, we want to estimate $\frac{x}{\varphi(x)}$ fom above and from below. (Now $\alpha = 1$.) Let

$$K(n) = \int_0^{\infty} t^{2n+1}(4+\sin t)e^{-t} dt, \quad K_0(n) = \int_0^{\infty} t^{2n} e^{-t} dt, \quad n \in \mathbb{N}.$$

At first we will compare $\frac{n}{K(n)^{\frac{1}{2n}}}$ and $\frac{n}{K_0(n)^{\frac{1}{2n}}}$. We have

$$K(n) = 4K_0\left(n + \frac{1}{2}\right) + \int_0^{\infty} t^{2n+1}(\sin t)e^{-t} dt.$$

By recurrence formulas for the integrals $I(m) = \int_0^{\infty} t^m (\sin t) e^{-t} dt$ and $J(m) = \int_0^{\infty} t^m (\cos t) e^{-t} dt$ (which are $I_m = \frac{m}{2}(I_{m-1} + J_{m-1})$; $J_m = \frac{m}{2}(J_{m-1} - I_{m-1})$), one can get that

$$I(2n+1) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^l \frac{(2n)!}{2^{n+1}}, & \text{if } n = 2l \end{cases},$$

that is

$$K(n) = 4K_0\left(n+\frac{1}{2}\right) + \varepsilon\frac{1}{2^{n+1}}K_0(n) = \left(4(2n+1)+\varepsilon\frac{1}{2^{n+1}}\right)K_0(n),$$

where $\varepsilon = 0$, or $\varepsilon = \pm 1$. It yields that $\frac{n}{K(n)^{\frac{1}{2n}}} \sim \frac{n}{K_0(n)^{\frac{1}{2n}}}$ (i.e. the quotient of the two quantities is between two positive constants). We have to show that the same holds for an arbitrary $x > 0$. At first we will show that $K(x)$ is increasing.

$$K^{'}(x) = 2\int_0^1 t^{2x+1}\log t(4+\sin t)e^{-t}dt + 2\int_1^\infty t^{2x+1}\log t(4+\sin t)e^{-t}dt = 2(N+P)$$

$$|N| \le -5\int_0^1 t^{2x+1}\log t = \frac{5}{(2x+2)^2},$$

$$P \ge 3\int_1^\infty t^{2x+1}e^{-t}dt \ge 3\Gamma(2x+2) - 3\int_0^1 t^{2x+1}dt,$$

that is

$$K^{'}(x) \ge 3\Gamma(2x+2) - \frac{3}{2x+2} - \frac{5}{(2x+2)^2} > 0, \quad x > 0.$$

Let $n \le x < n+1$! According to the previous calculation,

$$K(n+1) \sim nK_0(n+1) \sim n^3K_0(n) \sim n^2K(n).$$

Finally, if $K(n) > 1$,

$$c(K(n+1))^{\frac{1}{2(n+1)}} \le \left(\frac{K(n+1)}{cn^2}\right)^{\frac{1}{2(n+1)}} \le (K(n))^{\frac{1}{2(n+1)}} \le (K(x))^{\frac{1}{2x}}$$

$$\le (K(n+1))^{\frac{1}{2n}} \le (cn^2)^{\frac{1}{2n}}(K(n))^{\frac{1}{2n}} \le c(K(n))^{\frac{1}{2n}},$$

and if $K(n) < 1$, similarly we get the same. That is these computations yield that $\frac{n}{K(n)^{\frac{1}{2n}}} \sim \frac{x}{K(x)^{\frac{1}{2x}}}$, and similarly $\frac{n}{K_0(n)^{\frac{1}{2n}}} \sim \frac{x}{K_0(x)^{\frac{1}{2x}}}$. Because all the computations are valid with mK and mK_0, choosing an m such that $mK(0), mK_0(0) < 1$, we can finish our proof as in the classical case.

After these computations we have to note that for the formal generalization of the results of Fuchs (cf. eg. Lemma 1 in [15]), one can use $\varphi(x)^x$ instead of x^x. We have to note here, that this function x^x appears in several papers on momentum problems and density theorems, and in weighted cases $\varphi(x)^x$ seems to be the suitable expression.

3 Proofs

Before starting the proofs, we have to remark, that

$$K(x) = \mathcal{M}(w^2, 2x+1),$$

that is $K(x)$ is the Mellin transform of w^2, where

$$\mathcal{M}(f,s) = \tilde{f}(s) = \int_0^\infty t^{s-1} f(t)dt. \tag{16}$$

We will use the Mellin transform technique in the proofs, so some important properties of the Mellin transform will be enumerated here.

If the integral in the definition of $\mathcal{M}(f,s)$ converges absolutely on the line $\Re s = c$, and $f(t)$ is of bounded variation in a neighborhood of a $t_0 > 0$, then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s) t_0^{-s} ds = \frac{f(t_0+) + f(t_0-)}{2} \tag{17}$$

in principal value sense (see [27], p. 246. Th. 9.a).

A Parseval's formula also can be derived by the inversion formula. Computing on the domain of absolute convergence

$$\int_0^\infty t^{2z+1} h(t)\bar{g}(t)dt = \int_0^\infty t^z h(t) \frac{1}{2\pi} \int_{\mathbb{R}} \bar{\tilde{g}}(\bar{s} + \bar{z} + 1) t^{-s} d\tau dt,$$

where $s = \sigma + i\tau$. If the suitable conditions are fulfiled, changing the order of integration, and substituting $z - s = w = u + iv$, the computations can be continued as follows. Let $z = \Re w$, that is $z = x = u$. With $g = h$, if the inversion formula on $\bar{g}$ is valid on $\Re s = 0$, one can get

$$\int_0^\infty t^{2x+1} |h(t)|^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\tilde{h}(w+1)|^2 dv \tag{18}$$

(see [15] Lemma 1 (6)).

For the proof of the first theorem, at first we need a lemma:

Lemma 1 *Let $a = m$ be a positive integer. If w^2 is a continuous, positive, normal weight function on $[0,\infty)$ with finite moments, then there is a function $b(z)$ such that $b(z)$, $\frac{1}{b(z)}$ are regular on $\Re z > -a-1$, and it fulfils the inequality on $\Re z \geq -\frac{1}{2}$:*

$$\sqrt{\frac{K(x+a)}{K(x)}} \leq |b(z)|, \tag{19}$$

where $z = x + iy$.

Proof:

At first let x also be a positive integer, $x = n$. Then, using the Gaussian quadrature formula on the zeros of the N^{th} orthogonal polynomials ($x_{1,N} > \ldots > x_{k,N} > \ldots > x_{N,N}$) with respect to w^2, where $N = n + m + 1$, we get, that

$$\frac{K(n+m)}{K(n)} = \frac{\int_0^\infty t^{2(n+m)} w^2(t)dt}{\int_0^\infty t^{2n} w^2(t)dt} = \frac{\sum_{k=1}^N \lambda_{k,N} x_{k,N}^{2(n+m)}}{\sum_{k=1}^N \lambda_{k,N} x_{k,N}^{2n}} \leq x_{1,N}^{2m},$$

that is, by the condition of "normality"

$$\sqrt{\frac{K(n+m)}{K(n)}} \leq e^{cNm}.$$

Now we can show, that $\frac{K(x+a)}{K(x)}$ is increasing on $x > -\frac{1}{2}$, that is

$$\left(\frac{K(x+a)}{K(x)}\right)' = \frac{K(x+a)}{K(x)}\left(\frac{K'}{K}(x+a) - \frac{K'}{K}(x)\right), \tag{20}$$

which is nonnegative, because $\frac{K'}{K}$ is increasing. The last statement can be seen by the Cauchy-Schwarz inequality. It yields that the derivative of $\frac{K'}{K}$ is nonnegative:

$$\left(K'(x)\right)^2 \leq \left(2\int_0^\infty t^{2x}|\log t| w^2(t)dt\right)^2$$

$$\leq \int_0^\infty t^{2x} w^2(t)dt \int_0^\infty t^{2x} 4\log^2 t\ w^2(t)dt = K(x)K''(x). \tag{21}$$

So with $a = m$ and $x \geq -\frac{1}{2}$,

$$\sqrt{\frac{K(x+a)}{K(x)}} \leq e^{ca(a+1+\lceil x \rceil)} \leq e^{ca(a+2+x)} = C(a)\left|e^{caz}\right|. \tag{22}$$

Remark:

(1) $|b(z)| > e^{2ca}$, on $\Re z \geq -a$.

(2) If $\left(\frac{\varphi(2x)}{\varphi(x)}\right)^x$ does not grow too quickly, then one can choose $b(z) = c(a)b_1(z)$, where $b_1(z)$ is independent of a, because
$\sqrt{\frac{K(x+a)}{K(x)}} \leq K^{\frac{1}{4}}(2a)\frac{K^{\frac{1}{4}}(2x)}{K^{\frac{1}{2}}(x)} = c(a)\left(\frac{\varphi(2x)}{\varphi(x)}\right)^x.$

(3) Usually we can give $b(z)$ in polynomial form, for instance if $w(t) = e^{-Dt^\alpha}$ then $\sqrt{\frac{K(x+a)}{K(x)}} = \frac{1}{(2D)^{\frac{a}{\alpha}}}\sqrt{\frac{\Gamma\left(\frac{2x+1}{\alpha}+\frac{2a}{\alpha}\right)}{\Gamma\left(\frac{2x+1}{\alpha}\right)}} \leq c(2x+1+2a)^{\frac{a}{\alpha}}$, and so $b(z) = c(z+2a)^n$, where $n > \frac{a}{\alpha}$ is an integer. (By this choice of $b(z)$, on the interval $[-a, -\frac{1}{2}]$, $|b(z)| > c$.)

Proof: of Theorem 1.

Let us extend $f(r)$ to $\mathbb{R}$ as $f(-r) = f(r)$. Let $a \geq 1$ be as in (2). Furthermore let a be an integer. Because $\int_1^\infty \frac{f(r)}{r^2}dr < \infty$, the function

$$p(z) = p(x+iy) = p(re^{i\vartheta}) = \frac{2}{\pi}(x+a+1)\int_{-\infty}^\infty \frac{f(t)}{(x+a+1)^2+(t-y)^2}dt \tag{23}$$

is harmonic on $\Re z > -a-1$. Since $f(t)$ is increasing, and $x^2+y^2=r^2$

$$p(z) \geq \frac{2}{\pi} f(r) \int_{|t|>r} \frac{x+a+1}{(x+a+1)^2+(t-y)^2} dt$$

$$= f(r)\frac{2}{\pi}\left(\pi - \left(\arctan\frac{r-y}{x+a+1} + \arctan\frac{r+y}{x+a+1}\right)\right) > f(r).$$

(In the last inequality we applied the height theorem of a triangle, which height is $x+a+1$, and it divides the corresponding side to two pieces, $r-y$ and $r+y$. That is, because $x+a+1 > \sqrt{(r-y)(r+y)}$, the angle lying opposite to the side must be less than $\frac{\pi}{2}$.)

Let us choose a function $q(z)$ so that $-p(z)+iq(z)$, and hence $g(z)=g_a(z)=e^{-p(z)+iq(z)}$, be regular on $\Re z > -a-1$. According to the assumptions of Theorem 1, for this $g(z) \not\equiv 0$ on $\Re z > -a-1$ we have that

$$|g(z)| \leq e^{-f(r)} \leq \left(\frac{\varphi(x)}{\Psi(r)}\right)^x \qquad \Re z \geq 0. \tag{24}$$

We will show that in this case S is not dense. For this let us define a regular function on the half plane $\Re z \geq 0$ by

$$H(z) = \prod_{k=1}^{\infty} \frac{a_k - z}{a_k + z} e^{\frac{2z}{a_k}}. \tag{25}$$

According to a Lemma of Fuchs ([15] Lemma 3)

$$|H(z)| \leq (C\Psi(r))^x \quad \text{on} \quad \Re z \geq 0. \tag{26}$$

Let us replace the a_k-s in the definition of $H(z)$ by a_k+a, and let us denote the new function by $H^*(z)$. Now, with the help of g and H^* we can define a function $G(z)=G_a(z)$ which is regular on $\Re z > -a-1$ (recalling that $a \geq 1$), with $G(a_k)=0$, $k=1,2,\ldots$:

$$G(z) = \frac{g(z+a)H^*(z+a)}{b(z)C_1^{z+a}}, \tag{27}$$

where C_1 is a suitable constant, and according to Lemma 1, $\frac{1}{b(z)}$ is regular on $\Re z > -a-1$, and on $\Re z \geq -\frac{1}{2}$ we have (19).

Because for an $a>0$ $\frac{K(x+a)}{K(x)}$ is positive, and it is bounded, when x tends to $-\frac{1}{2}+$, according to Lemma 1, we can suppose that $|b(z)| > \delta > 0$ on $\Re z \in \left[-a, -\frac{1}{2}\right]$.

We have to estimate $|H^*(z+a)|$. Because the sequence $\{a_k+a\}$ has the same properties as the sequence $\{a_k\}$, as [15] p. 95. after Lemma 5, on $\Re z \geq 0$

$$|H^*(z)| \leq (C\Psi(r))^x,$$

and

$$H^*(z+a)| \leq (C\Psi(|z+a|))^{x+a} \tag{28}$$

$(x \geq -a)$. This inequality implies that if C_1 is large enough, then according to (19) and (24)

$$|G(z)| \leq \sqrt{K(x)} \quad \text{on} \quad \Re z > -\frac{1}{2}, \tag{29}$$

and because $a > \frac{1}{2}$, on $\Re z \in \left[-a, -\frac{1}{2}\right]$:

$$|G(z)| \leq \frac{(\varphi(x+a))^{x+a}}{|b(z)|} \leq \frac{1}{\delta} \max_{x \in \left[-a,-\frac{1}{2}\right]} \sqrt{K(x+a)} = M. \tag{30}$$

In the followings we will show that if there exists a function G which is not identically zero, and is regular on $\Re z > -a-1$, and fulfils the equations $G(a_k) = 0$ $(k = 1, 2, \ldots)$, and the inequalities (29) and (30) are valid, then S is not dense.

For the purpose of showing this, we need to construct a function $0 \not\equiv k(t) \in L^2_w(0,\infty)$ such that $\int_0^\infty t^{a_k} k(t) w^2(t) = 0$ for $k = 1, 2, \ldots$. We give $k(t)$ by the inversion formula for the Mellin transform (see (17) and [27] p. 247. Th. 9.c; p. 238. Th.2) of the regular function: $\frac{G(z)}{(1+a+z)^2}$, on $\Re z > -a-1$. Let us define the function $u(t)$ by an (absolute convergent) integral along a line parallel with the imaginary axis

$$t\nu(t)u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(z)}{(1+a+z)^2} t^{-z} dy. \tag{31}$$

It can be easily seen (by taking the integral round a rectangle $x_k \pm iL$ $k = 1, 2,$ where $L \to \infty$) that the integral is independent of x. Let us choose

$$k(t) = \frac{\nu(Ct)u(Ct)}{w^2(t)}, \tag{32}$$

where C is the same as in (4). Using that

$$\frac{G(z)}{(1+a+z)^2} = \int_0^\infty \nu(t)u(t)t^z dt,$$

we have

$$\begin{aligned}\int_0^\infty t^{a_k} k(t) w^2(t) dt &= \frac{1}{C_1^{a_k+1}} \int_0^\infty v^{a_k-1} v u(v) \nu(v) dv \\ &= \frac{1}{C_1^{a_k+1}} \frac{G(a_k)}{(1+a+a_k)^2} = 0.\end{aligned} \tag{33}$$

We have to show, that $k(t) \in L^2_w(0,\infty)$.

$$\|k\|^2_{2,w} = \int_0^\infty \frac{u^2(Ct)\nu^2(Ct)}{w^2(t)} dt = \int_0^{\frac{A}{C}} (\cdot) + \int_{\frac{A}{C}}^\infty (\cdot) = I + II, \tag{34}$$

where $A = \max\{1, C\}$.

According to (3), and by the positivity of the coefficients in γ,

$$II \le \int_{\frac{A}{C}}^{\infty} \nu^2(Ct)u^2(Ct)\sum_{k=0}^{\infty} c_k t^{\gamma_k} dt \le \sum_{k=0}^{\infty} \frac{c_k}{C^{\gamma_k+1}} \int_A^{\infty} t^{\gamma_k}\nu^2(t)u^2(t)dt$$
$$= \sum_{\substack{k \\ \gamma_k < \frac{1}{3}}} (\cdot) + \sum_{\substack{k \\ \gamma_k \ge \frac{1}{3}}} (\cdot) = S_1 + S_2. \tag{35}$$

By the notation $h(t) = \nu(t)u(t)$, and $\tilde{h}(z+1) = \frac{G(z)}{(1+a+z)^2}$, using Parseval's formula for the Mellin transform (see (18))

$$\int_0^{\infty} t^{2x+1}\nu^2(t)u^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{G(z)}{(1+a+z)^2}\right|^2 dy$$
$$\le cK(x) \le cK\left(x+\frac{1}{2}\right), \tag{36}$$

where the equality is valid on $\Re z \ge -a$, the first inequality is on $\Re z > -\frac{1}{2}$, and the last inequality is on $\Re z \ge -\frac{1}{3}$ say, where we used again that $\frac{K(x+a)}{K(x)}$ is increasing, that is

$$0 < c \le \frac{K(\frac{1}{6})}{K(-\frac{1}{3})} \le \frac{K(x+\frac{1}{2})}{K(x)}.$$

Therefore, by (35) and (4)

$$S_2 \le c \sum_{\substack{k \\ \gamma_k \ge \frac{1}{3}}} \frac{c_k}{C^{\gamma_k+1}} \int_0^{\infty} t^{\gamma_k}\nu^2(t)u^2(t)dt \le c\sum_{k=0}^{\infty} \frac{c_k}{C^{\gamma_k+1}} \left(\varphi\left(\frac{\gamma_k}{2}\right)\right)^{\gamma_k}$$
$$\le c\sum_{k=0}^{\infty} \frac{c_k}{C^{\gamma_k+1}} \int_0^{\infty} t^{\gamma_k} w^2(t)dt \le c\sum_{k=0}^{\infty} c_k \int_0^{\infty} \left(\frac{t}{C}\right)^{\gamma_k} w^2(t)dt$$
$$= c\int_0^{\infty} \gamma\left(\frac{t}{C}\right) w^2(t)dt < \infty. \tag{37}$$

To estimate S_1 and I, we will use that by (29) and (31), with $x = -\frac{1}{3}$

$$\nu^2(t)u^2(t) \le ct^{-\frac{4}{3}}\left(\varphi\left(-\frac{1}{3}\right)\right)^{-\frac{2}{3}}\left(\int_{-\infty}^{\infty} \frac{1}{\left|\left(\frac{2}{3}+a+iy\right)^2\right|}dy\right)^2 = ct^{-\frac{4}{3}}. \tag{38}$$

That is

$$t^{\gamma_k}\nu^2(t)u^2(t) \le ct^{\beta_k}, \quad \text{where} \quad \beta_k < -1,$$

and therefore the finite sum: S_1 is bounded. Similarly, if instead of $x = -\frac{1}{3}$ we use $x = -a$ in (31), we obtain by (30) that

$$t\nu(t)u(t) \le Mt^a \int_{\mathbb{R}} \frac{1}{|(1+iy)^2|}dy$$

that is

$$\nu^2(t)u^2(t) \leq cM^2t^{2a-2},$$

and so by (2)

$$I = \int_0^{\frac{A}{C}} \frac{u^2(Ct)\nu^2(Ct)}{\nu^2(t)\mu^2(t)}dt \leq c\int_0^{\frac{A}{C}} \frac{t^{2(a-1)}}{\nu^2(t)}dt < \infty. \tag{39}$$

This proves Theorem 1.

We now turn to the proof of Theorem 2. We will need a technical lemma. Following carefully the proof of Lemma 7 – Lemma 11 in [15], actually W. Fuchs proved the following:

Lemma 2 *[15] If there is a nonnegative, monotone increasing function h on $(0, \infty)$, which fulfils (13), and*

$$\int_1^{\infty} \frac{h(r)}{r^2}dr = \infty, \tag{40}$$

and if there is a function g regular on $\Re z \geq 0$ such that there are $C, c > 0$, $\alpha > 0$

$$|g(z)| \leq C\left(\frac{cx}{h(r)}\right)^{\frac{x}{\alpha}}, \tag{41}$$

then

$$g \equiv 0 \quad on \quad \Re z \geq 0. \tag{42}$$

Remark:

In Lemma 2 C and c means that instead of a regular function g another regular function: $bA^z g(z)$ can be considered (A, b are positive constants). It means that $\Psi(r)$ can be replaced by a function $\Psi_1(r)$ such that $\frac{\Psi}{\Psi_1}$ lies between finite positive bounds, and $\Psi_1(r)$ has a continuous derivative. Therefore in the followings we will assume that $\Psi(r)$, that is $m(r)$, is continuously differentiable, if it is necessary. Furthermore since $m(r)$ is increasing, we will assume that the derivative of m is nonnegative. If it is necessary, we can assume the same on h.

Proof: of Theorem 2.

From (14), and the previous lemma it follows that if a function $g(z)$ is regular on $\Re z \geq 0$, and it satisfies the inequality

$$|g(z)| \leq \left(\frac{c}{\Psi(r)}\right)^x \sqrt{K(x)}, \tag{43}$$

then $g \equiv 0$. Namely, if $r \geq x > 0$, then (14) and (45) together gives (43), and by the definition of φ and Ψ,

$$\lim_{x \to 0+} \left(c\frac{\varphi(x)}{\Psi(r)}\right)^x = \|w\|_{2,(0,\infty)}, \tag{44}$$

so we can choose a constant C, such that

$$\left(\frac{\varphi(x)}{\Psi(r)}\right)^{x} \leq C\left(\frac{cx}{h(r)}\right)^{\frac{x}{\alpha}} \quad \text{on } \Re z \geq 0. \tag{45}$$

Now let us assume, by contradiction, that S is not dense in L^2_w. In this case there is a function $f \not\equiv 0$ in L^2_w, such that the function

$$G(z) = \int_0^\infty t^z f(t) w^2(t) dt \tag{46}$$

defined on $\Re z \geq 0$, satisfies the equalities

$$G(a_k) = 0 \quad k = 1, 2, \ldots \tag{47}$$

and we can estimate its modulus by the Cauchy-Schwarz inequality

$$|G(z)| \leq \|f\|_{2,w}\sqrt{K(x)}. \tag{48}$$

Let us define now on $\Re z \geq 0$

$$g(z) = \frac{G(z)}{H(z)C_1^{z+1}}, \tag{49}$$

where H is as in (25). By Lemma 4 [15]

$$|H(z)| \geq (C_2\Psi(r))^x \tag{50}$$

on $\mathbb{C}\setminus\cup_{k=1}^\infty B\left(a_k, \hat{d}\right)$, where $B\left(a_k, \hat{d}\right)$ are open balls around a_k with fixed radius $\hat{d} < \min\{\frac{1}{5}, \frac{d}{4}\}$, and on the imaginary axis without exception. This implies that

$$|g(z)| \leq C\left(c\frac{\varphi(x)}{\Psi(r)}\right)^{x} \tag{51}$$

on $\Re z \geq 0 \setminus \cup_{k=1}^\infty B\left(a_k, \hat{d}\right)$ (and on the imaginary axis (see (44)). We will show that because g is regular on $\Re z \geq 0$, (51) holds on the whole half-plane. According to (47) it will imply by Lemma 2 that $g \equiv 0$, and hence $G \equiv 0$, which is a contradiction.

At first we deal with $\Psi(r)$. Around a_k $\Psi(r) = \Psi(a_k)$ or $\Psi(r) = \Psi(a_{k+1})$. Since $m(a_k) < m(a_{k+1}) < m(a_k) + \frac{1}{a_1} \leq m(a_k) + \frac{1}{d}$ (see (7)), that is

$$\Psi(a_k) < \Psi(a_{k+1}) < c(d)\Psi(a_k). \tag{52}$$

Then, by the same chain of ideas, we will deal with $K(x)$, when $x > 0$. According to Lemma 1, (see (22))

$$K(x+1) \leq c_1 e^{cx} K(x), \tag{53}$$

where c_1 and c are absolute constants. Let $0 \leq \delta \leq \frac{2}{5}$! By the Lagrange's theorem

$$0 < K(x+\delta)$$

$$= K(x) + 2\delta \int_0^\infty t^{2x} t^{2\delta_1} \log t \ w^2(t)dt \le K(x) + 2\delta \int_1^\infty t^{2x} t^{2\delta_1} \log t \ w^2(t)dt$$

$$\le K(x) + 2\delta \int_0^\infty t^{2(x+1)} w^2(t)dt \le K(x) + 2\delta c_1 e^{cx} K(x) \le c_2^x K(x), \tag{54}$$

where $0 \le \delta_1 \le \delta$, and we used (53). Similarly, with the notations above, because $x > 0$, $0 < \delta < \frac{1}{2}$, $K(x-\delta)$ exists, and

$$K(x-\delta) = K(x) - 2\delta \int_0^\infty t^{2x} t^{-2\delta_1} \log t \ w^2(t)dt$$

$$= K(x) + 2\delta \int_0^1 t^{2x-2\delta_1} |\log t| w^2(t)dt - 2\delta \int_1^\infty t^{2x-2\delta_1} \log t \ w^2(t)dt$$

$$= K(x) + 2\delta(A - B).$$

Now if $A \le B$,

$$0 < K(x-\delta) \le K(x). \tag{55}$$

If $A > B$, then, because $2\delta_1 < 1$ and $x > 0$

$$0 < A - B < A \le \max_{t\in[0,1]} w^2(t) \int_0^1 t^{2(x-\delta_1)} (-\log t)dt$$

$$= \frac{\max_{t\in[0,1]} w^2(t)}{2x - 2\delta_1 + 1} \int_0^1 t^{2(x-\delta_1)} dt \le \frac{\max_{t\in[0,1]} w^2(t)}{(2x-2\delta_1+1)^2} < \frac{c}{(x+\varepsilon)^2},$$

where $\varepsilon > \frac{1}{10}$. Because

$$K(x) > \int_1^2 t^{2x} w^2(t)dt \ge \min_{t\in[1,2]} w^2(t) \int_1^2 t^{2x} dt \ge c\frac{2^{2x+1}}{2x+1},$$

there is an absolute constant c_3 such that

$$K(x-\delta) \le K(x) + 2\delta A \le c_3 K(x). \tag{56}$$

Returning to the estimation of $|g(z)|$ on $\Re z \ge 0$, it is enough to show that

$$|g^2(z)| \le C \left(\frac{c}{\Psi(r)}\right)^{2x} K(x) \quad \text{on} \quad \Re z \ge 0. \tag{57}$$

The inequality above holds on the circle $C(a_k, \hat{d})$. So if ζ is in the ball $B(a_k, \hat{d})$, then

$$|g^2(\zeta)| \le \sup_{z\in C(a_k,\hat{d})} C \left(\frac{c}{\Psi(r)}\right)^{2x} K(x)$$

$$\le C \max\left\{ \max_{\substack{z\in C(a_k,\hat{d}) \\ |z|\le a_k}} \left(\frac{c}{\Psi(a_k)}\right)^{2x} K(x), \max_{\substack{z\in C(a_k,\hat{d}) \\ |z|\ge a_k}} \left(\frac{c}{\Psi(a_{k+1})}\right)^{2x} K(x) \right\}.$$

So there is an $x_0 \in [a_k - \hat{d}, a_k + \hat{d}]$, such that

$$|g^2(\zeta)| \le C \left(\frac{c}{\Psi(a)}\right)^{2x_0} K(x_0), \tag{58}$$

where $a = a_k$ or $a = a_{k+1}$. Let $|\zeta| = \varrho$ and $\Re\zeta = \sigma$! Now $\delta := |\sigma - x_0| < 2\hat{d} \le \frac{2}{5}$. By the previous calculations

$$\left(\frac{c}{\Psi(a)}\right)^{2x_0} K(x_0) = \left(\frac{c}{\Psi(a)}\right)^{\pm 2\delta} \left(\frac{c}{\Psi(a)}\right)^{2\sigma} K(\sigma \pm \delta)$$

$$\le \left(\frac{c}{\Psi(a)}\right)^{\pm 2\delta} \left(\frac{c(d)c}{\Psi(\varrho)}\right)^{2\sigma} c_4^{\sigma} K(\sigma) \le C \left(\frac{c}{\Psi(\varrho)}\right)^{2\sigma} K(\sigma), \tag{59}$$

where in the last inequality, in the case of $+2\delta$, we used that $\Psi(r) > 1$. When the exponent is -2δ, we used that by the well-separated property of the sequence $\{a_k\}$, (recalling that $a = a_k$ or $a = a_{k+1}$) $\left(\frac{\Psi(a)}{c}\right)^{2\delta} \le c_5(d) k^{c_6(d)}$, that is there is a $c_7(d)$ such that $c_5(d) k^{c_6(d)} \le (c_7(d))^{kd} \le C(c_7(d))^{\sigma}$. Finally in the last term of (59), C is an absolute constant, and c is a constant which depends only on d. That is (59) (and so (51)) is valid on $\Re z \ge 0$ with constants depending only on d, which proves the theorem.

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