

# TRANSLATION BEYOND DELSARTE

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ABSTRACT. We introduce general translations as solutions to Cauchy or Dirichlet problems. This point of view allows us to handle for instance the heat-diffusion semigroup as a translation. With the given examples Kolmogorov-Riesz characterization of compact sets in certain  $L^p_\mu$  spaces are given. Pego-type characterizations are also derived. Finally for some examples the equivalence of the corresponding modulus of smoothness and K-functional is pointed out.

## 1. INTRODUCTION

In 1938 Delsarte introduced the notion of generalized translation, see [13]. His starting point was the modification of Taylor's formula by the eigenfunctions of an ordinary differential operator:  $D_x f(\lambda, \cdot) = \lambda f(\lambda, x)$ . The Taylor series of  $f = f(\lambda, z)$  around  $z_0$  at  $y$ , with the notation  $x = y - z_0$ , can be expressed as

$$f(\lambda, x) = \sum_{k=0}^{\infty} \lambda^k \varphi_k(x),$$

where  $D_x \varphi_0(x) = 0$  and  $D_x \varphi_k(x) = \varphi_{k-1}(x)$  ( $\varphi_{-1} = 0$ ). Moreover  $\varphi_k(0) = 0$  if  $k > 0$ , and  $\varphi_0(0) = 1$ .

He illustrated his idea with the following two examples:

$$D_x(f) = f', \quad f(\lambda, x) = e^{\lambda x}, \quad \varphi_k(x) = \frac{x^k}{k!} \text{ and}$$

$$D_x(f) = f'' + \frac{2\alpha+1}{x} f', \quad f(\lambda, x) = j_\alpha(i\sqrt{\lambda}x), \quad \varphi_k(x) = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{2k};$$

and introduced the translation operator below:

$$T_x^t := \sum_{k=0}^{\infty} \varphi_k(t) D_x^k.$$

The convergence of the series above implies that  $T_x^t f(\lambda, \cdot) = f(\lambda, x) f(\lambda, t)$ . Examination of the second example leads to the next chain of ideas. Denoting by  $u(x, t) := T_x^t f$  the construction ensures that the translated functions are solutions to the next Cauchy problem.

$$D_t u = D_x u, \quad u(x, 0) = f(x), \quad \frac{\partial}{\partial t} u(x, 0) = 0.$$

The initial study of the properties of generalized translation is due to Delsarte [13] and Levitan see e.g. [34]. Braaksma and Snoo in a series of papers dealt with the problem of introducing general translations via certain hyperbolic Cauchy problems, see e.g. [10], [11]. Examination of product formulae with respect to classical

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orthogonal polynomials led to definition of Laguerre and Jacobi translations, see e.g. [20], [18], [27].

General translation has widespread applications. Estimation of the operator norm leads to a maximum principle with respect to the hyperbolic problem in question. It gives certain convolution structures, see e.g. [6], [33]. It also leads to estimation of  $p$ -Christoffel function, see e.g. [4], [5]. As in the standard case approximation theoretic problems, as smoothness, best approximation or approximation by Cesaro means can be examined by general translation, see e.g. [28], [37]. By general translation compactness criteria can be derived too, see [30].

Here we extend the notion of translation as solutions to Cauchy or Dirichlet problems in parabolic and elliptic cases as well. This approach allows to handle e.g. the heat-diffusion semigroup as a translation. Although several different examples are listed and investigated below, this one is highlighted, because unlike the old and the other new examples this one inherits the comfortable semigroup property of the standard translation, i.e.  $T^{t_1}T^{t_2} = T^{t_1+t_2}$ . Moreover parabolicity allows to draw conclusions in  $d$ -dimension while the variable of the translation is one-dimensional. We show that the general translations introduced below possess the properties of the standard one.

The paper is organized as follows. In the next section different types of examples of translation are given. In the third section the notion of "regularity with respect to compactness" is introduced and is applied to derive Kolmogorov-Riesz type characterizations in certain  $L^p$ -spaces. The rest of this section is devoted to show that all the listed translations are regular. In the fourth section Pego-type characterizations of compact sets of  $L^2$  are derived by convolution method. The fifth section deals with the approximation theoretic aspect of translation that is with modulus of smoothness and K-functional.

## 2. THE TRANSLATION OPERATOR

We define the translation operator as follows.

**Definition 1.** *Let  $\Omega = I \times J \subset \mathbb{R}^d \times \mathbb{R}_+$ . Let  $L$  be a linear partial differential operator of order at most two and take a function  $f(x) \in C(I)$  or  $f \in L^p_\mu(I)$ , where  $\mu$  is a Radon measure on  $I$ . Define translation as the next linear operator: the translation of  $f$*

$$(1) \quad T_x^t f := T^t f(x) = u(x, t),$$

where  $u(x, t)$  is the solution to

$$(2) \quad Lu = 0, \quad (x, t) \in \Omega,$$

$$(3) \quad u(x, 0) = f(x), \quad x \in I,$$

where the last equality is meant in sup-norm or in  $p$ -norm.

**Remark.**

All of our examples can be given by appropriate integral transformations as well. The corresponding kernel functions are denoted by  $W_t(x, y)$ ,  $K_t(x, y)$ , etc. according to the standard notation. Here  $t > 0$ , and  $x, y \in \mathbb{R}^d$ ,  $d \geq 1$ . That is

$$(4) \quad T^t f(x) := \int_I W_t(x, y) f(y) dy.$$

It also makes the solution well-defined.

Below different types of examples are introduced. Of course, by the same chain of ideas several further examples can be constructed. Here are the ones we study subsequently.

**2.1. Heat-diffusion semigroup associated with Hermite functions.** This semigroup, from different point of views, is investigated by several authors, see e.g. [41], [1], [21] and the references therein.

In  $\mathbb{R}^d$  the eigenfunctions of the  $d$ -dimensional harmonic oscillator (Hermite operator),

$$(5) \quad D_x = \Delta_x - |x|^2,$$

are the  $d$ -dimensional Hermite functions,

$$\tilde{h}_\nu = \tilde{h}_{n_1} \dots \tilde{h}_{n_d},$$

where  $\nu = (n_1, \dots, n_d)$   $n_i \in \mathbb{N}$  and

$$\tilde{h}_k(x) = h_k(x)e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2^k k! \sqrt{\pi}}} H_k(x)e^{-\frac{x^2}{2}}.$$

( $H_k$  are the Hermite polynomials cf. [42].) The associated heat-diffusion semigroup is given by its kernel function defined on  $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$ ,

$$(6) \quad W_t(x, y) = \sum_{n=0}^{\infty} e^{-(2n+d)t} \sum_{|\nu|=n} \tilde{h}_\nu(x) \tilde{h}_\nu(y).$$

Thus for and appropriate  $f$ , denoting by  $\tilde{f}(y) := f(y)e^{-\frac{y^2}{2}}$ ,

**Example 1.**

$$T^t f(x) := \int_{\mathbb{R}^d} W_t(x, y) \tilde{f}(y) dy.$$

Denoting by  $u(x, t) := T^t f(x)$ , if  $\tilde{f} \in L^p(\mathbb{R}^d)$ ,  $1 \leq p \leq \infty$ , then

$$(7) \quad \left( \frac{\partial}{\partial t} - D_x \right) u(x, t) = 0$$

and

$$(8) \quad \lim_{t \rightarrow 0^+} \|T^t f - \tilde{f}\|_p = 0,$$

cf. [41, Proposition 2.5 and Theorem 2.6].

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**2.2. Further Poisson integrals - elliptic and parabolic equations.** Following the previous track of thoughts translation of a function in  $f \in L^p(\mathbb{R})$  can be defined by Poisson integrals. First we take the simplest elliptic and parabolic equations on the upper half-plane and define translations by standard convolution.

In the next example  $T_x^t f$  is a harmonic function on the (open) upper half-plane with limit  $f$  on the real line and with limit zero at infinity cf. e.g. [31], that is the solution to this Dirichlet problem

$$\Delta u(x, t) = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+, \quad u(x, 0) = f(x).$$

It is given by the Poisson integral below.

**Example 2.** Let  $f \in L^p(\mathbb{R})$ ,  $1 \leq p < \infty$  or  $f \in C_b(\mathbb{R})$ .

$$(9) \quad T^t f(x) := \frac{1}{\pi} \int_{\mathbb{R}} f(\xi) \frac{t}{(\xi - x)^2 + t^2} dt = f * K_{t,e}(x),$$

where  $C_b(\mathbb{R})$  stands for continuous and bounded functions on the real line.

— . . —

Considering the Cauchy problem

$$u_t(x, t) = b^2 \Delta u(x, t) \quad (x, t) \in \mathbb{R} \times (0, \infty), \quad u(x, 0) = f(x),$$

the next translation can be defined by the corresponding Poisson integral.

**Example 3.** Let  $f \in L^p(\mathbb{R})$  ( $1 \leq p \leq \infty$ ).

$$(10) \quad T^t f(x) := \int_{\mathbb{R}} f(\xi) \frac{e^{-\frac{(x-\xi)^2}{4b^2t}}}{2b\sqrt{\pi}\sqrt{t}} dt = f * K_{t,p}(x).$$

— . . —

We continue with an elliptic example again, where the Poisson integral is not of convolution type.

Let  $P_n^{(\lambda)}(\cos \vartheta)$  be the  $n$ th ultraspherical polynomial ( $\lambda > -\frac{1}{2}$ ,  $\vartheta \in [0, \pi]$ ) orthogonal with respect to  $d\mu(\vartheta) = \sin^{2\lambda} \vartheta d\vartheta$ , see e.g. [42]. An  $f \in L^p_\mu$  with some  $1 \leq p \leq \infty([0, \pi])$  has an expansion  $f \sim \sum_{k=0}^{\infty} a_k P_k^{(\lambda)}(\cos \vartheta)$  with  $a_k = \gamma_k \int_0^\pi f(\vartheta) P_k^{(\lambda)}(\cos \vartheta) d\mu(\vartheta)$ . In [35] the authors examined the next Poisson integral of  $f$ ,

$$u(x, t) = f(r, \vartheta) = \sum_{k=0}^{\infty} a_k r^k P_k^{(\lambda)}(\cos \vartheta) = \int_0^\pi P(r, \vartheta, \varphi) f(\varphi) d\mu(\varphi),$$

where  $0 < r < 1$ ,  $x = r \cos \vartheta$ ,  $t = r \sin \vartheta$ . The Poisson kernel is

$$P(r, \vartheta, \varphi) = \sum_{k=0}^{\infty} \gamma_k r^k P_k^{(\lambda)}(\cos \vartheta) P_k^{(\lambda)}(\cos \varphi).$$

Then  $u(x, t)$  satisfies the differential equation

$$u_{xx} + u_{tt} + \frac{2\lambda}{t} u_t = 0.$$

Since  $\|f(r, \vartheta) - f(\vartheta)\|_{\mu,p} \rightarrow 0$  as  $r \rightarrow 1$  if  $1 \leq p < \infty$  and also for  $p = \infty$  if  $f$  is continuous, in view of Definition 1 we have the next example. As above, the closed form of the Poisson kernel implies the definition below.

**Example 4.** Let  $D = [0, \pi] \times [0, 1)$ ,  $f \in L^p_\mu([0, \pi])$ .

$$T^t f(x) = u(x, t) = T^r f(\vartheta) = f(r, \vartheta) = \int_0^\pi P(r, \vartheta, \varphi) f(\varphi) d\mu(\varphi),$$

where

$$(11) \quad P(r, \vartheta, \varphi) = \frac{\lambda(1-r^2)}{\pi} \int_0^\pi \frac{\sin^{2\lambda-1} \xi}{(1 - 2r(\cos \vartheta \cos \varphi + \sin \vartheta \sin \varphi \cos \xi) + r^2)^{\lambda+1}} d\xi.$$

— . . —

Considering the next differential operator

$$D_x = L^\alpha := \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 + \frac{\alpha^2 - \frac{1}{4}}{x^2} \right)$$

with eigenfunctions

$$L^\alpha \varphi_n^{(\alpha)} = (2n + \alpha + 1) \varphi_n^{(\alpha)},$$

where

$$(12) \quad \varphi_n^{(\alpha)}(x) = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}} e^{-\frac{x^2}{2}} L_n^{(\alpha)}(x^2) \sqrt{2x}$$

( $L_n^{(\alpha)}$  is the standard Laguerre polynomial, see e.g. [42]), we define a translation by the solution to the parabolic Cauchy problem

$$u_t + L^\alpha u = u_t - \frac{1}{2} u_{xx} + \frac{1}{2} \left( x^2 + \frac{\alpha^2 - \frac{1}{4}}{x^2} \right) u = 0, \quad u(x, 0) = g(x),$$

$x, t \in (0, \infty)$ ,  $\alpha \geq -\frac{1}{2}$ , cf. [12].

**Example 5.**

$$T^t g(x) := \int_0^\infty W_t^\alpha(x, y) g(y) dy,$$

where

$$W_t^\alpha(x, y) = 2\sqrt{xy} \frac{e^{-t}}{1 - e^{-2t}} I_\alpha \left( \frac{2xye^{-t}}{1 - e^{-2t}} \right) e^{-\frac{1}{2}(x^2 + y^2) \frac{1+e^{-2t}}{1-e^{-2t}}},$$

where  $I_\alpha(z) = \sum_{k=0}^\infty \frac{(\frac{z}{2})^{2k+\alpha}}{\Gamma(k+1)\Gamma(k+\alpha+1)}$  is the modified Bessel function.

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**2.3. Fourier method - hyperbolic equation.** The first - well known - examples are given by hyperbolic equations generated by Sturm-Liouville type operators, see e.g. [13], [34]. Consider the operator

$$D_x = \frac{\partial^2}{\partial x^2} + q(x) \frac{\partial}{\partial x} - r(x).$$

If

$$D_x \varphi(x) = a\varphi(x),$$

denoting by  $u(x, t) = \varphi(x)\varphi(t)$ ,  $u(x, t)$  fulfils the equation below.

$$(13) \quad u_{xx} - u_{tt} + q(x)u_x - q(t)u_t - (r(x) - r(t))u = 0.$$

In Laguerre and Bessel cases  $q(x) = \frac{2\alpha+1}{x}$ , where  $\alpha \geq -\frac{1}{2}$  and  $r(x) = x^2$  or  $r(x) = 0$ , respectively. The eigenfunctions of  $D_x$  are the Laguerre functions  $\mathcal{L}_n^{(\alpha)}(x) = \frac{n!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x^2) e^{-\frac{x^2}{2}}$ , and the Bessel functions  $j_\alpha(\lambda x)$ , where  $L_n^{(\alpha)}(x)$  and  $j_\alpha(\lambda x)$  stand for the classical Laguerre polynomials and the entire Bessel functions, respectively. These examples are studied by several authors, see e.g. [6], [4], [27], [37], [5].

In Jacobi case with  $\alpha \geq \beta \geq -\frac{1}{2}$ ,  $\alpha > -\frac{1}{2}$  by the same argument,  $q(x) = \frac{(\beta+1)\cos x - \beta}{\sin x}$  and  $r(x) = \frac{\alpha(\alpha-2\beta+1)\cos x}{4(1-\cos x)}$ . The eigenfunctions are the Jacobi functions,  $\mathcal{P}_n^{(\alpha, \beta)}(x) = c_n P_n^{(\alpha, \beta)}(\cos x) \sin^\alpha \frac{x}{2}$ , where  $P_n^{(\alpha, \beta)}(x)$ -s are the classical Jacobi polynomials. For Jacobi translation see e.g. [20], [3].

In the listed cases the eigenfunctions of the Sturm-Liouville operators  $\{u_k\}_{k=0}^{\infty}$  form Schauder bases in the corresponding (weighted) spaces, that is if the initial condition is given by  $f(x) \sim \sum_{k=0}^{\infty} a_k u_k(x)$ , then the solution to the Cauchy problem can be given as  $T^t f(x) = u(x, t) \sim \sum_{k=0}^{\infty} a_k u_k(x) u_k(t)$ .

Thus translation generated by a hyperbolic equation is as follows.

$$T^t f(x) = u(x, t)$$

if

$$(14) \quad L_h u = 0, \quad (x, t) \in I \times I; \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad x \in I.$$

Here  $L_h$  is given by (13), where  $I = \mathbb{R}_+$  in Laguerre and Bessel cases and in Jacobi case  $I = (0, \pi)$ .

It is pointed out in a series of papers that Bessel, Laguerre and Jacobi translations are bounded operators, see the cited papers above and the references therein. The method described above implies the symmetry of the translation derived by a Sturm-Liouville equation, see [10], [11].

Next we give an example by Jacobi functions. Let  $\alpha \geq \beta \geq -\frac{1}{2}$ ,  $\alpha > -\frac{1}{2}$  again and let us denote by  $\varrho = \alpha + \beta + 1$  and now let  $\tau \in \mathbb{R}$ . Let

$$(15) \quad w(x) := 2^{2\varrho} \sinh^{2\alpha+1} x \cosh^{2\beta+1} x.$$

With  $q(x) = \frac{w'}{w}(x)$ ,  $\varphi_{\tau}^{(\alpha, \beta)}(x)$  fulfils the differential equation:

$$y'' + qy' + (\varrho^2 + \tau^2)y = 0,$$

where

$$\varphi_{\tau}^{(\alpha, \beta)}(x) := \varphi_{\tau}(x) = {}_2F_1 \left( \frac{\varrho + i\tau}{2}, \frac{\varrho - i\tau}{2}; \alpha + 1; -\sinh^2 x \right)$$

are the Jacobi functions. Thus the translation is defined as it follows, cf. e.g. [38], [39], [22] and the references therein.

**Example 6.** Let  $f$  be a suitable even function on  $\mathbb{R}$ .  $T^t f(x) = u(x, t)$  if

$$u_{xx} - u_{tt} + q(x)u_x - q(t)u_t = 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad x \in \mathbb{R},$$

where

$$q(x) = (2\alpha + 1) \coth x + (2\beta + 1) \tanh x.$$

### 3. REGULARITY WITH RESPECT TO COMPACTNESS

In this section we derive Kolmogorov-Riesz-type compactness criteria in certain  $L^p$ -spaces. For the original theorem see e.g. [2]. This theorem has several extensions to different function spaces with standard translation, see e.g. [7], [15], [25], [26], [40], [29] and with Bessel and Laguerre translations, see [30]. In all but one of our examples the translation is not symmetric. This implies that one of the criteria of compactness has to be splitted into two different assumptions, see Definition 4 below.

We start with some notation. Let  $\mu$  and  $\nu$  be Radon measures on  $I$  and  $J$ , respectively, and  $1 \leq p < \infty$ .  $\|f\|_{\mu(x), p} = \left( \int_I |f|^p d\mu \right)^{\frac{1}{p}}$ .  $T^0 f(x) = f(x)$ .

We define the "norm" of the translation operator  $M_T$  as

$$(16) \quad M_T := M_{T, p} = \sup_t \sup_{\|f\|_{\mu, p} \leq 1} \|T^t f(x)\|_{\mu(x), p}, \quad 1 \leq p \leq \infty.$$

We introduce the next notation: Let  $a > 0$ ,  $M_0 > 0$  fixed,  $B_a := \{\|x\| \leq a\} \cap I$ , or  $B_a := (-a, a) \cap J$ ;  $A := \int_{B_a} 1 d\nu(t)$ .

$$(17) \quad M_{a,R}f(x) := \begin{cases} \frac{1}{A} \int_{B_a} T^t f(x) d\nu(t) & |x| < R, \\ 0 & |x| \geq R. \end{cases}$$

**Definition 2.** A translation,  $T_x^t$  is regular with respect to  $p$ ,  $\mu$ ,  $\nu$  if it fulfils the properties below.

(T1) There is a dense set  $\mathcal{E}_1 \subset L_\mu^p(I)$  such that for all  $g \in \mathcal{E}_1$ ,  $0 < t < M_0$  and  $\varepsilon > 0$  there is a  $\delta = \delta(\varepsilon, M_0, g) > 0$  such that for all  $0 \leq |h| \leq \delta$

$$(18) \quad \left( \int_I |(T^{t+h}g(x) - T^t g(x))|^p d\mu(x) \right)^{\frac{1}{p}} < \varepsilon.$$

(T2) There is a dense set  $\mathcal{E}_2 \subset L_\mu^p(I)$  such that for all  $g \in \mathcal{E}_2$ , and positive numbers  $\varepsilon$ ,  $a$ ,  $R$  there is a  $\delta = \delta(g, \varepsilon, a, R) > 0$  such that if  $0 \leq |h| \leq \delta$ , then

$$(19) \quad |M_{a,R}g(x+h) - M_{a,R}g(x)| < \varepsilon, \quad x, x+h \in B_R.$$

(T3) There is a finite constant,  $c(a, R)$  so that for all  $f \in L_\mu^p(I)$ ,

$$(20) \quad |M_{a,R}f(x)| \leq c(a, R) \|f\|_{\mu,p}, \quad \forall x \in I.$$

**Remark.**

(R1) If  $T_x^y = T_y^x$  and  $\mu = \nu$ , then (18) implies (19) and if  $M_T < \infty$  it also implies (20). Indeed

$$|M_a g(x+h) - M_a g(x)| \leq \frac{1}{A} \left( \int_{B_a} 1 d\mu(y) \right)^{\frac{1}{p'}} \left( \int_{B_a} |T^{x+h}g(y) - T^x g(y)|^p d\mu(y) \right)^{\frac{1}{p}},$$

which ensures (19). (20) can be obtained in the same way.

**Definition 3.** A set  $K \subset L_\mu^p(I)$  is equivanishing, or we say it fulfils property  $\mathbf{P}_a$  if for all  $\varepsilon > 0$  there is an  $R > 0$  such that for all  $f \in K$

$$\left( \int_{I \setminus B_R} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} < \varepsilon.$$

**Definition 4.** A set  $K \subset L_\mu^p(I)$  is equicontinuous in mean if it fulfils the next properties

$\mathbf{P}_{b_1}$ : For all  $\varepsilon$  and  $M_0$  positive numbers there is a  $\delta > 0$  (independent of  $f$ ) such that for all  $t \in B_{M_0}$ ,  $0 \leq |h| \leq \delta$  and  $f \in K$  (18) is satisfied.

$\mathbf{P}_{b_2}$ : For all  $\varepsilon > 0$  there is a  $\delta = \delta(a, R) > 0$  (independent of  $f$ ) such that for all  $0 \leq |h| \leq \delta$  and  $f \in K$  (19) is satisfied.

**Theorem 1.** With the notation above, let  $T = T_x^t$  be a bounded translation in (16)-sense and suppose that it is regular with respect to  $p$ ,  $\mu$ ,  $\nu$ . Let  $K \subset L_\mu^p(I)$  be a bounded set. Then  $K$  is precompact if and only if it is equivanishing and equicontinuous in mean.

*Proof.* First we prove that if  $K$  is bounded, equivanishing and equicontinuous in mean and the translation has the property **T3**, then  $K$  is precompact.

For an arbitrary  $\varepsilon > 0$  let  $R$  be chosen according to **P<sub>a</sub>**. Let  $a > 0$  be fixed and will be chosen later, and define

$$F_{a,R} := \{M_{a,R}f(x) : f \in K, x \in B_R\}.$$

Then **T3** and the boundedness of  $K$  imply that  $F_{a,R}$  is (uniformly) bounded. According to **P<sub>b2</sub>**  $F_{a,R}$  is equicontinuous, thus it is precompact. Let  $M_a f_1, \dots, M_a f_n$  be an  $\varepsilon_1$ -net in  $F_{a,R}$ , where  $\varepsilon_1 = \frac{\varepsilon}{(\mu(B_R))^{\frac{1}{p}}}$ . We show that  $f_1, \dots, f_n$  is a  $5\varepsilon$ -net in  $K$ . Recalling the definition of  $R$  and  $A$

$$\begin{aligned} \|M_{a,R}f - f\|_{\mu,p} &\leq \|M_{a,R}f - f|_{B_R}\|_{\mu,p} + \left( \int_{I \setminus B_R} |f|^p d\mu \right)^{\frac{1}{p}} \\ &\leq \left( \int_{B_R} \left| \frac{1}{A} \int_{B_a} (T^t f(x) - f(x)) d\nu(t) \right|^p d\mu(x) \right)^{\frac{1}{p}} + \varepsilon \\ &\leq \frac{1}{A} \int_{B_a} \left( \int_I |T^t f(x) - f(x)|^p d\mu(x) \right)^{\frac{1}{p}} d\nu(t) + \varepsilon \\ &\leq \sup_{0 \leq |t| \leq a} \|T^t f(x) - f(x)\|_{\mu,p} + \varepsilon. \end{aligned}$$

Considering **P<sub>b1</sub>**, if  $a$  is small enough,  $\|M_{a,R}f - f\|_{\mu,p} \leq 2\varepsilon$  for all  $f \in K$ . Let  $f \in K$  be arbitrary. Selecting  $f_i$  such that  $|M_{a,R}f(x) - M_{a,R}f_i(x)| \leq \varepsilon_1$ ,

$$\|M_{a,R}f - M_{a,R}f_i\|_{\mu,p} \leq \varepsilon_1 (\mu(B_R))^{\frac{1}{p}} = \varepsilon.$$

Thus by triangle inequality

$$\|f - f_i\|_{\mu,p} \leq 5\varepsilon.$$

On the other hand suppose that  $K$  is precompact. Let  $\varepsilon > 0$  be arbitrary and  $\varphi_1, \dots, \varphi_n$  be an  $\frac{\varepsilon}{2}$ -net in  $K$ . As  $C_0$ , that is the set of compactly supported continuous functions, is dense in  $L^p_\mu(I)$ , there are  $g_1, \dots, g_n \in C_0$  so that for all  $f \in K$  there is an  $i$  such that  $\|g_i - f\|_{\mu,p} < \varepsilon$ . Then  $R$  is appropriate with respect to **P<sub>a</sub>** if  $B_R$  contains the supports of  $g_i$ ,  $i = 1, \dots, n$ .

Since  $T$  is bounded, selecting  $g_1, \dots, g_n \in \mathcal{E}_1$  as above, triangle inequality together with **T1** ensures **P<sub>b1</sub>**.

Similarly, selecting  $g_1, \dots, g_n \in \mathcal{E}_2$  as above, **T2**, **T3** and the triangle inequality imply **P<sub>b2</sub>**.

**Remark.** (R2) For compactness it is enough to assume that the initial condition, (3) is fulfilled uniformly. The importance of the extension above will be shown in the next section.

In the rest of this section we show that all the translations listed above are regular in sense of Definition 2. At first we make some observations which will be useful in parabolic cases below and in the last section.

**Remark.** (R3) For the next observation let us write the differential equations of the listed examples in the next form:

$$(21) \quad D_{(1),t}u(x,t) = D_{(2),x}u(x,t), \quad u(x,0) = f(x).$$



$T^t f(x) = u(x, t)$  can be defined by an integral transformation, that is

$$T^t f(x) = \int_I f(\xi) K(x, t, \xi) w(\xi) d\xi.$$

Thus the kernel function fulfils the differential equation above. As the kernel function is symmetric at least in  $x$  and  $\xi$ , i.e.

$$K(x, t, \xi) = K(\xi, t, x),$$

we have

$$D_{(1),t} K(x, t, \xi) = D_{(2),x} K(x, t, \xi) = D_{(2),\xi} K(x, t, \xi).$$

These observations imply the next lemma.

**Lemma 1.** *Let  $g$  be a smooth function on  $I$  and assume that its derivatives disappears quickly enough at the boundary of  $I$ . Then in the listed cases*

$$(22) \quad D_{(2)} T^t g = T^t D_{(2)} g.$$

*Proof.* Recall, that in our one dimensional examples

$$D_{(2)} h = h'' + \frac{w'}{w} h' + r h.$$

Integrating by parts and considering the boundary condition we have

$$\begin{aligned} D_{(2),x} T^t g(x) &= \int_I g(\xi) D_{(2),x} K(x, t, \xi) w(\xi) d\xi = \int_I g(\xi) D_{(2),\xi} K(x, t, \xi) w(\xi) d\xi \\ &= \int_I -\frac{\partial}{\partial \xi} K(x, t, \xi) (g w)'(\xi) + (g w)(\xi) \left( \frac{w'(\xi)}{w(\xi)} \frac{\partial}{\partial \xi} K(x, t, \xi) + r(\xi) \right) w(\xi) d\xi \\ &= \int_I K(x, t, \xi) D_{(2),\xi} g(\xi) w(\xi) d\xi = T^t D_{(2)} g(x). \end{aligned}$$

If  $w \equiv 1$ , it is more direct. Let us see for instance the heat diffusion semigroup. Recalling that  $I = \mathbb{R}^d$

$$\int_I g(\xi) (\Delta_\xi W_t(x, \xi) - |\xi|^2 W_t(x, \xi)) d\xi = \int_I (\Delta_\xi g(\xi) - g(\xi) |\xi|^2) W_t(x, \xi) d\xi.$$

**Corollary 1.** *Let  $D_{(1)} = \frac{\partial}{\partial t}$  and let  $g$  be as above. Then*

$$(23) \quad \|T^{t+h} g - T^t g\|_{p,\mu} \leq h M_T \|D_{(2)} g\|_{p,\mu}.$$

*Proof.*

$$T^{t+h} g(x) - T^t g(x) = h \frac{\partial}{\partial t} T^{t+\eta} g(x) = h D_{(2),x} T^{t+\eta} g(x) = h T^{t+\eta} D_{(2)} g(x).$$

Taking into consideration (16) we get the estimation.

**Lemma 2.** *The translation given in Example 1 is regular and  $M_T = 1$ .*

*Proof.* The boundedness of the operator by a constant  $C$  is proved in [41, Theorem 2.6]. It can be shown similarly that  $M_T = 1$ . Indeed, according to Mehler's formula and [21, Remark 2.5]

$$\begin{aligned} (24) \quad 0 < W_t(x, y) &= (2\pi \sinh 2t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2}} \coth 2t - \langle x, y \rangle \tanh t \\ &\leq (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4t}} = K_t(x-y). \end{aligned}$$

Thus

$$|T^t f| \leq |\tilde{f}| * K_t$$

and since  $\|K_t\|_1 = 1$

$$\|T^t f\|_p \leq \| |\tilde{f}| * K_t \|_p \leq \|\tilde{f}\|_p,$$

$1 \leq p \leq \infty$ .

Let  $\mathcal{E}_1 = C_0^2 \subset L^p(\mathbb{R}^d)$ . In view of (23) (T1) is fulfilled.

We prove property (T2) by similar arguments. Considering the uniform convergence of (6) for a  $t > 0$ , the recurrence and derivation formulae for Hermite functions imply that

$$\frac{\partial}{\partial x_j} W_t(x, y) = (y_j e^{-2t} - x_j) W_t(x, y) - e^{-2t} \frac{\partial}{\partial y_j} W_t(x, y).$$

Thus

$$W_t(x + \chi, y) - W_t(x, y) = \langle e^{-2t} y - (x + \xi), \chi \rangle W_t(x + \xi, y) - e^{-2t} \langle \nabla_y W_t(x + \xi, y), \chi \rangle.$$

Let  $f \in C_0^1(\mathbb{R}^d)$ ,  $\text{supp } f \subset B_r$ ,  $x \in B_R$ .

$$\begin{aligned} & \left| \frac{1}{a} \int_0^a \int_{\mathbb{R}^d} (W_t(x + \chi, y) - W_t(x, y)) \tilde{f}(y) dy dt \right| \\ & \leq \frac{1}{a} \int_0^a \left( 2|\chi|(r + R) \|T^t |f|\|_p + \int_{\mathbb{R}^d} W_t(x + \chi, y) \langle \chi, \nabla_y \tilde{f}(y) \rangle dy \right) dt \\ & \leq |\chi| \left( 2(r + R) \|\tilde{f}\|_p + \|\nabla_y \tilde{f}\|_p \right). \end{aligned}$$

To prove (T3) let us recalling (24). Then we have

$$\left| \frac{1}{a} \int_0^a T^t f dt \right| \leq \frac{1}{a} \int_0^a \int_{\mathbb{R}^d} |\tilde{f}(y)| K_t(x - y) dy dt \leq \|\tilde{f}\|_p.$$

**Lemma 3.** *The translation given in Example 6 is regular and  $M_T = 1$ .*

*Proof.* For boundedness see [18, Lemma 5.2].

As it is mentioned above, in [11] it is pointed out that a translation generated by a hyperbolic equation is necessarily symmetric. It can be expressed by an integral transform whose kernel is symmetric in its three variables, see [18]. Thus according to remark (R1) it is enough to prove property (T1). Since  $f$  is even, we can take  $I = \mathbb{R}_+$ . Let  $\mathcal{D} = C_0^1(\mathbb{R}_+) \subset L_\mu^p(\mathbb{R}_+)$  the dense set in question, where

$$(25) \quad d\mu(x) = \frac{1}{\sqrt{2\pi}} w(x) dx,$$

cf. (15). Let  $f \in C_0^1(\mathbb{R}_+)$  and let  $f(z) = g(\cosh 2z)$ . In view of [18, (5.1)] and [18, (4.16)]

$$T^t f(x) = \int_{\mathbb{R}_+} f(z) K(x, t, z) d\mu(z) = \int_0^1 \int_0^\pi g(x, t, r, \psi) dm(r, \psi),$$

where

$$(x, t, r, \psi)$$

$$= 2(\cosh^2 x \cosh^2 t + r^2 \sinh^2 x \sinh^2 t + 2r \cos \psi \sinh x \sinh t \cosh x \cosh t) - 1,$$

and

$$dm(r, \psi) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha - \beta)\Gamma(\beta + \frac{1}{2})} (1 - r^2)^{\alpha - \beta - 1} r^{2\beta + 1} \sin^{2\beta} \psi dr d\psi.$$

It can be readily seen that

$$2 \cosh^2(x-t) - 1 \leq (x, t, r, \psi) \leq 2 \cosh^2(x+t) - 1.$$

Since  $g \in C_0^1(\mathbb{R}_+)$  too, suppose that  $\text{supp}g(z) \subset B_R$ . Recalling that  $0 < t < M_0$   $\text{supp}g_{t,r,\psi}(x) \subset B_{C(M_0,R)}$ . Considering that  $\frac{\partial}{\partial y}g(x, t, r, \psi)$  is bounded and  $m(r, \psi)$  is a bounded measure one can conclude that **(T1)** fulfils with  $\delta = c\varepsilon$ .

**Lemma 4.** *The translations given by Examples 2 and 3 are regular and  $M_T = 1$  in both cases.*

*Proof.* The proof is based on the standard convolution structure.

First let us observe that the kernel functions are positive and  $\int_{\mathbb{R}} K_{e,p}(x, t) dx = 1$  for all  $t > 0$ , which gives the operator norm.

In both cases let  $\mathcal{E}_1 = \mathcal{E}_2 = C_0(\mathbb{R})$ . In harmonic case by the standard substitution  $\frac{\xi-x}{t+h} = \tan \varphi$ , where  $h > 0$  and  $h = 0$  in the different integrals, we have

$$\begin{aligned} & \|T^{t+h}g(x) - T^tg(x)\|_p \\ &= \left( \int_{\mathbb{R}} \left| \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g(x + (t+h)\tan \varphi) - g(x + t\tan \varphi)) d\varphi \right|^p dx \right)^{\frac{1}{p}} \\ &\leq \left( \int_{\mathbb{R}} \left| \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{-\alpha} (\cdot) d\varphi \right|^p dx \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}} \left| \frac{1}{\pi} \int_{-\alpha}^{\alpha} (\cdot) d\varphi \right|^p dx \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}} \left| \frac{1}{\pi} \int_{\alpha}^{\frac{\pi}{2}} (\cdot) d\varphi \right|^p dx \right)^{\frac{1}{p}} \\ &= I + II + III. \\ &I, III \leq 2\|g\|_p^{\frac{\frac{\pi}{2}-\alpha}{\pi}}, \end{aligned}$$

which is small if  $\alpha$  is close enough to  $\frac{\pi}{2}$ .

Since  $g$  is uniformly continuous for fixed  $\alpha$  and  $\varepsilon$  we can choose  $h, |h| < 1$  such that  $|h \tan \varphi|$  is small enough to be the integrand in  $II$  small, less than  $\varepsilon$ , say.

Then recalling that  $g$  is compactly supported and  $|t| \leq M_0$ , there is an  $R$  such that the support of  $g(x + (t+h)\tan \varphi) - g(x + t\tan \varphi)$  is in  $[-R, R]$ . Thus  $II \leq \varepsilon(2R)^{\frac{1}{p}}$ .

Similarly in parabolic case let  $u = \frac{\xi-x}{2b\sqrt{t+h}}$ . Then

$$\begin{aligned} & \|T^{t+h}g(x) - T^tg(x)\|_p \\ &= \left( \int_{\mathbb{R}} \left| \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} (g(x + 2b\sqrt{t+h}u) - g(x + 2b\sqrt{t}u)) e^{-u^2} du \right|^p dx \right)^{\frac{1}{p}} \\ &\leq \frac{1}{\sqrt{\pi}} \int_{|u|>R} e^{-u^2} \left( \int_{\mathbb{R}} |g(x + 2b\sqrt{t+h}u) - g(x + 2b\sqrt{t}u)|^p dx \right)^{\frac{1}{p}} du + \frac{1}{\sqrt{\pi}} \int_{|u|\leq R} (\cdot) \\ &= I + II. \end{aligned}$$

Recalling that  $g$  is compactly supported, the  $p$ -norm inside is bounded, thus if  $R$  is large enough,  $I$  is small. The uniform continuity of  $g$  implies that  $II$  is small if  $h$  is small enough.

Turning to the proof of (19) and (20) we define  $\nu(t)$ . If  $1 < p < \infty$  and in parabolic case also for  $p = 1$  let  $d\nu(t) = dt$ , and if  $p = 1$ , in the elliptic case let  $d\nu(t) = \sqrt{t}dt$ . Then the same replacements imply the results. Indeed, let us see the elliptic case first.

$$|M_ag(x+h) - M_ag(x)|$$

$$\leq \frac{1}{A\pi} \int_{B_a} \left( \int_{-\frac{\pi}{2}}^{-\alpha} (\cdot) + \int_{-\alpha}^{\alpha} |g(x + (t+h)\tan\varphi) - g(x + t\tan\varphi)| d\varphi + \int_{\alpha}^{\frac{\pi}{2}} (\cdot) \right) d\nu(t).$$

In the first and third integrals the boundedness of the integrand, in the second one the uniform continuity of  $g$  ensures the required estimates.

To prove (20) let  $1 < p < \infty$ .

$$|M_a f(x)| \leq \frac{1}{A} \int_{B_a} \|f\|_p \frac{1}{\pi t} \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2(p'-1)}(\varphi) t d\varphi \right)^{\frac{1}{p'}} dt \leq ca^{-\frac{1}{p}} \|f\|_p.$$

If  $p = 1$ ,

$$|M_a f(x)| \leq \|f\|_1 \frac{1}{A} \int_{B_a} \sqrt{t} \frac{1}{t} dt = \frac{c}{a} \|f\|_1.$$

In the parabolic case

$$\begin{aligned} & |M_a g(x+h) - M_a g(x)| \\ & \leq \frac{1}{a\sqrt{\pi}} \int_{B_a} \left( \int_{|u|>R} (\cdot) + \int_{|u|\leq R} |g(x+h+2b\sqrt{t}u) - g(x+2b\sqrt{t}u)| e^{-u^2} du \right) dt. \end{aligned}$$

Again in the first integral we refer to boundedness, in the second one to the uniform continuity of  $g$ .

Let  $1 \leq p < \infty$ . Then

$$|M_a f(x)| \leq \frac{1}{\sqrt{\pi}} \|f\|_p \|e^{-u^2}\|_{p'} \frac{1}{A} \int_{B_a} (2b\sqrt{t})^{\frac{1}{p'}-1} dt \leq ca^{-\frac{1}{2p}} \|f\|_p,$$

which is (20).

**Lemma 5.** *The translation defined in Example 4 is regular and  $M_T = 1$ .*

*Proof.* For boundedness see [35, Theorem 2].

Let our dense sets  $\mathcal{E} = \mathcal{E}_i$ ,  $i = 1, 2$  be the polynomials, that is

$$\mathcal{E} := \left\{ \sum_{k=0}^N a_k P_k^{(\lambda)}(\cos \vartheta) : a_k \in \mathbb{R}, N \in \mathbb{N} \right\}.$$

Let  $P \in \mathcal{E}$ .

$$\begin{aligned} \|T^{r+h}P(\vartheta) - T^r p(\vartheta)\|_{\mu,p} &= \left\| \sum_{k=0}^N a_k ((r+h)^k - r^k) P_k^{(\lambda)}(\cos \vartheta) \right\|_{\mu,p} \\ &\leq hc(N) \sum_{k=0}^N |a_k| \|P_k^{(\lambda)}(\cos \vartheta)\|_{\mu,p} = C(P)h, \end{aligned}$$

which is (T1). (T2) can be shown similarly. Let  $\nu(r) = \lambda(r)$ , the Lebesgue measure on  $[0, 1)$  and  $P \in \mathcal{D}$ .

$$\begin{aligned} |M_{a,R}P(\vartheta+h) - M_{a,R}P(\vartheta)| &= \left| \sum_{k=0}^N a_k \left( P_k^{(\lambda)}(\cos(\vartheta+h)) - P_k^{(\lambda)}(\cos \vartheta) \right) \frac{1}{a} \int_0^a r^k dr \right| \\ &= \left| -2h\lambda \sin \tilde{\vartheta} \sum_{k=0}^N \frac{a_k}{k+1} a^k P_{k-1}^{(\lambda+1)}(\cos \tilde{\vartheta}) \right| \leq c(P, a)|h|. \end{aligned}$$

In view of (11)  $P(r, \vartheta, \varphi) > 0$ , and if  $r \leq \frac{1}{8}$ , say, then it is also bounded. Thus if  $a \leq \frac{1}{8}$ ,  $1 \leq p < \infty$

$$|M_a f(\vartheta)| \leq \frac{1}{a} \int_0^a \|f\|_{\mu, p} \|P(r, \vartheta, \varphi)\|_{\mu, p'} dr \leq c \|f\|_{\mu, p},$$

where  $\|\cdot\|_{\mu, \infty} := \|\cdot\|_{\infty}$ .

**Lemma 6.** *The translation defined in Example 5 is regular and bounded.*

*Proof.* For boundedness see [12, Theorem 2.2].

The dense sets are the set of polynomials. By the notation (12) it is as follows

$$\mathcal{E}_{1,2} = \mathcal{E} = \left\{ \sum_{k=0}^n c_k \varphi_k^{(\alpha)} : n \in \mathbb{N}, c_k \in \mathbb{R} \right\}.$$

Let  $f \in \mathcal{E}$ ,  $f(x) = \sum_{k=0}^n c_k \varphi_k^{(\alpha)}(x)$ . Then  $T^t f(x) = \sum_{k=0}^n e^{-(2k+\alpha+1)t} c_k \varphi_k^{(\alpha)}(x)$ . Thus

$$\|T^{t+h} f(x) - T^t f(x)\|_p = \left\| \sum_{k=0}^n e^{-(2k+\alpha+1)t} \left( e^{-(2k+\alpha+1)h} - 1 \right) c_k \varphi_k^{(\alpha)}(x) \right\|_p,$$

that is  $h$  can be chosen appropriately depending on  $f$ .

Similarly, since  $x \in B_R$

$$|M_{a,R} f(x+h) - M_{a,R} f(x)| = \left| \sum_{k=0}^n \frac{e^{-(2k+\alpha+1)a} - 1}{(2k+\alpha+1)a} c_k \left( \varphi_k^{(\alpha)}(x+h) - \varphi_k^{(\alpha)}(x) \right) \right|$$

is small with an appropriate  $h$  depending on  $f$ .

To prove the third property we consider the next estimation of the kernel function, see [16, (2.4)]

$$0 \geq W_t^\alpha(x, y) \leq C \frac{e^{-c \frac{(x-y)^2}{t}}}{\sqrt{t}}.$$

Let  $g \in L^p(I)$ .

$$|M_{a,R} f(x)| \leq \left| \int_0^{x-1} g(y) \frac{1}{a} \int_0^a W_t^\alpha(x, y) dt dy \right| + \left| \int_{x-1}^{x+1} (\cdot) \right| + \left| \int_{x+1}^\infty (\cdot) \right| = I + II + III.$$

$$II \leq \|g\|_p \frac{c(p)}{\sqrt{a}}.$$

$$I, III \leq \|g\|_p \frac{1}{\sqrt{a}} \left\| e^{-c \frac{(x-\cdot)^2}{a}} \right\|_{p'}.$$

The consequence of Theorem 1 is the next one.

**Corollary 2.** *Let  $K \subset L_\mu^p(I)$  be bounded.  $K$  is precompact if and only if it is equivanishing and equicontinuous in mean with  $T$  defined in Examples 1, 2, 3, 4, 5 and 6.*

#### 4. REGULARITY WITH RESPECT TO AN INTEGRAL TRANSFORM I - PEGO TYPE THEOREMS

In [36] Pego characterized the compact sets of  $L^2(\mathbb{R}^d)$  by Fourier transform. This result was extended to different spaces and to Abelian groups, see e.g. [24], [23] and the references therein. A Pego-type theorem by Laplace transform was proved in [32]. These results based on standard translation. The extension of Pego's theorem to Bessel and Laguerre translations and transforms is given in [30]. The main ingredients of this type of theorems are a translation, a generated convolution and a corresponding integral transform.

**4.1. Convolution.** The study of convolution structures with general translations dates back to the 70s and it is a widely studied topic, see e.g. [6], [20], [18], [19], [9] for Jacobi, [27] and [33]. These general convolutions, similarly to the standard ones are defined as follows. For  $f$  and  $g$  appropriate functions

$$f * g(t) := \int_I T^t f(x)g(x)d\mu(x).$$

All the convolutions in the listed papers are based on symmetric translations which are related to hyperbolic equations. This symmetry ensures algebraic structures with respect to the convolution in question since the symmetry of the kernel of the translation in its three variables implies the commutativity and associativity of the convolution, see the references above. In parabolic and elliptic cases the kernels of the translations are symmetric only in two variables thus the associativity fails.

To prove Pego-type theorems the next relations of convolution, integral transform and translation are necessary.

$$\mathcal{J}(f * g) = \mathcal{J}(f)\mathcal{J}(g),$$

and

$$\mathcal{J}(T^t f)(z) = \psi(t, z)\mathcal{J}(f)(z), \quad T^t \mathcal{J}(f)(z) = \mathcal{J}(\psi f)(z),$$

where  $\mathcal{J}$  is the integral transform and  $\psi$  is an appropriate function. The translation is regular with respect to the integral transform if it possesses the properties above.

**4.2. Pego type theorems by standard convolution.** Pego type characterization of compactness can be given in cases of Examples 2 and 3.

In elliptic case we characterize compactness of sets in  $L^2(\mathbb{R}_+)$  and consider cosine transformation, in parabolic case the sets are in  $L^2(\mathbb{R})$  and the corresponding transformation is the standard Fourier one.

The transformation pairs are normalized as follows.

$$\mathcal{J}(f)(z) := \hat{f}(z) = \int_{\mathbb{R}} f(x)k(xz)dx, \quad \mathcal{J}(\hat{f})(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(z)k(xz)dz,$$

where  $\mathcal{J} = C$  or  $\mathcal{J} = \mathcal{F}$ , the cosine or the Fourier transform with  $k(xz) = \cos(xz)$  or  $k(xz) = e^{-ixz}$ , respectively. First the actions between translations and transformations are derived. Suppose that  $f$  is in the Schwartz class and in elliptic case we assume that it is even.

$$\begin{aligned} T^t C(f)(z) &= \frac{1}{\pi} \int_{\mathbb{R}} 2 \int_{\mathbb{R}_+} f(u) \cos u\eta du \frac{t}{(\eta - z)^2 + t^2} d\eta \\ &= 2 \int_{\mathbb{R}_+} f(u) \frac{1}{\pi} \int_{\mathbb{R}} \cos u\eta du \frac{t}{(\eta - z)^2 + t^2} d\eta du = 2 \int_{\mathbb{R}_+} e^{-ut} f(u) \cos uz du \end{aligned}$$

$$= C(e^{-|\cdot|^t} f(\cdot))(z),$$

cf. [8, 1.2 (13)]. By similar calculation, supposing that  $z \geq 0$ ,

$$C(T^t f(x))(z) = e^{-zt} C(f)(z).$$

In the parabolic case we have

$$\begin{aligned} T^t \mathcal{F}(f)(z) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-\frac{(z-\eta)^2}{4b^2t}}}{2b\sqrt{\pi}\sqrt{t}} e^{-i\eta u} f(u) dud\eta = \int_{\mathbb{R}} T_z^t(e^{-i(\cdot)u}) f(u) du \\ &= \int_{\mathbb{R}} f(u) e^{-b^2tu^2} e^{-izu} du = \mathcal{F}\left(f(\cdot) e^{-b^2t(\cdot)^2}\right)(z). \end{aligned}$$

On the other hand

$$(26) \quad \mathcal{F}(T^t f(x))(z) = e^{-b^2tz^2} \mathcal{F}(f)(z).$$

As it is well-known,

$$\mathcal{J} : L^p \longrightarrow L^{p'}, \quad 1 \leq p \leq 2.$$

Let  $K \subset L^p(I)$  ( $1 \leq p \leq 2$ ). Denote  $\hat{K} \subset L^p(0, \infty)$ ,  $\hat{K} = \mathcal{J}(K)$ . With this notation we have the next Pego-type theorem.

**Theorem 2.** *Let  $1 \leq p \leq 2$ ,  $K \subset L^p(I)$  a bounded set.*

*Let us consider Examples 2 and 3.*

*Assume that  $K$  satisfies  $\mathbf{P}_a$ . Then  $\hat{K}$  fulfils  $\mathbf{P}_{b_1}$  and  $\mathbf{P}_{b_2}$ .*

*On the other hand if  $K$  satisfies  $\mathbf{P}_{b_1}$ , then  $\hat{K}$  fulfils  $\mathbf{P}_a$ .*

*Proof.* Let  $g(\cdot, t) = e^{-|\cdot|^t}$  or  $g(t, \cdot) = e^{-b^2t(\cdot)^2}$ , and  $f$  as above.

$$\begin{aligned} \|T^{t+h} \mathcal{J}(f)(z) - T^t \mathcal{J}(f)(z)\|_{p', x} &= \|\mathcal{J}(g(\cdot, t+h)f(\cdot))(z) - \mathcal{J}(g(\cdot, t)f(\cdot))(z)\|_{p', z} \\ &\leq c \|g(x, t+h)f(x) - g(x, t)f(x)\|_p \\ &\leq \left( \int_{B_R} |f(x)|^p |g(x, t+h) - g(x, t)|^p dx \right)^{\frac{1}{p}} + 2 \left( \int_R^\infty |f(x)|^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

In view of  $\mathbf{P}_a$ , the second term is small. Since  $x \in B_R$ ,  $g(x, \cdot)$  is uniformly continuous. Thus considering that  $K$  is bounded,  $\mathbf{P}_{b_1}$  is satisfied for dense set uniformly and so for  $\hat{K}$ .

$$\begin{aligned} &|M_{a,R} \mathcal{J}(f)(z+h) - M_{a,R} \mathcal{J}(f)(z)| \\ &= \left| \frac{1}{A} \int_{B_a} \int_{\mathbb{R}} g(u, t) f(u) (k(u(z+h)) - k(uz)) dud\nu(t) \right| \\ &\leq \int_{\mathbb{R}} \frac{1}{A} \int_{B_a} g(u, t) d\nu(t) |f(u)| |k(u(z+h)) - k(uz)| du \\ &\leq c \left( \int_{B_R} h(u) |f(u)| |k(u(z+h)) - k(uz)| du + \int_{\mathbb{R} \setminus B_R} (\cdot) \right) = c(III + IV), \end{aligned}$$

where  $h(u) = \frac{1}{A} \int_{B_a} g(u, t) d\nu(t)$  is uniformly bounded in  $u$ , and if  $u \in \mathbb{R} \setminus B_R$ ,

$$h(u) \leq \frac{c(a)}{|u|}$$

in both cases. Thus

$$IV \leq \left( \int_{\mathbb{R} \setminus B_R} |f|^p \right)^{\frac{1}{p}} c(a) \left( \int_{\mathbb{R} \setminus B_R} u^{-p'} du \right)^{\frac{1}{p'}},$$

which is small by  $\mathbf{P}_a$ , considering that  $p' > 1$ , and

$$III \leq c \|f\|_p \|k(u(z+h)) - k(uz)\|_{p', [0, R]}.$$

Since the interval is bounded,  $h$  can be chosen such that the third factor be small and so, as above,  $\hat{K}$  fulfils  $\mathbf{P}_{b_2}$ .

To prove the opposite direction we use the convolution below in cosine transform case, and the standard one in Fourier transform case. For cosine transform, supposing that  $f, g$  are even

$$f * g(x) := \int_{\mathbb{R}} f(\xi) \frac{1}{2} (g(x+\xi) + g(x-\xi)) d\xi.$$

Assume that  $K$  fulfils  $\mathbf{P}_{b_1}$ . Let  $k_t(x) = K_{t,e}(x)$  or  $k_t(x) = K_{t,p}(x)$ ,  $x \in \mathbb{R}$ ,  $y > 0$ . Choose  $R$  so large that  $|\mathcal{J}(k_t)(\eta)| \leq \frac{1}{2}$  if  $\eta \in \mathbb{R} \setminus B_R$ .

$$\begin{aligned} \left( \int_{\mathbb{R} \setminus B_R} |\mathcal{J}(f)(\eta)|^{p'} d\eta \right)^{\frac{1}{p'}} &\leq 2 \left( \int_{\mathbb{R} \setminus B_R} |\mathcal{J}(f)(\eta)| (1 - |\mathcal{J}(k_t)(\eta)|)^{p'} d\eta \right)^{\frac{1}{p'}} \\ &\leq c \|f - f * k_t\|_p = c \|f - T^t f(x)\|_p. \end{aligned}$$

Thus  $\mathbf{P}_{b_1}$  implies  $\mathbf{P}_a$ .

The next corollary implies that equicontinuity in Examples 2 and 3 sense is equivalent with the standard one from this point of view.

**Corollary 3.** *Let  $K \subset L^2(I)$  be a bounded set and  $\hat{K}$  the Fourier or the cosine transform of  $K$ , respectively. Then  $K$  is precompact if and only if  $K$  and  $\hat{K}$  are equicontinuous in mean in sense of Examples 3 and 2, respectively.*

**4.3. A Pego type theorem by general convolution.** In Laguerre and Bessel cases Pego-type theorems are proved in [30]. Below we prove a similar theorem with Jacobi transform. As the translation in the original and dual spaces are different, this situation is more complex than the Bessel case.

Jacobi transform is defined as it follows, see [17, Proposition 3]. Recalling the measure  $d\mu(x) = \frac{1}{\sqrt{2\pi}} w(x) dx$  (see (15) and (25)), let  $f \in L^2_\mu(\mathbb{R}_+)$  and  $\lambda \in \mathbb{R}_+$ .

$$(27) \quad \mathcal{J}(f)(\lambda) := \hat{f}(\lambda) = \int_0^\infty f(x) \varphi_\lambda^{(\alpha, \beta)}(x) d\mu(x),$$

where the integral is convergent in  $L^2_\nu(\mathbb{R}_+)$  with  $d\nu(\lambda) = \frac{1}{\sqrt{2\pi}} v(\lambda) d\lambda$ , where

$$(28) \quad v(\lambda) = \left| \frac{2^{e-i\lambda} \Gamma(i\lambda) \Gamma(\alpha+1)}{\Gamma\left(\frac{e+i\lambda}{2}\right) \Gamma\left(\frac{e+i\lambda}{2} - \beta\right)} \right|^{-2}.$$

$\mathcal{J} : f \mapsto \hat{f}$  is a linear and normpreserving map of  $L^2_\mu(\mathbb{R}_+)$  onto  $L^2_\nu(\mathbb{R}_+)$ . The inverse transform is given by

$$(29) \quad \mathcal{J}^{-1}(f)(x) = f(x) = \int_0^\infty \hat{f}(\lambda) \varphi_\lambda^{(\alpha, \beta)}(x) d\nu(\lambda),$$



where the integral is convergent in  $L^2_\mu(\mathbb{R}_+)$ .

Since  $\mathcal{J}$  maps  $L^1_\mu(\mathbb{R}_+)$  into  $L^\infty(\mathbb{R}_+)$ , see [18, (3.2)], Riesz-Thorin theorem ensures that  $\mathcal{J}$  maps  $L^p_\mu(\mathbb{R}_+)$  to  $L^{p'}_\nu(\mathbb{R}_+)$  continuously, where  $1 \leq p \leq 2$ .

As the range of the Jacobi transform is different from its domain, introduction of a dual translation is necessary, see [19]. It is given by the next formula.

$$(30) \quad T_d^\eta g(\xi) = \int_0^\infty a(\xi, \eta, \zeta) g(\zeta) d\nu(\zeta),$$

where the dual kernel is

$$a(\xi, \eta, \zeta) = \mathcal{J}(\varphi_\xi^{(\alpha, \beta)} \varphi_\eta^{(\alpha, \beta)})(\zeta) = \int_0^\infty \varphi_\xi^{(\alpha, \beta)}(x) \varphi_\eta^{(\alpha, \beta)}(x) \varphi_\zeta^{(\alpha, \beta)}(x) d\mu(x),$$

see [19, (4.14)]. That is the kernel is symmetric in its three variables. Moreover

$$a(\xi, \eta, \zeta) \geq 0, \quad (\xi, \eta, \zeta) \in \mathbb{R}^3; \quad \int_0^\infty a(\xi, \eta, \zeta) d\nu(\zeta) = 1,$$

see [19, Theorem 4.4 and (4.17)]. This ensures, in standard way, that

$$(31) \quad \|T_d^\eta g\|_{p, \nu} \leq \|g\|_{p, \nu}, \quad 1 \leq p \leq \infty.$$

Subsequently we need the next properties (see also [22, Section 2]).

**Lemma 7.** *Let  $f \in C_0(\mathbb{R}_+)$ . Then*

$$(32) \quad \mathcal{J}(T^t f)(\lambda) = \varphi_\lambda^{(\alpha, \beta)}(t) \mathcal{J}(f)(\lambda)$$

and

$$(33) \quad T_d^\eta \hat{f}(\lambda) = \mathcal{J}(f \varphi_\eta^{(\alpha, \beta)})(\lambda).$$

*Proof.* By [18, (4.2)]

$$\int_0^\infty K(x, t, z) \varphi_\lambda^{(\alpha, \beta)}(z) d\mu(z) = \varphi_\lambda^{(\alpha, \beta)}(t) \varphi_\lambda^{(\alpha, \beta)}(x).$$

Thus

$$\begin{aligned} \mathcal{J}(T^t f)(\lambda) &= \int_0^\infty \int_0^\infty f(z) K(x, t, z) d\mu(z) \varphi_\lambda^{(\alpha, \beta)}(x) d\mu(x) \\ &= \int_0^\infty f(z) \varphi_\lambda^{(\alpha, \beta)}(t) \varphi_\lambda^{(\alpha, \beta)}(z) d\mu(z), \end{aligned}$$

where by the assumption on  $f$  Fubini theorem can be applied and the last integral is convergent. It proves (32).

Similarly, in view of [19, (4.16)]

$$\int_0^\infty a(\xi, \eta, \zeta) \varphi_\zeta^{(\alpha, \beta)}(x) d\nu(\zeta) = \varphi_\xi^{(\alpha, \beta)}(x) \varphi_\eta^{(\alpha, \beta)}(x),$$

which implies that

$$\begin{aligned} T_d^\eta \hat{f}(\lambda) &= \int_0^\infty \int_0^\infty f(x) a(\lambda, \eta, \zeta) \varphi_\zeta^{(\alpha, \beta)}(x) d\mu(x) d\nu(\zeta) \\ &= \int_0^\infty f(x) \varphi_\eta^{(\alpha, \beta)}(x) \varphi_\lambda^{(\alpha, \beta)}(x) d\mu(x) \end{aligned}$$

which, with the remark above, ensures (33).

Before stating the next theorem we introduce the corresponding convolution. For appropriate functions

$$f * g(t) := \int_0^\infty T^t f(x)g(x)d\mu(x)$$

and

$$\mathcal{J}(f * g) = \mathcal{J}(f)\mathcal{J}(g),$$

furthermore the convolution is commutative and associative, see [18, (5.3), (5.4)].

**Theorem 3.** *Let  $1 \leq p \leq 2$ ,  $K \subset L_\mu^p(\mathbb{R}_+)$  a bounded set. Let  $\hat{K} := \mathcal{J}(K)$ .*

*Let us consider Example 6.*

*If  $K$  satisfies  $\mathbf{P}_a$ , then  $\hat{K}$  fulfils  $\mathbf{P}_{b_1}$  and  $\mathbf{P}_{b_2}$ .*

*If  $K$  satisfies  $\mathbf{P}_{b_1}$ , then  $\hat{K}$  fulfils  $\mathbf{P}_a$ .*

*Proof.* Let  $f \in K$ . In view of (33)

$$\|T_d^{\eta+h} \hat{f}(\lambda) - T_d^\eta \hat{f}(\lambda)\|_{p',\nu} = \|\mathcal{J}(f(\varphi_{\eta+h}^{(\alpha,\beta)} - \varphi_\eta^{(\alpha,\beta)}))\|_{p',\nu} \leq c\|f(\varphi_{\eta+h}^{(\alpha,\beta)} - \varphi_\eta^{(\alpha,\beta)})\|_{p,\mu}.$$

According to [17, Theorem 2, (ib)] if  $\eta \in \mathbb{R}$ ,

$$\frac{\partial}{\partial \eta} \varphi_\eta^{(\alpha,\beta)}(x) \leq K(1+x)^2 e^{-\varrho x}.$$

Thus, since  $\varrho > 0$ ,  $\|T_d^{\eta+h} \hat{f}(\lambda) - T_d^\eta \hat{f}(\lambda)\|_{p',\nu} \leq c(\varrho)h\|f\|_{p,\mu}$ , so  $\mathbf{P}_{b_1}$  is satisfied for  $\hat{K}$ .

Repeating the standard arguments, for an arbitrary  $\delta > 0$  take a function  $g_\delta$  such that  $\text{supp}g_\delta \subset [-\delta, \delta]$ ,  $\int_{\mathbb{R}_+} g_\delta(t)d\mu(t) = 1$ ,  $\hat{g}_\delta > 0$ , cf. [19, Lemma 4.2]. As  $\hat{g}_\delta(\lambda)$  tends to zero when  $\lambda$  tends to infinity, choose  $R$  so large that  $\hat{g}_\delta(\lambda) < \frac{1}{2}$  if  $\lambda > R$ . Then

$$\begin{aligned} \left( \int_R^\infty |\hat{f}|^{p'} d\nu \right)^{\frac{1}{p'}} &\leq \left( \int_R^\infty |\hat{f}(1 - \hat{g}_\delta)|^{p'} d\nu \right)^{\frac{1}{p'}} \leq c(p)\|f - f * g_\delta\|_{p,\mu} \\ &= c(p) \left( \int_0^\infty \left| \int_0^\infty (f(x) - T^x f(t))g_\delta(t)d\mu(t) \right|^p d\mu(x) \right)^{\frac{1}{p}} \\ &\leq c(p) \left( \int_0^\infty \int_0^\infty |f(x) - T^t f(x)|^p d\mu(x)g_\delta(t)d\mu(t) \right)^{\frac{1}{p}} \\ &\leq c(p) \sup_{0 \leq |t| \leq \delta} \|f(x) - T^t f(x)\|_{p,\mu}. \end{aligned}$$

## 5. MODULUS OF SMOOTHNESS AND K-FUNCTIONAL - REGULARITY WITH RESPECT TO AN INTEGRAL TRANSFORM II

As it is well-known, the modulus of smoothness generated by the standard translation is equivalent with the Peetre's K-functional, see e.g. [14, page 171]. This property is extended to Bessel translation, see [37]. Below we derive the same equivalence for heat-diffusion semigroup, where the semigroup property implies arguments very similar to the standard case, and for Example 3, where the corresponding integral transform proves to be the right tool.

Recalling the notation of (21) we can define the Sobolev space generated by  $D_{(2)}$  as follows.

$$(34) \quad W_{D,r,\mu}^p = \{f(x) \in L_\mu^p : D_{(2),x}^l f \in L_\mu^p, l = 1, \dots, r\}.$$

Then the corresponding K-functional is

$$K_r(f, t, L_\mu^p, W_{D,r,\mu}^p) := K_r(f, t)_p = \inf_{g \in W_{D,r,\mu}^p} \{\|f - g\|_{p,\mu} + t\|(D_{(2)}^r g)\|_{p,\mu}\}.$$

The moduli of smoothness generated by general translations are defined as follows.

**Definition 5.** *Let*

$$\Delta_t f(x) := T^t f(x) - f(x), \quad \Delta_t^r f(x) := \Delta_t(\Delta_t^{r-1} f)(x).$$

*Let  $1 \leq p \leq \infty$ . The  $p$ -modulus of smoothness is*

$$\omega_r(f, t)_p := \sup_{0 < h \leq t} \|\Delta_h^r f\|_{p,\mu}.$$

**Remark.** (R4)

$$(35) \quad \Delta_t^r f(x) = (I - T^t)^r f = \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} (T^t)^k f(x),$$

where  $(T^t)^k = T^t \circ \dots \circ T^t$ . For the standard and the heat-diffusion translation semigroups  $(T^t)^k$  becomes  $T^{kt}$ .

**Theorem 4.** *There are positive constants  $M = M(r, p)$  independent of  $f$  such that for Examples 1 and 3*

$$\frac{1}{M} \omega_r(f, t)_p \leq K_r(f, t^{cr})_p \leq M \omega_r(f, t)_p,$$

$r \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ ,  $c = \text{ord}(D_{(1)})$ .

**Remark.**

(R5) In our parabolic cases  $c = 1$ , in Bessel case  $c = 2$ , cf. [37].

*Proof.* By standard arguments

$$\omega_r(f, t)_p \leq \omega_r(f - g, t)_p + \omega_r(g, t)_p \leq 2^r \|f - g\|_p + t^r \|D_{(2)} g\|_p.$$

In view of (23) and Remark (R4),

$$\|\Delta_t^r g\|_p \leq t^r \|D_{(2)} g\|_p.$$

The second estimation needs different methods in the two cases.

First we deal with the heat semigroup. Let us introduce the next notation.

$$(36) \quad \begin{aligned} g_{r,t}(x) &:= f(x) + (-1)^{r+1} r^r \int_{(0, \frac{1}{r})^r} \Delta_t^r(\sum_{i=1}^r z_i) f(x) dz_1 \dots dz_r \\ &= r^r \int_{(0, \frac{1}{r})^r} \sum_{k=1}^r (-1)^{k+1} \binom{r}{k} u(x, kt(\sum_{i=1}^r z_i)) dz_1 \dots dz_r. \end{aligned}$$

Thus, according to (7)

$$(37) \quad D_{(2),x}^r g_{r,t}(x) = \frac{r^r}{t^r} \sum_{k=1}^r (-1)^{k+1} \binom{r}{k} \frac{1}{k^r} \Delta_{\frac{kt}{r}}^r f(x).$$

Considering the operator norm, (36) and (37)  $g_{r,t} \in W_{r,L}^p$ . By Minkowski's inequality

$$t^r \|D_{(2),x}^r g_{r,t}(x)\|_p \leq (2r)^r \omega_r(f, t)_p.$$

Recalling the definition of  $g_{r,t}$ , as above

$$\|f - g_{r,t}\|_p = \left\| r^r \int_{(0, \frac{1}{r})^r} \Delta_t^r(\sum_{i=1}^r u_i) f(x) du_1 \dots du_r \right\|_p \leq M(r,p) \omega_r(f,t)_p.$$

To prove the second inequality for Example 3 we apply the integral transform method again. According to (35) and (26)

$$(\mathcal{F}(\Delta_t^r f))(z) = (1 - e^{-b^2 tz^2})^r \hat{f}(z).$$

Let  $I_1 := I \cap [-1, 1]$  and  $I_2 := I \setminus (-2, 2)$ . Let us define

$$\eta(x) = \begin{cases} 1, & x \in I_1, \\ 0, & x \in I_2 \\ \in C_0^\infty & \text{otherwise,} \end{cases}$$

such that  $0 \leq \eta \leq 1$ . Let  $\varepsilon > 0$  and define  $\eta_\varepsilon(x) := \eta(\frac{x}{\varepsilon})$ . Let

$$\varrho_\varepsilon := \mathcal{F}^{-1}(\eta_\varepsilon).$$

It is enough to take an  $f \in L^1 \cap L^p$ . Define

$$g := g_\varepsilon := (\varrho_\varepsilon * f).$$

Let  $0 < t < \frac{1}{b^2 \varepsilon^2}$ . Then we have

$$\begin{aligned} \mathcal{F}(D_{(2)}^r g_\varepsilon)(z) &= (-1)^r b^{2r} z^{2r} \eta_\varepsilon(z) \hat{g}(z) \\ &= \frac{1}{t^r} \frac{t^r (-1)^r b^{2r} z^{2r} \eta_\varepsilon(z)}{(1 - e^{-b^2 tz^2})^r} (\mathcal{F}(\Delta_t^r g))(z) = \frac{1}{t^r} \frac{t^r (-1)^r b^{2r} z^{2r} \eta(bz\sqrt{t})}{(1 - e^{-b^2 tz^2})^r} (\mathcal{F}(\Delta_t^r g))(z). \end{aligned}$$

Let

$$h(u) = \left( \frac{-u^2}{1 - e^{-u^2}} \right)^r \eta(u).$$

Then

$$D_{(2)}^r g_\varepsilon = \frac{1}{t^r} \Delta_t^r g \mathcal{F}^{-1}(h(bz\sqrt{t})).$$

Since

$$\|\mathcal{F}^{-1}(h(bz\sqrt{t}))\|_1 = \|\mathcal{F}^{-1}(h)\|_1 =: c,$$

we have

$$(38) \quad \|D_{(2)}^r g_\varepsilon\|_p \leq c \frac{1}{t^r} \|\Delta_t^r g\|_p.$$

We proceed similarly to estimate the first term of  $K_r(f, t^r)_p$ .

$$\mathcal{F}(f - g_\varepsilon)(z) = (1 - \eta_\varepsilon) \hat{g}(z) = \frac{1 - \eta_\varepsilon}{(1 - e^{-b^2 tz^2})^r} \widehat{\Delta_t^r g}(z).$$

Let

$$\frac{1}{(1-x)^r} = \sum_{k=0}^{\infty} a_k x^k = Q_N(x) + \sum_{k=N+1}^{\infty} a_k x^k, \quad |x| < 1.$$

We decompose

$$\frac{1}{(1 - e^{-b^2 tz^2})^r} = \left( \frac{1}{(1 - e^{-b^2 tz^2})^r} - Q_N(e^{-b^2 tz^2}) \right) + Q_N(e^{-b^2 tz^2}),$$

that is

$$\begin{aligned} \mathcal{F}(f - g_\varepsilon)(z) &= U_1 + U_2 \\ &= (1 - \eta_\varepsilon) Q_N(e^{-b^2 tz^2}) \widehat{\Delta_t^r g}(z) \end{aligned}$$

$$+ (1 - \eta_\varepsilon) \left( \frac{1}{(1 - e^{-b^2tz^2})^r} - Q_N(e^{-b^2tz^2}) \right) \widehat{\Delta_t^r g}(z).$$

Let  $\varepsilon = \frac{1}{b\sqrt{t}}$  and

$$k_N(z) = (1 - \eta(z)) \left( \frac{1}{1 - e^{-z^2}} - Q_N(z) \right).$$

Then

$$\mathcal{F}^{-1}U_2 = \mathcal{F}^{-1} \left( k_N(bz\sqrt{t}) \right) * \Delta_t^r g.$$

If  $N$  is large enough we have

$$(39) \quad \|\mathcal{F}^{-1}U_2\|_p \leq \|\mathcal{F}^{-1}k_N\|_1 \|\Delta_t^r g\|_p.$$

Turning our attention to the first term we have

$$\mathcal{F}^{-1}U_1 = \sum_{k=0}^N a_k \left( (T^t)^k (\Delta_t^r g) - \varrho_{\frac{1}{bz\sqrt{t}}} * ((T^t)^k (\Delta_t^r g)) \right).$$

Since  $T^t$  is bounded

$$(40) \quad \|\mathcal{J}^{-1}U_1\|_p \leq C(Q)(1 + \|\varrho_1\|_1) \|\Delta_t^r f\|_p.$$

Thus according to (39) and (40)

$$\|f - g_\varepsilon\|_p \leq c\omega_r(f, t)_p.$$

Recalling (38) the second inequality is proved.

**Remark.** (R6)

This integral transform method works because considering the two expressions  $\mathcal{J}(\Delta_t f) = (1 - \psi(t, \cdot))\hat{f}$  and  $\mathcal{J}D_{(2)}f = h(\cdot)\hat{f}$  we have

$$h(\cdot) \sim 1 - \psi(t, \cdot)$$

around zero. This property is the second regularity property of translation with respect to the integral transform. This is the situation e.g. in Bessel case, see [37]. For instance Example 2 does not possess second regularity property with respect to the cosine transform.

## REFERENCES

- [1] I. Abu-Falahah, J. L. Torrea, Hermite function expansions versus Hermite polynomial expansions, *Glasgow Math. J.* **48** (2006) 203-215.
- [2] R. A. Adams, *Sobolev spaces* Academic Press, New York (1975)
- [3] V. Arestov and M. Deikalova, Nikol'skii inequality between the uniform norm and  $L_q$ -norm with Jacobi weight of algebraic polynomials on an interval, *Analysis Math.*, **42:2** (2016), 91-120.
- [4] V. Arestov, M. Deikalova, Á. Horváth, On Nikol'skii type inequality between the uniform norm and the integral q-norm with Laguerre weight of algebraic polynomials on the half-line, *J. of Approx. Theory*, **222** (2017) 40-54.
- [5] V. Arestov, A. Babenko, M. Deikalova, Á. Horváth, Nikol'skii inequality between the uniform norm and the integral norm with Bessel weight for entire functions of exponential type on the half-line, *Analysis Math.*, **44** (2018) 21-42.
- [6] R. Askey, G. Gasper, Convolution structures for Laguerre polynomials, *J. Analyse Math.* **31** (1977) 48-68.
- [7] I. Aydin, C. Unal, The Kolmogorov-Riesz theorem and some compactness criterions of bounded subsets in weighted variable exponent amalgam and Sobolev spaces, *Collectanea Mathematica* **71** (2020) 349-367.
- [8] H. Bateman, A. Erdélyi, *Higher Transcendental Functions, II*, McGraw-Hill (New York 1953).

- [9] H. Bavinck, A special class of Jacobi series and some applications, *J. Math. Anal. Appl.*, **37**(1972), 767-797.
- [10] B. L. J. Braaksma, A singular Cauchy problem and generalized translations, in *Proceedings of an International Conference on Differential Equations Held at the University of Southern California* edited by H. A. Antosiewicz (1974), 40-52.
- [11] B. L. J. Braaksma and H. S. V. de Snoo, Generalized translation operators associated with a singular differential operator, *Proc. Conf. Theory of ordin. and part. diff. equat. in Dundee 1974* Springer Lecture Notes. **415** B. D. Sleeman, I. M. Michael (Eds.).
- [12] A. Chicco Ruiz, E. Harboure, Weighted norm inequalities for heat diffusion Laguerre's semi-groups. *Math. Z.* **257** (2007), 329-354.
- [13] J. Delsarte, Sur une extension de la formule de Taylor, *Journ. de Math. pures et appliquees* **17** 3 (1938), 213-231.
- [14] R. A. DeVore, G. Lorentz, *Constructive Approximation* Springer-Verlag Berlin Heidelberg 1993.
- [15] M. Dörfler, H. G. Feichtinger, K. Gröchenig, Compactness criteria in function spaces, *Colloquium Mathematicae* **94** (2002) 37-50.
- [16] Huiying Fan, Tao Ma, Parabolic equations involving Laguerre operators and weighted mixed-norm estimates, *Communications on Pure and Applied Analysis* **19** (2020) 5487-5508.
- [17] M. Flensted-Jensen, Paley-Wiener type theorems for a differential operator connected with symmetric spaces, *Ark. Mat.* **10** (1972), 143-162.
- [18] M. Flensted-Jensen, T. H. Koornwinder, The convolution structure for Jacobi function expansions *Ark. Mat.* **11** (1973), 245-262.
- [19] M. Flensted-Jensen, T. H. Koornwinder, Jacobi functions: the addition formula and the positivity of the dual convolution structure, *Ark. Mat.* **17** (1979), 139-151.
- [20] G. Gasper, Positivity and the convolution structure for Jacobi series, *Ann. Math.*, Second Series, **93**:1(1971), 112-118.
- [21] J. Gosselin, K. Stempak, Conjugate expansions for Hermite functions, *Illinois J. Math.* **38** (1994) 177-197.
- [22] D.V. Gorbachev, V.I. Ivanov, S.Yu. Tikhonov, Logan's problem for Jacobi transform, arXiv:2112.05802 (2021)
- [23] P. Górká, Pego theorem on locally compact Abelian groups, *Journal of Algebra and Its Applications*, **13**(2014) 1350143.
- [24] P. Górká, T. Kostrzewa, Pego everywhere, *Journal of Algebra and Its Applications*, **15**(2016) 1650074.
- [25] P. Górká, A. Macios, Almost everything you need to know about relatively compact sets in variable Lebesgue spaces, *Journal of Functional Analysis* **269** (2015)1925-1949.
- [26] P. Górká, H. Rafeiro, From Arzelá-Ascoli to Riesz-Kolmogorov, *Nonlinear Analysis* **144** (2016) 23-31.
- [27] E. Görlich, C. Markett, A convolution structure for Laguerre series, *Indag. Math.*, **44** (1982) 161-171.
- [28] E. Görlich, C. Markett, On approximation by Cesaro means of the Laguerre expansion and best approximation, *Resultate der Mathematik* **2** (1979) 124-150.
- [29] H. Hanche-Olsen, H. Holden, E. Malinnikova, An improvement of the Kolmogorov-Riesz compactness theorem, *Expositiones Mathematicae* **37** (2019) 84-91.
- [30] Á. P. Horváth, Compactness criteria via Laguerre and Hankel transformations, *J. Math. Anal. Appl.* **507** (2022) 125852.
- [31] K. Hoffman, *Banach Spaces of Analytic Functions* Prentice-Hall, Inc., Englewood Cliffs, N.J., (1962)
- [32] M. Krukowski, Characterizing compact families via Laplace transform, *Annales Acad. Sci. Fennicae Math.* **45** (2020) 991-1002.
- [33] Y. Kanjin, On algebras with convolution structures for Laguerre polynomials, *Trans. of the Amer. Math. Soc.*, **295** (1986) 783-794.
- [34] B. M. Levitan, Expansion in Fourier series and integrals with Bessel functions, *Uspekhi Mat. Nauk* (in Russian) **6** ( 1951) 102-143.
- [35] B. Muckenhoupt, E. M. Stein, Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.* **118** (1965), 17-92.
- [36] R. L. Pego, Compactness in  $L^2$  and the Fourier transform, *Proc. of the Amer. Math. Soc* **95** (1985) 252-254.

- [37] S. S. Platonov, Bessel harmonic analysis and approximation of functions on the half-line, *Izv. RAN, Ser. Mat.*, **71** (2007), 149-196 (in Russian); translated in *Izv. Math.*, **71**:5 (2007), 1001-1048.
- [38] S. S. Platonov, Fourier-Jacobi harmonic analysis and some problems of approximation of functions on the half-axis in  $L^2$  metric: Jackson's type direct theorems, *Integral Transforms and Special Functions* **30** (2019) 264-281.
- [39] S. S. Platonov, Fourier-Jacobi harmonic analysis and some problems of approximation of functions on the half-axis in  $L^2$  metric: Nikol'skii-Besov type function spaces. *Integral Transforms and Special Functions* **31** (2020) 281-298.
- [40] H. Rafeiro, S. Samko, Dominated compactness theorem in Banach function spaces and its applications, *Complex Anal. Oper. Theory* **2** (4) (2008) 669-681.
- [41] K. Stempak, J. L. Torrea, Poisson integrals and Riesz transforms for Hermite function expansions with weights *J. of Functional Analysis* **202** (2003) 443-472.
- [42] G. Szegő, *Orthogonal Polynomials*, AMS. AMS Coll. Publ. Vol. XXXIII (New York, 1959).

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