Weighted Hermite-Fejér Interpolation on the Real Line: L_∞ Case

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Abstract

We give a weighted Hermite-Fejér-type interpolatory method on the real line, which is a positive operator on "good" matrices. We give an example on "good" interpolatory matrix by weighted Fekete points. To prove the convergence theorem we need the generalization of "Rodrigues' property".

1 Introduction, definitions, notations

Hermite-Fejér interpolation on the real line in infinity norm with Freud weights, in the last 15 years was investigated by several authors. We can mention here eg. the works of S. B. Damelin, H. S. Jung, K, H. Kwon and P. Vértesi (see [3], [10], [9], [20]) The immediate prelimiraies of present investigation can be found in the papers of D. S. Lubinsky [13], J. Szabados [18] and S. Szabó [19]. Their investigations based on the root system ($\{y_{k,n}\}$) of orthogonal polynomials ($p_n(w)$) with respect to some Freud weight: w. These theorems can be formulated in the following form:

Let w_1 and w_2 be Freud weights. If f is a continuous function on the real line, such that $\lim_{|x|\longrightarrow\infty}(fw_1)(x) = 0$, then with $n \longrightarrow \infty$:

$$\left\| w_2(x)\left(f(x) - \sum_{k=1}^{n}\left(1 - \frac{p_n''(w)}{p_n'(w)}(y_{k,n})(x - y_{k,n})\right) l_k^2(x) f(y_{k,n})\right)\right\| \longrightarrow 0. \tag{1}$$

That is in such type of theorems, three different Freud weights appears.

*supported by Hungarian National Foundation for Scientific Research, Grant No. T049301.
Key words: weighted interpolation, positive operator, weighted Fekete points, Rodrigues' property
2000 MS Classification: 65D05, 41A10, 41A36

As usual, l_k is the fundamental polynomial of Lagrange interpolation, $\|\cdot\|$ is the infinity norm of a function, and we call a weight function on $\mathbf{R}$ as Freud weight, if $w(x) = e^{-Q(x)}$, where Q is even, $Q^{'} > 0$ on $(0, \infty)$, and $Q^{''}$ is continuous on $(0, \infty)$, furthermore for some $A, B > 1$ on $(0, \infty)$

$$A \leq \frac{(xQ^{'}(x))^{'}}{Q^{'}(x)} \leq B$$

is valid.

D. S. Lubinsky [13] gave an estimation on the ratio of w_1 and w_2, which in the worst situation is:

$$\frac{w_1}{w_2}(x) > (1 + |Q^{'}(x)|)^{3+\varepsilon}(1 + |x|)^3$$

and J. Szabados [18] improved it as:

$$\frac{w_1}{w_2}(x) \sim (1 + |Q^{'}(x)|)(1 + |x|)^{\frac{1}{3}}.$$

The ideal situation would be $w_1(x) = w_2(x)$. We are not able to reach it at present, but to move in this direction we try to choose another system of nodes, and an Hermite-Fejér-type process which depends on the weight. On finite interval this new process has very nice convergence properties (with Jacobi weights see [7]), and the same is valid on the half line (with Laguerre weights see [8]).

Our starting point is the so-called ϱ-normal point systems of L. Fejér. At first, Hermite-Fejér interpolation on ϱ-normal point systems is a positive operator, and as P. Erdős and P. Turán proved [5], the placing of these points is uniform in some sense.

The weighted analog of Fejér's definition is the following [7]:

$(a, b) \subset \mathbf{R}, w$ is a positive, differentiable weight function on (a, b), such that the Mhaskar-Rahmanov-Saff sets are intervals: $I_n = (a_{2n-1}(w), b_{2n-1}(w))$. Let $X = \{x_{k,n}, k = 1, \dots, n; n = 1, 2 \dots\} \subset (a, b)$ is a point system such that $\{x_{k,n}, k = 1, \dots, n\} \subset cI_n$, (Here, and the followings c is a positive constant, generally is different on different places.) $\omega_n(x) = \prod_{k=1}^{n}(x - x_{k,n})$. X is a $\varrho(w)-$normal matrix for some $\varrho \in (0, 1]$, if for every $n \in \mathbf{N}$

$$w(x) \sum_{k=1}^{n} \frac{(1 - C_k(x - x_{k,n}))l_k^2(x)}{w(x_{k,n})} \leq 1; \quad x \in I_n \tag{2}$$

and

$$1 - C_k(x - x_{k,n}) \geq \varrho > 0; \quad x \in I_n, \quad k = 1, \dots, n \tag{3}$$

where

$$C_k = \frac{\omega''}{\omega'}(x_{k,n}) + \frac{w'}{w}(x_{k,n}). \tag{4}$$

And as in [7] [8], we will use the following weighted Hermite-Fejér operator:

$$H_{n,w}(f,x) = \sum_{k=1}^{n}(1 - C_k(x - x_{k,n}))l_k^2(x)f(x_{k,n}) \tag{5}$$

We have to mention that

$$w(x)H_{n,w}(f,x)|_{x=x_{k,n}} = (wf)(x_{k,n})$$

$$(w(x)H_{n,w}(f,x))'|_{x=x_{k,n}} = 0, \quad k = 1,\ldots,n.$$

So this weighted operator is also positive, and the nice placing of fundamental points was proved by P. Vértesi [21].

To give some explanation how this weighted operator appears, let us remark, that it is only a rearrengement of the original one in weighted space:

$$w(x)H_n(f,f',x) = w(x)\sum_{k=1}^{n}\frac{\left(1 - \frac{\omega_n''(x_{k,n})}{\omega_n'(x_{k,n})}(x - x_{k,n})\right)l_k^2(x)}{w(x_{k,n})}(fw)(x_{k,n})$$

$$+w(x)\sum_{k=1}^{n}\frac{l_k^2(x)(x - x_{k,n})}{w(x_{k,n})}(f'w)(x_{k,n})$$

$$= w(x)\sum_{k=1}^{n}\frac{(1 - C_k(x - x_{k,n}))\,l_k^2(x)}{w(x_{k,n})}(fw)(x_{k,n})$$

$$+w(x)\sum_{k=1}^{n}\frac{l_k^2(x)(x - x_{k,n})}{w(x_{k,n})}(fw)'(x_{k,n}).$$

We will call as kernel functions of the Hermite interpolation in weighted space:

$$K_n(x) = w(x)\sum_{k=1}^{n}\frac{|1 - C_k(x - x_{k,n})|\,l_k^2(x)}{w(x_{k,n})}$$

and

$$\hat{K}_n(x) = w(x)\sum_{k=1}^{n}\frac{l_k^2(x)|x - x_{k,n}|}{w(x_{k,n})}.$$

We want to prove a convergence theorem on weighted Hermite-Fejér interpolation on some kind of systems of nodes. For it we need to investigate the difference of an integral of a weighted polynomial and its weighted Hermite interpolatory polynomial.

Definition 1 *w is a weight function and X is an interpolatory matrix on an interval $I = (a, b)$. We say that w has the Rodrigues' property with respect to X, if for every polynomials q_n with degree n, for which*

$$\int_I q_n w = 0 \tag{6}$$

(therefore $q_n = \sum_{j=1}^{n} c_{j,n} p_n(w)$), for which there is a constant K such that

$$|c_{j,n}| = O(K^n) \quad (j = 1, \dots, n); \tag{7}$$

we have that there is a $c = c(K, w) \in \mathbf{R}$, such that for every $\alpha \in \mathbf{R}$

$$\left\| n^\alpha w(x) \left(\frac{\int_a^x q_n w}{w} - H_{cn} \left(\frac{\int_a^x q_n w}{w}, \left(\frac{\int_a^x q_n w}{w} \right)', x \right) \right) \right\| \longrightarrow 0, \tag{8}$$

when $n \longrightarrow \infty$.

Remark: We call the previous property as "Rodrigues'", because that weight functions, for wich the Rodrigues' formula is valid (eg. Jacobi-, Laguerre-, Hermite weights), and for q_n-s with property (6), we have that

$$\int_a^x q_n w = p_m w,$$

where $m \sim n$, and because of the projection property of Hermite interpolation, the difference above (8)is equal to zero.

Definition 2 *An interpolatory matrix $X = (\{x_{k,n}\}, k = 1, \dots, n; n \in \mathbf{N})$ is $1(w)-$normal, if $C_k = 0 (k = 0, 1, \dots)$ in (2)(3)(4), that is*

$$|x_{k,n}| < c a_{2n-1}(w), \tag{9}$$

$$w(x) \sum_{k=1}^{n} \frac{l_k^2(x)}{w(x_k)} \leq 1, \quad x \in \mathbf{R} \tag{10}$$

Remark:$1(w)-$normality means that the lines $1 - C_k(x - x_k)$ are horizontal for all $k-$s. This is the situation, when we are on the roots of Hermite polynomials in Hermite-weighted space [7]. The weighted Hermite-Fejér interpolatory operator on $1(w)-$normal systems coincides with the Grünwald-operator. For Grünwald-operator, on the roots of orthogonal polynomials S. Szabó [19] proved a convergence theorem with $w_1 = w_2$. So this theorem is "stronger" then the one we will prove in the general case. It is not clear yet, that on infinite intervals the point systems, which are the root systems of orthogonal polynomials, and are $\varrho(w)-$normal simultaneously are better, or the estimations on orthogonal polynomials by Christoffel-function are stronger then in our case.

It is necessary to give an example on $1(w)-$normal matrices when w is not the Hermite weight. We can do it by the help of weighted Fekete points.

Example

Let X be an interpolatory matrix such that the n^{th} row of $X:\ \{x_{k,n}\}_{k=1}^{n}$ consists of the weighted Fekete points (see eg [17, p. 142.]) with respect to the weight $w^{\frac{1}{n-1}}$, where $w=e^{-2Q}$ is an arbitrary Freud weight. Abbreviating by $x_k=x_{k,n}$, according to standard arguments:

$$\left(w(x)\frac{l_k(x)}{w(x_k)}\right)^{\frac{2}{n(n-1)}}=\left(\prod_{\substack{1\le i\le n\\ i\neq k}}\frac{(x-x_i)w^{\frac{1}{n-1}}(x)}{(x_k-x_i)w^{\frac{1}{n-1}}(x_k)}\frac{w^{\frac{1}{n-1}}(x_i)}{w^{\frac{1}{n-1}}(x_i)}\right.$$

$$\left.\times\prod_{\substack{1\le i<j\le n\\ i,j\neq k}}\frac{(x_i-x_j)w^{\frac{1}{n-1}}(x_i)w^{\frac{1}{n-1}}(x_j)}{(x_i-x_j)w^{\frac{1}{n-1}}(x_i)w^{\frac{1}{n-1}}(x_j)}\right)^{\frac{2}{n(n-1)}}\le 1$$

where the inequality follows from the extremality of Fekete points. It means that

$$\left\|w(x)\frac{l_k(x)}{w(x_k)}\right\|=w(x_k)\frac{l_k(x_k)}{w(x_k)},$$

that is

$$\frac{w^{'}(x_k)}{w(x_k)}+l_k^{'}(x_k)=\frac{1}{2}\left(-2Q^{'}(x_k)+\frac{\omega^{''}(x_k)}{\omega^{'}(x_k)}\right)=\frac{1}{2}C_k=0.$$

Now, as in [7], Remark 2. it is enough to prove that $(w^{-1})^{(2m)}\ge 0$, which can be easily seen by induction when $w(x)=e^{-x^{2k}}$.

2 Results

Definition 3 *Let $w=e^{-Q}$ be a Freud weight.*

$$w_1(x):=\frac{w^2(x)}{\hat{\log}(|x|)}h(x), \tag{11}$$

where

$$\hat{\ln}(|x|) = \begin{cases} 1 & if\ \ |x| < e-\delta \\ \ln(|x|) & if\ \ |x| > e+\delta \\ p_3(x) & if\ \ e-\delta \leq |x| \leq e+\delta \end{cases} \tag{12}$$

Where $p_3(x)$ is given such that $\hat{\ln}(|x|)$ be twice differentiable everywhere, and $\hat{\log}(|x|) = c\hat{\ln}(|x|)$ with a suitable constant. Let h be a positive function such that $h(x) \longrightarrow 0$ arbitrary slowly, when $|x| \longrightarrow \infty$ such that w_1 be a Freud weight again.

Remark: One can see by easy calcuation that with $w(x) = e^{-\frac{x^2}{2}}$ eg. $w_1 = \frac{e^{-x^2}}{\hat{\ln}^2|x|}$ will be a proper weight.

Definition 4 *w is a weight function on (a,b).*

$$f \in C_w$$

if f is continuous on (a,b), and $\lim_{x\longrightarrow a+0} fw = \lim_{x\longrightarrow b-0} = 0$.

Theorem 1 *w is a Freud weight such that $A > 2$ and for w^2 the Rodrigues' property is valid on X, which is a $1(w^2)-$ normal matrix, then*

$$\|(f - H_{n,w^2}(f))w_1\| \longrightarrow 0$$

for every $f \in C_{w^2}$.

We gave an example on $1(w)$-normal matrices with some kind of Freud weights. From the definition of the kernels of Hermite interpolation, by (10) we can see that the kernels have polynomial growth, and in this case, according to Theorem 2, we have examples on Freud weights with Rodrigues' property with respect to $1(w)$-normal matrices.

For the proof we need some lemmas:

Lemma 1 *If X is a $1(w^2)-$ normal matrix such that*

$$\left\| w_1(x) \sum_{\substack{1\leq k\leq n \\ |x_k| < a_n^\lambda}} \frac{l_k^2(x)}{w^2(x_k)} |x - x_k| e^{-|x_k|} \right\| \longrightarrow 0, \tag{13}$$

where $\lambda \in (0,1)$ and $a_n := a_{2n-1}(w^2)$, then

$$\|(f - H_{n,w^2}(f))w_1\| \longrightarrow 0$$

$(n \longrightarrow \infty)$ for every $f \in C_{w^2}$.

Proof:

$$|(f-H_{n,w^2}(f))w_1| \le |(f-p_m)w^2| + \left| w^2(x)\sum_{k=1}^{n}\frac{l_k^2(x)}{w^2(x_k)}(f-p_m)w^2(x_k)\right|$$

$$+\left| w_1(x)\sum_{k=1}^{n}\frac{l_k^2(x)}{w^2(x_k)}(x-x_k)(p_mw^2)^{'}(x_k)\right| = I+II+III,$$

where $p_m \in P_m$ is the uniformly best approximating polynomial with degree m, where $m \le 2n-1$, and so $\|(f-p_m)w^2\| =: E_{m,w^2}(f)$. Therefore

$$I+II = O(E_{m,w^2}(f)).$$

Because of the Ahiezer-Babenko theorem ([1]) we can choose for an arbitrary $\varepsilon > 0$ an $m = m(\varepsilon)$ such that $E_{m,w^2}(f) < \varepsilon$. Fixing this m we have that

$$\|(p_mw^2)^{'}(x)e^{|x|}\| = \|(p_m^{'} - 2Q^{'}p_m)(x)e^{-2Q(x)+|x|}\|$$

$$\le C(m)|x|^{m+B-1}e^{-C|x|^A} \le \tilde{C}(m),$$

where A and B are in the definition of Freud weights, and the estimation of $Q^{'}$ uses [11] Lemma 5.1.. So we can estimate

$$III \le \tilde{C}(m)w_1(x)\sum_{k=1}^{n}\frac{l_k^2(x)}{w^2(x_k)}|x-x_k|e^{-|x_k|}$$

$$=\tilde{C}(m)\left(\sum_{\substack{k\\ |x_k|\ge a_n^{\lambda}}}(\cdot)+\sum_{\substack{k\\ |x_k|< a_n^{\lambda}}}(\cdot)\right) = III_1 + III_2.$$

Since III_1 is a sum of absolute values of weighted polynomials

$$\|III_1\| = III_1(x_0),$$

where

$$x_0 \in (-a_{2n-1}(w_1), a_{2n-1}(w_1)),$$

see [18] Lemma 1.. Therefore by the assumptions of the lemma and using that $a_{2n-1}(w_1) \le ca_n$, from (10) and (11) we have that

$$\|III_1\| \le C(m)a_ne^{-a_n^{\lambda}},$$

where $C(m)$ depends on m and K only, so indeed, the relation $III_2 \longrightarrow 0$ what we had to prove.

So it is enough to prove (13). To get a similar estimation in unweighted case on finite interval G. Grünwald had the following idea [6] : he defined auxiliary functions depending on a : g_a and g_a, which is equal to zero if $x \leq a$ or $x \geq a$, and for which the following inequality is valid:

$$H_n(g_a, g_a^{'}, a) \geq c \sum_{\substack{1 \leq k \leq n \\ x_k > a}} l_k^2(a)(x_k - a)$$

(for g^a the same with $x_k < a$) Because $g_a(a) = 0$, if $|g_a(a) - H_n(g_a, g_a^{'}, a)|$ tends to zero, then the "tail part" of the Hermite interpolation tends to zero as well. We will do the same, when $|a| \leq ca_n^\lambda$, and a variant of it, when $|a| \geq ca_n^\lambda$. For this purpose let us define:

Definition 5 *Let a be a parameter such that $-ca_n < a < ca_n$. If $a \geq -ca_n^\lambda$, let*

$$g_a(x) = g_{a,n}(x) = (x - a)^{1-\mu} f_a(x), \tag{14}$$

where

$$f_a(x) = \begin{cases} 0 & x < a \\ e^{-x} & -2 \leq a \leq x \\ e^{-\frac{x}{|a|}} & a \leq x, -ca_n^\lambda < a < -2 \end{cases} \tag{15}$$

if $a \leq -ca_n^\lambda$, then we need two new functions:

$$g_{a,1}(x) = \begin{cases} \frac{x-a}{|a|} e^{-|x| - \frac{1}{d|x|}} sgn(x) & \text{if } a < -ca_n^\lambda,\ a \leq x,\ x \neq 0 \\ 0 & x < a \text{ or } x = 0 \end{cases} \tag{16}$$

where $d > 1$ is a constant, and

$$g_{a,2}(x) = \begin{cases} 0 & x < a \\ \frac{e^{-\frac{x}{|a|}}}{|a|^{1+\nu}}(x - a)\arctan(-x) & a \leq x \leq -1 \\ Ce^{-Bx+D} & x > -1 \end{cases}, \tag{17}$$

where

$$B = \frac{2}{\pi} - \frac{1}{a(1 + a)}$$

$$C = \frac{|1 + a|\pi}{4|a|^{1+\nu}},$$

$$D = \frac{1}{|1 + a|} - \frac{2}{\pi}.$$

If $a \le ca_n^\lambda$

$$g^a(x) = g^{a,n}(x) = (a-x)^{1-\mu} f^a(x), \tag{18}$$

where

$$f^a(x) = f_a(-x), \tag{19}$$

and $g^{a,i}$ *similarly.*

Lemma 2 *If*

$$\left| g_a(a) - w^2(a) H_n\left(\frac{g_a}{w^2}, \left(\frac{g_a}{w^2}\right)', a\right)\right| \longrightarrow 0 \tag{20}$$

"enough quickly", and the same is valid for $g_{a,i},\ g^a,\ g^{a,i}$*, then*

$$\left\| w_1(x) \sum_{\substack{1\le k\le n \\ |x_k|<a_n^\lambda}} \frac{l_k^2(x)}{w^2(x_k)} |x-x_k| e^{-|x_k|} \right\| \longrightarrow 0. \tag{21}$$

We will see that if $|x| \le ca_n^\lambda$, then (21) is true with w^2 instead of w_1, that is we need w_1 only if $ca_n^\lambda < |x| < ca_n$.

Proof: We will prove only for g_a, because for g^a the proof works on the same line.

If $a > -ca_n^\lambda$, then

$$g_a(x) + (a-x)g_a'(x) = (x-a)^{1-\mu}(\mu f_a(x) - (x-a)f_a'(x))$$

$$= (x-a)^{1-\mu} \begin{cases} 0 & x<a \\ e^{-x}(\mu + x - a) & -2 \le a \le x \\ e^{-\frac{x}{|a|}}\left(\mu + \frac{x-a}{|a|}\right) & a \le x, -ca_n^\lambda < a < -2 \end{cases} \tag{22}$$

$$\ge c(a)(x-a)e^{-|x|} \tag{23}$$

where

$$c(a) = \begin{cases} 0 & x<a \\ c & -2 \le a \le x \\ c|a|^{-\mu} & a \le x, -ca_n^\lambda < a < -2 \end{cases} \tag{24}$$

In the upper estimation we used that if $x - a \le 1$, then $(x-a)^{1-\mu} \ge x - a$, and the other positive term is omitted; and when $x - a > 1$, we take only the second term in consideration. So thus if $a > -ca_n^\lambda$, and

$$\varepsilon_n > \left| g_a(a) - w^2(a) H_n\left(\frac{g_a}{w^2}, \left(\frac{g_a}{w^2}\right)', a\right)\right|$$

$$\geq c(a)w^2(a) \sum_{\substack{1\leq k\leq n \\ a<x_k<ca_n^\lambda}} \frac{l_k^2(a)}{w^2(x_k)}|a-x_k|e^{-|x_k|} \tag{25}$$

then (21) $\longrightarrow 0$ if $\frac{\varepsilon_n}{c(a)} \longrightarrow 0$ (it means eg that $\varepsilon_n a_n^{\lambda\mu} \longrightarrow 0$).

And if $a < -ca_n^\lambda$:

$$g_{a,1}(x) + (a-x)g_{a,1}^{'}(x) = \frac{(x-a)^2}{|a|} e^{-|x|-\frac{1}{d|x|}} \left(1 - \frac{1}{dx^2}\right), \tag{26}$$

$(d > 1)$ which is positive, when $dx^2 > 1$, and is negative around 0. We also have that

$$|g_{a,1}(x) + (a-x)g_{a,1}^{'}(x)|$$

$$\begin{cases} \geq C(\delta)(x-a)e^{-|x|} & \text{if } x > ca \text{ and } |x| > \delta, ||x| - d^{-\frac{1}{2}}| > \delta \\ \leq c(x-a)e^{-|x|} & |x| \leq d^{-\frac{1}{2}} \text{ or } d^{-\frac{1}{2}} + \delta < |x| < ca_n^\lambda, \end{cases} \tag{27}$$

So thus if

$$\left| g_{a,1}(a) - w^2(a)H_n\left(\frac{g_{a,1}}{w^2}, \left(\frac{g_{a,1}}{w^2}\right)^{'}, a\right)\right| < \varepsilon_{n,1} \tag{28}$$

say, it means that

$$\left| w^2(a) \sum_{|x_k|>\frac{1}{\sqrt{d}}} \frac{l_k^2(a)}{w^2(x_k)} |g_{a,1}(x_k) + (a-x_k)g_{a,1}^{'}(x_k)| \right.$$

$$\left. -w^2(a) \sum_{|x_k|\leq\frac{1}{\sqrt{d}}} \frac{l_k^2(a)}{w^2(x_k)} |g_{a,1}(x_k) + (a-x_k)g_{a,1}^{'}(x_k)| \right|$$

$$= |H_1 - H_2| < \varepsilon_{n,1}. \tag{29}$$

That is in this case it is enough to prove that $|H_2|$ is "small". For this purpose we have to use $g_{a,2}$:

$$g_{a,2}(x) + (a-x)g_{a,2}^{'}(x)$$

$$= \begin{cases} 0 & x \leq a \\ \frac{e^{-\frac{x}{|a|}}}{|a|^{1+\nu}}(x-a)\left(\arctan(-x)\frac{x-a}{|a|} + \frac{x-a}{1+x^2}\right) & a \leq x \leq -1 \\ Ce^{-Bx+D}(1+B(x-a)) & x > -1 \end{cases} \tag{30}$$

Thus we can see that $g_{a,2}(x)+(a-x)g_{a,2}^{'}(x)\geq 0$ for all $x>a$, and if $|x|\leq 1$, then $g_{a,2}(x)+(a-x)g_{a,2}^{'}(x)>c(x-a)e^{-|x|}|a|^{-\nu}$. So if

$$\left|g_{a,2}(a)-w^2(a)H_n\left(\frac{g_{a,2}}{w^2},\left(\frac{g_{a,2}}{w^2}\right)^{'},a\right)\right|<\varepsilon_{n,2}$$

then

$$H_2\leq cw^2(a)\sum_{\substack{k\\|x_k|\leq 1}}\frac{l_k^2(a)}{w^2(x_k)}(x_k-a)e^{-|x_k|}$$

$$\leq c|a|^{\nu}\left|g_{a,2}(a)-w^2(a)H_n\left(\frac{g_{a,2}}{w^2},\left(\frac{g_{a,2}}{w^2}\right)^{'},a\right)\right|\leq c\varepsilon_{n,2}|a|^{\nu}$$

and from (27) and(29) we have (with $c=c(\delta)$) that

$$w_1(a)\sum_{\substack{k\\|a|>a_n^{\lambda}>|x_k|>1}}\frac{l_k^2(a)}{w^2(x_k)}(x_k-a)e^{-|x_k|}\leq c\frac{|H_1|}{\log(|a|)}h(a)$$

$$\leq c\frac{\varepsilon_{n,1}}{\log(|a|)}h(a)+\frac{|H_{n,2}|}{\log(|a|)}h(a)$$

$$\leq c\frac{\varepsilon_{n,1}}{\log(|a|)}h(a)+c\frac{\varepsilon_{n,2}|a|^{\nu}}{\log(|a|)}h(a) \tag{31}$$

So if (31) tends to zero, then (21) tends to zero as well, for $x<-a_n^{\lambda}$.

Summarizing, we get, that eg. $-a_n^{\lambda}<a<-2$ then we have to divide the "tail sum" to two parts:

$$w^2(a)\sum_{\substack{k\\|x_k|<a_n^{\lambda}}}\frac{l_k^2(a)}{w^2(x_k)}|a-x_k|e^{-|x_k|}=\sum_{x_k<a}(\cdot)+\sum_{x_k>a}(\cdot)=I_a+II_a$$

For estimating I_a we have to use g^a in that case when $a<2$, and for II_a we use g_a, when $-a_n^{\lambda}<a<-2$. On the other four intervals we have to do the same, noting that when $|a|>a_n^{\lambda}$, I_a or II_a is empty.

Lemma 3 *There exists a polynomial $q_n(x)$ with degree n, such that*

$$\|(x-a)g_a^{'}(x)-(x-a)q_nw^2(x)\|<\varepsilon_n^* \tag{32}$$

and

$$\|g_a(x)-\int_{-\infty}^{x}q_nw^2\|<c\varepsilon_n^*\log n, \tag{33}$$

and the same is valid for g_a, g^a, $g_{a,i}$ and $g^{a,i}$, with $\varepsilon_n^* = \varepsilon_n^*(a)$, such that if $|a| \le ca_n^\lambda$, then

$$\log n\varepsilon_n^*(a)c(a)^{-1} \longrightarrow 0 \text{ independently of } a, \text{ as } n \longrightarrow \infty \tag{34}$$

and if $|a| > ca_n^\lambda$, then

$$\varepsilon_n^*|a|^\nu = O(1)$$

Proof: We will prove only for g_a and $g_{a,i}$ again. We have to distinguish some cases according to the placing of a.

Case 1.

If $a > -2$, then

$$g_a^{'} = \begin{cases} e^{-x}(x-a)^{-\mu}(1-\mu-x+a) & x > a \\ 0 & x < a \end{cases} \tag{35}$$

Let us define with some $\delta = \delta(n)$

$$\Psi_\delta(x) = \begin{cases} 1 & |x-a| > \delta \\ 0 & |x-a| < \frac{\delta}{2} \\ \text{continuous, linear} & \text{elsewhere} \end{cases} \tag{36}$$

and we define now E_n as it follows:

$$E_n := E_{n,a,\delta,w} = \left\| \left(\frac{(\Psi_\delta g_a^{'})(x)}{w^2(x)} - q_n(x) \right) |\hat{a}| w^2(x) e^{\frac{|x|}{2}} \right\| \tag{37}$$

where $|\hat{a}| = \max\{1, |a|\}$,$q_n$ is the best approximating polynomial of $\frac{(\Psi_\delta g_a^{'})(x)}{w^2(x)}$ with respect to $w_a(x) = |\hat{a}|w^2(x)e^{\frac{|x|}{2}}$. This w_a is also a Freud weight, and the function above is in C_{w_a}. With this notation we have that

$$|(x-a)g_a^{'}(x) - (x-a)q_n w^2(x)||\hat{a}|e^{\frac{|x|}{4}} \le |(x-a)g_a^{'}(x) - (x-a)g_a^{'}(x)\Psi_\delta(x)|\hat{a}|e^{\frac{|x|}{4}}$$

$$+|g_a^{'}(x)\Psi_\delta(x) - q_n w^2(x)||a|e^{\frac{|x|}{2}} \left| \frac{x-a}{|\hat{a}|} e^{-\frac{|x|}{4}} \right| = I + II.$$

$$I \le \|(x-a)g_a^{'}(x)|\hat{a}|e^{\frac{|x|}{4}}\|_{[a-\delta,a+\delta]}$$

$$= \left\| (x-a)^{1-\mu}(1-\mu-x+a)|\hat{a}|e^{-x+\frac{|x|}{4}} \right\|_{[a-\delta,a+\delta]}$$

$$\le c|\hat{a}|\delta^{1-\mu}e^{-a+\frac{|a|}{4}} \le c\delta^{1-\mu}. \tag{38}$$

$$II \leq cE_n \tag{39}$$

So thus we have to estimate E_n. According to [4] Theorem 1.2. and Corollary 1.5.,we have to investigate the following modulus of smoothness:

$$E_n \leq c$$

$$\times \sup_{0<h\leq \frac{a_n(w_a)}{n}} \left\| \left(\frac{(\Psi_\delta g_a^{'})}{w^2}\left(x+\frac{h}{2}\right) - \frac{(\Psi_\delta g_a^{'})}{w^2}\left(x-\frac{h}{2}\right)\right)|\hat{a}|w^2(x)e^{\frac{|x|}{2}}\right\|_{|x|<\kappa a_n(w_a)}$$

$$+\||\hat{a}|e^{\frac{|x|}{2}}\Psi_\delta(x)g_a^{'}(x)\|_{|x|>\kappa a_n(w_a)} = II_1 + II_2$$

($\kappa \leq 1$ but fixed constant and $a_n(w_a) \sim a_n$.)

At first we start the estimation of II_2: Because in $x < -ca_n$ the function is identically zero, we have to deal with the positive part of the real line. Because $a+1 \leq \kappa a_n(w_a)$,

$$||a|e^{-\frac{|x|}{2}}(x-a)^{-\mu}(1-\mu-x+a)| \leq |a|(x-a)^{1-\mu}e^{-\frac{|x|}{2}} \leq ca_n^{2-\mu}e^{-ca_n}.$$

If $|a-\kappa a_n(w_a)| \leq 1$, then

$$||a|e^{\frac{|x|}{2}}\Psi_\delta(x)g_a^{'}(x)| < ca_n e^{-ca_n}\delta^{-\mu}.$$

If $\kappa a_n(w_a)+1 < a$, then

$$||a|e^{-\frac{|x|}{2}}(x-a)^{-\mu}(1-\mu-x+a)\Psi_\delta(x)| \leq \max\{a_n^{2-\mu}e^{-ca_n}, a_n e^{-ca_n}\delta^{-\mu}\}.$$

The main part will be II_1: Since when $|x| < ca_n$,

$$\frac{w^2(x)}{w^2\left(x \pm \frac{h}{2}\right)} \sim c, \tag{40}$$

we have to deal with the weighted modulus of $g_a^{'}\Psi_\delta$ with the weight: $|a|e^{\frac{|x|}{2}}$.

Around a we get that

$$\left||\hat{a}|e^{\frac{|x|}{2}}\left(\left(g_a^{'}\Psi_\delta\right)\left(x+\frac{h}{2}\right) - \left(g_a^{'}\Psi_\delta\right)\left(x-\frac{h}{2}\right)\right)\right| \leq c|\hat{a}|e^{-\frac{|a|}{2}}\delta^{-\mu-1}\frac{a_n}{n} \tag{41}$$

Here we used that if $-2 < a < 0$, then $|a|e^{\frac{3|a|}{2}}$ is at most a constant. Let $\delta = n^{-\gamma}$! Then using the estimation on a_n [11] Lemma 5.2. :

$$|\hat{a}|e^{-\frac{|a|}{2}}\delta^{-\mu-1}\frac{a_n}{n} \leq cn^{\gamma(\mu+1)+\frac{1}{A}-1} \tag{42}$$

which tends to zero, if $\gamma < \frac{A-1}{A(\mu+1)}$.

If $|x-a| > \delta, |x| < \kappa a_n(w_a)$ then

$$\left| |\hat{a}| e^{\frac{|x|}{2}} \left(\left(g_a^{'} \right) \left(x + \frac{h}{2} \right) - \left(g_a^{'} \right) \left(x - \frac{h}{2} \right) \right) \right| \le c \frac{a_n}{n} |\hat{a}|$$

$$\left| -e^{-x+\frac{|x|}{2}} (x-a)^{-\mu-1} ((x-a)(2-\mu-x+a) + \mu(1-\mu-x+a)) \right|$$

$$\le c \frac{a_n}{n} \begin{cases} |\hat{a}| e^{-\frac{|a|}{2}} \delta^{-\mu-1} & |x-a| \sim \delta \\ e^{-c|a|} & x-a \sim c \\ e^{-\frac{|x|}{2}} |\hat{a}| (x-a)^{1-\mu} & x-a > c \end{cases} \le c \varepsilon_n^* \tag{43}$$

where

$$\varepsilon_n^* < c n^{\gamma(\mu+1)} \frac{a_n}{n}$$

If we assume further that

$$\frac{1}{2} \left(1 - \frac{1}{A} \right) < \gamma < \frac{A-1}{A(\mu+1)},$$

then in (38) and (39) I and $II \longrightarrow 0$.

Case 2.

When $-c a_n^\lambda < a < -2$,

$$g_a^{'}(x) = \begin{cases} e^{-\frac{x}{|a|}} (x-a)^{-\mu} \left(1 - \mu - \frac{x-a}{|a|} \right) & x > a \\ 0 & x < a \end{cases} \tag{44}$$

So thus introducing

$$E_n := E_{n,a,\delta,w} = \left\| \left(\frac{(\Psi_\delta g_a^{'})(x)}{w^2(x)} - q_n(x) \right) |a| w^2(x) e^{\frac{|x|}{2|a|}} \right\|, \tag{45}$$

thinking on the same line as in *Case* 1. (36) - (39), we have to estimate E_n again. The computation also works on the same line, that is we have to estimate the two terms of the modulus of smoothness : II_1 and II_2. Because $-a_n^\lambda < a$, we have to deal with the positive part of the real axis again, and so

$$II_2 \le \left\| e^{-\frac{x}{c|a|}} |a| \frac{(x-a)^{1-\mu}}{|a|} \right\|_{x > \kappa a_n} \le c a_n^{1-\mu} e^{-a_n^{1-\lambda}} \tag{46}$$

As in the previous case, the main part will be II_1. According to (40) we have to deal with

$$\sup_{0 < h < \frac{a_n}{n}} \left\| |a| e^{\frac{|x|}{2|a|}} \left(\left(g_a^{'} \Psi_\delta \right) \left(x + \frac{h}{2} \right) - \left(g_a^{'} \Psi_\delta \right) \left(x - \frac{h}{2} \right) \right) \right\|_{|x| < \kappa a_n} \tag{47}$$

Around a with the previous notations (with some γ):

$$(47) \leq c|a|e^{-\frac{3|a|}{2|a|}}\delta^{-\mu-1}\frac{a_n}{n} \leq \frac{a_n^{\lambda+1}n^{\gamma(\mu+1)}}{n} \leq n^{\frac{\lambda+1}{A}+\gamma(\mu+1)-1}, \tag{48}$$

If $x > a + \delta$ with some $|\eta| \leq c\frac{a_n}{n}$:

$$(47) \leq \max\left\{c|a|e^{\frac{3|a|}{2|a|}}\delta^{-\mu-1}\frac{a_n}{n}; c|a|\frac{a_n}{n}\left\| e^{\frac{|x|}{2|a|}}e^{-\frac{|x+\eta|}{|a|}}(x+\eta-a)^{-\mu-1}\right.\right.$$

$$\left.\left.\times\left(\frac{(x+\eta-a)^2}{|a|} - \frac{x+\eta-a}{|a|}2(1-\mu) - \mu(1-\mu)\right)\right\|_{|x|<ca_n}\right\}$$

$$\leq c\begin{cases} \frac{a_n^{\lambda+1}n^{\gamma(\mu+1)}}{n} & x-a<1 \\ \frac{a_n}{n}\|(x-a)^{1-\mu}\| \leq \frac{a_n^{\lambda(1-\mu)+1}}{n} & x-a<c|a| \\ \frac{|a|^{1-\mu}a_n}{n}\left\|e^{-\frac{3|x|}{2|a|}}\left(\frac{x-a}{|a|}\right)^{1-\mu}\right\|_{x-a>c|a|} \leq \frac{a_n^{\lambda(1-\mu)+1}}{n} & x-a>c|a| \end{cases} \tag{49}$$

That is the worst situation is around a. By (38) and (43) we have that

$$\varepsilon_n^* \leq c\max\{n^{\frac{1}{A}-1+\gamma(\mu+1)}, n^{\frac{\lambda+1}{A}+\gamma(\mu+1)-1}, n^{-\gamma(1-\mu)}\}. \tag{50}$$

Let

$$\frac{\lambda(\mu+1)}{1-\mu} < A-1, \quad \frac{\lambda\mu}{A(1-\mu)} < \gamma < \frac{A-\lambda(\mu+1)-1}{A(1+\mu)} \tag{51}$$

With this choice of parameters (32) is valid, and we get that

$$\log n\frac{\varepsilon_n^*}{c(a)} \longrightarrow 0. \tag{52}$$

Case 3.

When $a < -a_n^\lambda$ we can prove only for $d = 1$, in the other cases the proofs are the same.

$$g'_{a,1}(x) = \begin{cases} \frac{1}{|a|}e^{-|x|-\frac{1}{|x|}}\left(sgn x + (x-a)\left(-1+\frac{1}{x^2}\right)\right) & x>a,\ x\neq 0 \\ 0 & x<a \ or \ x=0 \end{cases} \tag{53}$$

As in the first case we have to deal with I and II.

$$I \leq \|g'_{a,1}(x-a)e^{\frac{|x|}{4}}|a|\|_{[a-\delta,a+\delta]} = O\left(\delta e^{-|a|}\right) \tag{54}$$

For estimation of II, we introduce:

$$E_n := E_{n,\delta,w_a} = \left\| \left(\frac{(\Psi_\delta g_a^{'})(x)}{w^2(x)} - q_n(x) \right) |a| w^2(x) e^{\frac{|x|}{2}} \right\|, \tag{55}$$

and with the same notations as in Case 1, let us estimate the two parts of the modulus of smoothness: II_1 and II_2.

$$II_2 = \| \Psi_\delta e^{-\frac{|x|}{2} - \frac{1}{|x|}} (sgnx + (x-a)(-1+x^{-2})) \|_{|x|>\kappa a_n(w_a)} \le c a_n e^{-c a_n} \tag{56}$$

Estimating II_1 we don't need to deal with w^2 again, so

$$II_1 \le c \sup_{0<h<\frac{a_n}{n}} \left\| |a| e^{\frac{|x|}{2}} \left(\left(g_{a,1}^{'} \Psi_\delta \right) \left(x + \frac{h}{2} \right) - \left(g_{a,1}^{'} \Psi_\delta \right) \left(x - \frac{h}{2} \right) \right) \right\|_{|x|<\kappa a_n(w_a)} \tag{57}$$

Around a the difference above can be estimated by

$$c e^{-c a_n^\lambda} \delta^{-1} \frac{a_n}{n},$$

which tends to zero exponentially with n tends to infinity, because δ^{-1} grows polynomially. Away from a

$$II_1 \le c \frac{a_n}{n}$$

$$\left\| e^{-c|x| - \frac{1}{|x|}} (-2 + 2x^{-2} - 2(x-a)x^{-3} + sgnx(-1+x^{-2})^2 (x-a)) \right\|_{|x|<ca_n}$$

$$\le c|a| \frac{a_n}{n} \le c \frac{a_n^2}{n} \tag{58}$$

It means that $\varepsilon_n^* = O\left(n^{\frac{2}{A}-1}\right)$ and II_1 tends to zero only if $\frac{a_n^2}{n} \le n^{\frac{2}{A}-1} \longrightarrow 0$, iff $A > 2$.

Let us see now

$$g_{a,2}^{'}(x)$$

$$= \begin{cases} 0 & x < a \\ \frac{e^{-\frac{x}{|a|}}}{|a|^{1+\nu}} \left(-\frac{1}{|a|}(x-a)\arctan(-x) + \arctan(-x) - \frac{x-a}{1+x^2} \right) & a < x \le -1 \\ -BCe^{-Bx+D} & x > -1 \end{cases} \tag{59}$$

and

$$E_n := E_{n,\delta,w_a} = \left\| \left(\frac{(\Psi_\delta g_a^{'})(x)}{w^2(x)} - q_n(x) \right) |a| w^2(x) e^{\frac{|x|}{2|a|}} \right\|, \tag{60}$$

Similarly to the previous cases, at first we have to estimate

$$I \leq \|g_{a,1}^{'}(x-a)e^{\frac{|x|}{4|a|}}|a|\|_{[a-\delta,a+\delta]} = O\left(\delta|a|^{-\nu}\right) \tag{61}$$

then II:

$$\|g_{a,2}^{'}\Psi_{\delta}|a|e^{\frac{|x|}{2|a|}}\|_{|x|>\kappa a_n} = O(|a|^{-\nu}) \tag{62}$$

and

$$\sup_{0<h<\frac{a_n}{n}}\left\||a|e^{\frac{|x|}{|a|2}}\left(\left(g_{a,2}^{'}\Psi_{\delta}\right)\left(x+\frac{h}{2}\right)-\left(g_{a,2}^{'}\Psi_{\delta}\right)\left(x-\frac{h}{2}\right)\right)\right\|_{|x|<\kappa a_n} \tag{63}$$

Around a

$$(63) \leq c\frac{a_n}{\delta|a|^{\nu}n},$$

Away from a, when $x<-1$:

$$\left||a|\frac{a_n}{n}e^{\frac{|x|}{|a|2}}g_{a,2}^{''}(x)\right| \leq c\frac{a_n}{|a|^{\nu}n},$$

and if $x \geq -1$:

$$\left||a|\frac{a_n}{n}e^{\frac{|x|}{|a|2}}g_{a,2}^{''}(x)\right| \leq c\frac{|a|a_n}{|a|^{\nu}n}.$$

That is $E_n = O(|a|^{-\nu})$ and so $\varepsilon_n^* = O\left(n^{-\frac{\nu}{A}}\right)$.

Now we proved that

$$|(x-a)g_a^{'}(x)-(x-a)q_n w^2(x)||\hat{a}|e^{\frac{|x|}{4b}} \leq \varepsilon_n^* \tag{64}$$

where

$$b = \begin{cases} |a| & \text{if} -ca_n^{\lambda} < a < -2 \\ 1 \text{ otherwise} & \end{cases}, \tag{65}$$

and $\log n\varepsilon_n^* c(a)^{-1} \longrightarrow 0$,and the speed of convergence is independent of a, so thus (32) and (34) is proved. We prove (33).

If $x < a+n^{-\beta}, (\beta > 0)$

$$\left|g_a(x)-\int_{-\infty}^{x} q_n w^2\right| \leq |g_a(x)|+\left|\int_{-\infty}^{x} q_n w^2\right| \tag{66}$$

Here in the worst case:

$$|g_a(x)| \leq n^{-\beta(1-\mu)} \tag{67}$$

If $x < a - n^{-\beta}$, then according to (64)

$$\left|\int_{-\infty}^{x} q_n w^2\right| \leq \int_{-\infty}^{x} |q_n w^2|$$

$$\leq \varepsilon_n^* \int_{-\infty}^{x} \frac{e^{-\frac{|y|}{4b}}}{|\hat{a}|(y-a)} dy \leq c\varepsilon_n^* |\hat{a}|^{-1} \log n \tag{68}$$

If $x \in (a - n^{-\beta}, a + n^{-\beta})$:

$$\left|\int_{-\infty}^{x} q_n w^2\right| \leq \left|\int_{-\infty}^{a-n^{-\beta}} q_n w^2\right| + \left|\int_{a-n^{-\beta}}^{x} q_n w^2\right|$$

$$\leq c\varepsilon_n^* |\hat{a}|^{-1} \log n + cn^{-\beta} \|q_n w^2\|_{(a-n^{-\beta}, a+n^{-\beta})} \tag{69}$$

$$\|q_n w^2\|_{(a-n^{-\beta}, a+n^{-\beta})} \leq cE_n + \|\Psi_\delta g_a^{'}\|_{(a-n^{-\beta}, a+n^{-\beta})} \leq cE_n,$$

if $n^{-\beta} \leq \frac{1}{2} n^{-\gamma}$, that is

$$\left|\int_{-\infty}^{x} q_n w^2\right| = O(\varepsilon_n^* \log n) \tag{70}$$

If $x > a + n^{-\beta}$

$$\left|g_a(x) - \int_{-\infty}^{x} q_n w^2\right| \leq \left|\int_{-\infty}^{x} g_a^{'} - q_n w^2\right| \leq \left|\int_{-\infty}^{a+n^{-\beta}} g_a^{'} - q_n w^2\right|$$

$$+ \int_{a+n^{-\beta}}^{x} \left|((y-a) g_a^{'}(y) - (y-a) q_n w^2(y)) e^{\frac{|y|}{4b}}\right| \frac{1}{|y-a|} e^{-\frac{|y|}{4b}} = I + II$$

$$I \leq \left|\int_{-\infty}^{a+n^{-\beta}} g_a^{'} - g_a^{'} \Psi_\delta\right| + \left|\int_{-\infty}^{a+n^{-\beta}} ((g_a^{'} \Psi_\delta)(y) - q_n w^2(y)) e^{\frac{|y|}{2b}} e^{-\frac{|y|}{2b}}\right|$$

$$\leq cn^{-\beta(1-\mu)} + c\frac{E_n}{|\hat{a}|} b \leq \varepsilon_n^* \tag{71}$$

$$II \leq c\varepsilon_n^* \log n \tag{72}$$

Thus (33) is proved and Lemma 3 as well.

Lemma 4 *Denoted by $c_{j,n}$ the Fourier coefficients of q_n:*

$$q_n = \sum_{j=0}^{n} c_{j,n} p_n(w^2) \tag{73}$$

we have that

$$|c_{0,n}| = O(\varepsilon_n^* \log n), \ |c_{j,n}| = O\left(n a_n^{\frac{1}{2}} k^{cn}\right) \tag{74}$$

Proof: We will prove only for g_a again. Since $g_a \longrightarrow 0$ when $x \longrightarrow \infty$, from (33) follows that

$$\lim_{x\longrightarrow\infty}\left|g_a(x)-\int_{-\infty}^{x} q_n w^2\right| = \lim_{x\longrightarrow\infty}\left|\int_{-\infty}^{x} q_n w^2\right| = |c_{0,n}| = O(\varepsilon_n^* \log n)$$

Let us denote by

$$q_n^* := q_n - c_{0,n}p_0(w^2) \quad \text{and} \quad p_j := p_j(w^2)$$

$$|c_{j,n}| = \left|\int_{\mathbf{R}} q_n^* p_j w^2\right| = \left|\left[p_j \int_{-\infty}^{x} q_n^* w^2\right]_{\mathbf{R}} - \int_{\mathbf{R}} p_j' \int_{-\infty}^{x} q_n^* w^2\right|$$
$$\leq \left\|\frac{\int_{-\infty}^{x} q_n^* w^2}{w}\right\| \|p_j' w\|_1$$

If $|x| > ca_n(w)$ then

$$\left|\frac{\int_{-\infty}^{x} q_n^* w^2}{w}\right| = \left|\frac{\int_{x}^{\infty} q_n^* w^2}{w}\right| \leq \|q_n^* w\|_{|x|>ca_n(w)} \left|\frac{\int_{|x|}^{\infty} wQ'(Q')^{-1}}{w}\right|$$

$$\leq \|q_n^* w\|_{|x|>ca_n(w)} \|(Q')^{-1}\|_{|x|>ca_n(w)}$$

which is exponentially small ([16]). If $|x| \leq ca_n(w)$ then

$$\left|\frac{\int_{-\infty}^{x} q_n^* w^2}{w}\right| \leq \left|\int_{-\infty}^{x} (q_n - c_{0,n}p_0)w^2\right| w^{-1}(ca_n(w)) \leq c\|g_a\|k^n \leq ca_n k^n$$

According to [4] Theorem 1.4., and [14] Theorem 1.:

$$\|p_j' w\|_1 \leq c\frac{j}{a_j(w)}\|p_j w\|_1 \leq c\frac{j}{\sqrt{a_j(w)}}$$

So thus

$$|c_{j,n}| \leq cn\sqrt{a_n(w)}k^n \leq cn\sqrt{a_n}k^n = O(k^n)$$

Lemma 5 *If w^2 has the Rodrigues' property with respect to X, then*

$$\left|g_a(a) - w^2(a)H_n\left(\frac{g_a}{w^2}, \left(\frac{g_a}{w^2}\right)', a\right)\right| \longrightarrow 0 \tag{75}$$

"enough quickly" and the same is valid for $g_{a,i}, g^a, g^{a,i}$ too.

Proof:

$$\left|g_a(a) - w^2(a)H_n\left(\frac{g_a}{w^2}, \left(\frac{g_a}{w^2}\right)', a\right)\right| \leq \left|g_a(a) - \int_{-\infty}^{a} q_{\kappa n}w^2\right|$$

$$+\left|\int_{-\infty}^{a} q_{\kappa n}w^2 - w^2(a)H_n\left(\frac{\int_{-\infty}^{x} q_{\kappa n}w^2}{w^2}, \left(\frac{\int_{-\infty}^{x} q_{\kappa n}w^2}{w^2}\right)', a\right)\right|$$

$$+\left|w^2(a)H_n\left(\frac{\int_{-\infty}^{x} q_{\kappa n}w^2}{w^2}, \left(\frac{\int_{-\infty}^{x} q_{\kappa n}w^2}{w^2}\right)', a\right) - w^2(a)H_n\left(\frac{g_a}{w^2}, \left(\frac{g_a}{w^2}\right)', a\right)\right|$$

$$\leq \varepsilon_n^*(a)\log n + \delta_n + \left|w^2(a)\sum_{k=1}^{n}\frac{l_k^2(a)}{w^2(x_k)}\right.$$

$$\left.\left[\left(\int_{-\infty}^{x_k} q_{\kappa n}w^2 - g_a(x_k)\right) + \Big((a-x_k)(q_{\kappa n}w^2)(x_k) - (a-x_k)g_a'(x_k)\Big)\right]\right|$$

$$\leq c(\varepsilon_n^*(a)\log n + \delta_n)$$

Applying Lemma 1,2,3,4 we have that

$$\varepsilon_n^*(a)\log n c(a)^{-1} \longrightarrow 0 \text{ and } \delta_n n^{\frac{\mu}{A}} \longrightarrow 0 \tag{76}$$

and the statement is proved.

Proof of Theorem 1 :

Along the line of previous lemmas at first we choose an m large enough such that E_{m,w^2} is small, and then with fixed m and suitable λ, γ we choose κn ($\kappa \in (0,1)$) and Lemma 3 can be applied with $q_{\kappa n}$ and at the and because of Lemma 4 we can choose in the definition of "Rodrigues' property" $c = \kappa^{-1}$ and $\alpha = \frac{\mu}{A}$ and Lemma 5 shows that the convergence is "enough quickly" as it needs according to Lemma 2.

Remark: This proof can be carried out for $\varrho(w)-$normal systems without any changes. The difficulty is to give examples to such systems. It requires a very careful computation eg. with differential equation of orthogonal polynomials [15]. Namely we can have a conjecture (see [7]), that the root system of the orthogonal polynomials with respect to a Freud weight depending on ϱ and w has to be $\varrho(w)$-normal to another Freud weight: w.

Now we have to deal with the "Rodrigues' property" namely we have to show that there is some Freud-type weight functions for which this property is valid.

Theorem 2 *If* $w^2(x) = w_k^2(x) = e^{-x^{2k}}$ *where* $k = 2, 3, 4, 5$*; and* X *is an interpolatory matrix such that the kernels of Hermite interpolation have polynomial growth, then the "Rodrigues' property" is valid with respect to* X*.*

The conception of the proof is the following: We will express a polynomial with degree n and with property (6) as a sum of such terms, whose weighted integral functions are weighted polynomials, plus a linear combination of fix number (depending on k) of orthonormal polynomials (wrt the weight) with degree around n. Then we have to deal with the integral of this weighted orthogonal polynomials. We will show that this integral can be expressed by Fourier sums, in which the index of the leading coefficient is $O(n)$. Then we will cut a suitable piece with degree cn of this Fourier sum and we will show that the remainder part is small together with it's weighted Hermite interpolatory polynomial. Summarizing, we will decomposite the integral function of a weighted polynomial to two parts: to a weighted polynomial with degree cn, which is coincides with it's weigthed Hermite interpolatory polynomial with the same degree, and a small remainder term.

That is how we decomposite the proof to some lemmas.

Lemma 6 *If* q_n *is a polynomial with degree* n *and with property (6), and* $w^2 = w_k^2, k \in \mathbf{N}$*, then*

$$q_n = \sum_{i=0}^{n+2k} c_i^* f_i + \sum_{j=0}^{2k} c_j^* p_{n-j}(w^2), \tag{77}$$

and

$$|c_j^*| = O(K^n) \tag{78}$$

where

$$f_i = (p_i(w^2)w^2)' w^{-2} \tag{79}$$

($p_i(w^2)$ *is the* i*-th orthonormal polynomial with respect to* w^2*,* K *is an absolute constant.)*

Proof: Using the abreviation $p_l = p_l(w^2)$, at first we show that

$$f_l(x) = -2kx^{2k-1}p_l(x) + p_l'(x) = \sum_{m=1}^{k} \gamma_{m,l} p_{l-1+2m}$$

The upper bound of the summation is k, because $f_l \in \mathcal{P}_{l+2k-1}$, and the lower is 1, because

$$\gamma_{l,j} = \int_{\mathbf{R}} (w^2 p_l)' p_{l-1+2j} = -\int_{\mathbf{R}} w^2 p_l (p_{l-1+2j})' = 0, \text{ if } j \le 0 \tag{80}$$

Here we used an integration by parts, and the integrated part is zero because of the L'Hospital rule, and then we applied orthogonality. Because the weight function is even, the parity of f_l and $l-1$ are the same. So thus we have that

$$p_{l+1} = \frac{f_l}{\gamma_{l,1}} - \sum_{j_1=2}^{k} \frac{\gamma_{j_1,l}}{\gamma_{1,l}} p_{l-1+2j_1}$$

$$= \frac{f_l}{\gamma_{1,l}} - \sum_{j_1=2}^{k} \frac{\gamma_{j_1,l}}{\gamma_{1,l}} \frac{f_{l-2+2j_1}}{\gamma_{1,l-2+2j_1}} + \sum_{j_1=2}^{k} \sum_{j_2=2}^{k} \frac{\gamma_{j_1,l}}{\gamma_{1,l}} \frac{\gamma_{j_2,l-2+2j_1}}{\gamma_{1,l-2+2j_1}} p_{l-3+2j_1+2j_2}$$

$$= \cdots = \sum_{i=0}^{\left[\frac{n-l+1}{2}\right]+k} \hat{c}_{l+2i} f_{l+2i} + \sum_{j=k-2}^{0} \hat{c}_{j,l} p_{n^*-2j}, \tag{81}$$

where $n^* = n$ or $n^* = n-1$ according to the parity of l. We got this last formula on such way, that we until decomposite a p_m, till its decomposition contains only such p_ν-s for which $\nu \le n$. Thus

$$q_n = \sum_{l=1}^{n} c_l p_l = \sum_{l=0}^{n-2k-2} c_{l+1} p_{l+1} + \sum_{l=n-2k}^{n} c_l p_l$$

$$= \sum_{l=0}^{n-2k-2} c_{l+1} \sum_{i=0}^{\left[\frac{n-l+1}{2}\right]+k} \hat{c}_{l+2i} f_{l+2i} + \sum_{l=0}^{n-2k-2} c_{l+1} \sum_{j=k-2}^{0} \hat{c}_{j,l} p_{n^*-2j} + \sum_{l=n-2k}^{n} c_l p_l$$

$$= \sum_{i=0}^{n+2k} c_i^* f_i + \sum_{j=-2k}^{0} c_{j^*} p_{n-j}(w^2) \tag{82}$$

Now we have to estimate $|c_j^*|$. At first we can observe ((81),(82)) that c_j^* consists of a Fourier coefficient of q_n plus some c_l times a product of $\frac{\gamma_{m,l}}{\gamma_{1,l}}$, which product contains at most $\frac{n}{2}$ terms. According to (81), by a coarse estimation we have that

$$|\hat{c}_{j,l}| \le \left(\max_{\substack{2 \le m \le k \\ n-2k \le l \le n}} \left| \frac{\gamma_{m,l}}{\gamma_{1,l}} \right| \right)^n nk =: \mathcal{K}_{l,n}^n$$

and by this estimation and (7), we have that $|c_j^*| \le c \max\{K^n, \mathcal{K}_{l,n}^n\}$, that is we have to show only that $\mathcal{K}_{l,n}$ is bounded independently of l and n.

According to the formulas of S.B. Bonan and D.S. Clark ([2] Cor.5. and Th. 5.)

$$f_l = 2k\left\{-\sum_{j=0}^{2k-1}\binom{2k-1}{j}\tilde{\varrho}_l^{2k-1}p_{l+2j-2k+1} + \sum_{i=0}^{k-1}\tilde{\varrho}_l^{2i+1}\binom{2i}{i}x^{2k-2i-2}p_{l-1}\right.$$

$$\left.-\sum_{i=0}^{k-2}\tilde{\varrho}_l^{2i+2}\binom{2i+1}{i}x^{2k-2i-3}p_l\right\} \tag{83}$$

$$= 2k\left\{-\sum_{j=0}^{2k-1}\binom{2k-1}{j}\tilde{\varrho}_l^{2k-1}p_{l+2j-2k+1}\right.$$

$$+\sum_{i=0}^{k-1}\tilde{\varrho}_l^{2i+1}\binom{2i}{i}\sum_{j=0}^{2k-2i-2}\tilde{\varrho}_{l-1}^{2k-2i-2}\binom{2k-2i-2}{j}p_{l-1+2(i+j-(k-1))}$$

$$\left.-\sum_{i=0}^{k-2}\binom{2i+1}{i}\tilde{\varrho}_l^{2i+2}\sum_{j=0}^{2k-2i-3}\tilde{\varrho}_l^{2k-2i-3}\binom{2k-2i-3}{j}p_{l-1+2(i+j-(k-2))}\right\} \tag{84}$$

As usually, ϱ_l is the quotient of the leading coefficients of p_{l-1} and p_l. We know that $\varrho_l \sim a_l$ [15. prop.3.7.], where a_l is the M-R-S-number. We denote by $\tilde{\varrho}_l^m$ a product of m factors, in which every factors are in the set $\{\varrho_l, \varrho_{l+1}, \dots, \varrho_{l+2k-1}\}$.

Thus for any $l \ge 0$ and $1 \le m \le k$

$$\frac{\gamma_{m,l}}{2k\varrho_l^{2k-1}} = -u_{m,l}\binom{2k-1}{m+k-1} + \sum_{i=0}^{k-m-1}\binom{2i}{i}\binom{2k-2i-3}{m+k-i-2}$$

$$\times\left(v_{i,l}\frac{2k-2i-2}{m+k-i-1} - w_{i,l}\frac{2i+1}{i+1}\right) \tag{85}$$

where $u_{i,l}, v_{i,l}, w_{i,l} \longrightarrow 1$, when $l \longrightarrow \infty$, namely

$$\varrho_l = \beta(k)l^{\frac{1}{2k}} + O\left(l^{-\left(1-\frac{1}{2k}\right)}\right) \tag{86}$$

see [2.Th.4], where $\beta(k)$ is the Freud constant depending only on k. So $\left(\frac{\tilde{\varrho}_l}{\varrho_l}\right)^m \longrightarrow 1$ if $l \longrightarrow \infty$.

That is

$$\frac{\gamma_{m,l}}{\gamma_{1,l}} \le C(k,m)\ , \tag{87}$$

where $C(k,m)$ does not depend on l and n and so the lemma is proved.

We have to remark here, that however after Bonan and Clark (see [2. Cor. 4., Cor. 5. etc]) we denote such quantities uniformly by $\tilde{\varrho}_l^m$ which are around ϱ_l^m, but we will follow, until it is useful and possible, the difference of the limit and the proper value. When it is clear, that for a fixed k and for large n the error is negligible, we will calculate with the limit value.

Lemma 7 *Let* $w^2 = w_k^2, k = 1, \dots, 5, \quad p_l = p_l(w^2)$. *Let* $n \geq l \geq n-2(k-1)$ *and* $n > n_0 = n_0(k)$.

$$w^{-2}(x) \int_{-\infty}^{x} p_l w^2 \sim \sum_{i=0}^{\infty} \sum_{j=0}^{k-2} b_{l-1+2(i(k-1)+j),l} p_{l-1+2(i(k-1)+j)} \tag{88}$$

and

$$|b_{l-1+2(i(k-1)+j),l}| = O\left(\frac{1}{a_n^{2k-1}} \delta^i\right) \tag{89}$$

where a_n *is the Mhaskar-Rahmanov-Saff number and* $\delta \in (0,1)$.

Proof:

$$b_{m,n} = \int_{\mathbf{R}} p_m \int_{-\infty}^{x} p_n w^2 = -\int_{\mathbf{R}} P_m p_n w^2 = 0 \text{ if } m < n-1$$

Here $P_m \in \mathcal{P}_{m+1}$ is the primitive function of p_m, and as in (80) the integrated part is zero. The parity of $\int_{-\infty}^{x} p_l w^2$ and of $l-1$ are equal again. We have to estimate $|b_{l-1+2m,l}|$. For this we observe that

$$\int_{\mathbf{R}} (p_\nu)' \int_{-\infty}^{x} p_n w^2 = -\int_{\mathbf{R}} p_\nu p_n w^2 = -\delta_{\nu,n} \tag{90}$$

We will compute the coefficients of $p_\nu^{'}$ and by the help of it we can get some recurrence formula on $b_{m,l}$. Recalling the formulas of Bonan and Clark again, we have that

$$p_\nu^{'} = 2k\varrho_\nu^{2k-1} \sum_{m=0}^{k-1} g_{m,\nu} p_{\nu-1-2m}$$

$$= 2k\varrho_\nu^{2k-1} \left\{ \sum_{i=0}^{k-1} \binom{2i}{i} \sum_{j=0}^{2k-2i-2} \binom{2k-2i-2}{j} u_{i,j,\nu} p_{\nu-1-2(k-1-(i+j))} \right.$$

$$\left. - \sum_{i=0}^{k-2} \binom{2i+1}{i} \sum_{j=0}^{2k-2i-3} \binom{2k-2i-3}{j} v_{i,j,\nu} p_{\nu-1-2(k-2-(i+j))} \right\} \tag{91}$$

Here we lifted out ϱ_ν from every $\{\varrho_{\nu+1}, \varrho_\nu, \dots, \varrho_{\nu-2(k-1)}\}$ again, and this is the reason why some (not the same) $u_{i,j,\nu}$-s and $v_{i,j,\nu}$-s appear. According to (86) $\lim_{\nu\longrightarrow\infty} u_{i,j,\nu}, v_{i,j,\nu} = 1$. We have to remark that $p_\nu^{'} \in \mathcal{P}_{\nu-1}$, and

$$g_{k-1,\nu} = u_{0,0,\nu} \sim 1.$$

If $0 \le m \le k-2$, then

$$g_{m,\nu} = \sum_{i=0}^{k-m-2} \binom{2i}{i}\binom{2k-2i-3}{k-i-2-m}\left(u_{i,\nu}\frac{2k-2i-2}{k-1-i-m} - v_{i,\nu}\frac{2i+1}{i+1}\right)$$
$$+\binom{2(k-m-1)}{k-m-1}u_{1,\nu} \qquad (92)$$

with some other $u_{i,\nu},\ v_{i,\nu}-$s which tend to 1 in the same order. Also from (86) we know that

$$g_{m,\nu} = g_m\left(1 + O\left(\nu^{-\left(1-\frac{1}{k}\right)}\right)\right),$$

where we denote by

$$g_m := \lim_{\nu\longrightarrow\infty} g_{m,\nu}.$$

Let us remark that in the bracket we have a positive expression if $n > n_0$.

Our observation in (90) means that the scalar product of $w^{-2}\int_{-\infty}^{x} p_n w^2$ and $p_n^{'}$ is equal to -1, that is

$$b_{n-1,n} = -\frac{1}{g_{0,n}2k\tilde{\varrho}_n^{2k-1}} = O\left(a_n^{-2k+1}\right) \qquad (93)$$

and this scalar product is zero, when $\nu > n$, that is for an $l \ge 0$

$$b_{n+2l-1,n} = -\sum_{j=1}^{k-1} b_{n+2l-1-2j,n}\frac{g_{j,n+2l}}{g_{0,n+2l}} \qquad (94)$$

Now we can express every $b_{m,n}$-s as a sum of $b_{n-1,n}, \cdots, b_{n-1+2(k-2),n}$. For this purpose we will cut the sequence of $b_{m,n}-$s to several pieces, such that every pieces contain $k-1$ coefficients with consecutive indices. At first we give the formula which express a $b_{n-1+2i(k-1)+2j,n}$ as the sum of $b_{n-1+2i(k-1)-2l,n}$-s, that is the sum of some constants times the elements of the previous group of the coefficients:

$$b_{n-1+2i(k-1)+2j,n} = \sum_{l=1}^{k-j-1} b_{n-1+2i(k-1)-2l,n}\left(\frac{\sum_{\substack{i_1+\cdots+i_{j+1}=l+j \\ if\ \ i_s\le\frac{k-1}{2}\ \ then\ \ i_s\le j}} (g_{i_1,\cdot}\cdots g_{i_{j+1},\cdot})}{\tilde{g}_0^{j+1}}\right.$$

$$-\frac{\sum\limits_{\substack{i_1+\cdots+i_j=l+j \\ if\ \ i_s\le\frac{k-1}{2}\ \ then\ \ i_s\le j}} (g_{i_1,\cdot}\cdots g_{i_j,\cdot})}{\tilde{g}_0^j} + \cdots + (-1)^{j+1}\frac{g_{l+j},\cdot}{g_{0,\cdot}}\Bigg)$$

$$+\sum_{l=k-j}^{k-1} b_{n-1+2i(k-1)-2l,n}\left(\frac{\sum\limits_{\substack{i_1+\cdots+i_{j+1}=l+j \\ if\ \ i_s\le\frac{k-1}{2}\ \ then\ \ i_s\le j}} (g_{i_1,\cdot}\cdots g_{i_{j+1},\cdot})}{\tilde{g}_0^{j+1}} - \cdots\right.$$

$$\left. +(-1)^j\frac{\sum\limits_{\substack{i_1+i_2=l+j \\ if\ \ i_s\le\frac{k-1}{2}\ \ then\ \ i_s\le j}} (g_{i_1,\cdot}g_{i_2,\cdot})}{\tilde{g}_0^2}\right) = \tag{95}$$

$$= \sum_{l=1}^{k-1} b_{n-1+2i(k-1)-2l,n}m_{j,l,\cdot} \tag{96}$$

In the previous expression, for simplicity, we used $g_{m,\cdot}$ instead of $g_{m,\nu}$, and we denoted by $\tilde{g}_0^m$ a product of m factors, in which every factors are in the form: $\tilde{g}_{0,\nu}$. It was useful, because in the following calculations we will omit the second indices, that is we will use the limits of the coefficients. This is correct because of (86) and because we assumed that $n > n_0$, which means that the coefficients of the recurrence formula are depend on k and the error term is in order of $\nu^{\frac{1}{k}-1}$.

Now we will give an estimation on the dependence of the Fourier coefficients in one group on the coefficients in the previous group, when $1 \le k \le 5$.

Let us denote by

$$M_j := \sum_{l=1}^{k-1} |m_{j,l}| \quad j = 0, \ldots, k-2.$$

From (92) (with g_m-s) we can compute

$k=2$	$g_0 = 3$	$g_1 = 1$			
	$M_0 = \frac{1}{3}$				
$k=3$	$g_0 = 10$	$g_1 = 5$	$g_2 = 1$		
	$M_0 = \frac{3}{5}$	$M_1 = \frac{1}{5}$			
$k=4$	$g_0 = 45$	$g_1 = 17$	$g_2 = 7$	$g_3 = 1$	
	$M_0 = \frac{5}{9}$	$M_1 = \frac{13}{225}$	$M_2 = \frac{6763}{91125}$		
$k=5$	$g_0 = 144$	$g_1 = 84$	$9_2 = 36$	$g_3 = 9$	$g_4 = 1$
	$M_0 = \frac{65}{72}$	$M_1 = \frac{179}{864}$	$M_2 = \frac{8087}{10368}$	$M_3 = \frac{30019}{124416}$	

In the enumerated cases we got that

$$|b_{n-1+2i(k-1)+2j}| \leq \max_{l=1}^{k-1} |b_{n-1+2i(k-1)-2l}| M_j \leq \delta \max_{l=1}^{k-1} |b_{n-1+2i(k-1)-2l}| \quad (97)$$

where

$$\delta = \max_{j=0}^{k-2} M_j \in (0,1)$$

and it depends only on k. By iteration we get that

$$|b_{n-1+2i(k-1)+2j}| \leq \delta^i \max_{l=1}^{k-1} |b_{n-1+2(k-1)-2l}| \leq c|b_{n-1}|\delta^i \quad (98)$$

The last inequality follows from the recurrence formula: if $l < k-1$

$$b_{n+1} = -\left(\frac{g_1}{g_0} + \varepsilon_n\right) b_{n-1}, \quad b_{n+3} = b_{n-1}\left(\frac{g_1^2}{g_0^2} - \frac{g_2}{g_0} + \varepsilon_n\right), \ldots,$$

$$b_{n-1+2l} = b_{n-1}\left(\frac{g_1^l}{g_0^l} - \frac{\sum_{i_1+\ldots+i_{l-1}=l}(g_{i_1}\cdots g_{i_{l-1}})}{g_0^{l-1}}\right.$$
$$\left. + \frac{\sum_{i_1+\ldots+i_{l-2}=l}(g_{i_1}\cdots g_{i_{l-2}})}{g_0^{l-2}} - \cdots + (-1)^l \frac{g_l}{g_0} + \varepsilon_n\right),$$

where

$$\varepsilon_n = O\left(n^{\frac{1}{k}-1}\right).$$

According to [15] $\varrho_n \sim a_n$ and the lemma is proved.

Lemma 8 *With the assumptions of Lemma 7, there exist absolute constants c d; and $\delta_1 \in (0,1)$, such that*

$$\int_{-\infty}^{x} q_n w^2 = P_{cn}(x) w^2(x) + M_{dn}(x), \quad (99)$$

where

$$\|M_{dn}\| = O\left(\delta_1^n\right), \quad \|M_{dn}^{'}\| = O\left(\delta_1^n\right) \quad (100)$$

and P_{cn} is a polynomial with degree cn.

Proof: According to Lemma 6 and 7,

$$\int_{-\infty}^{x} q_n w^2 = \sum_{i=0}^{n+2k} c_i^* \int_{-\infty}^{x} f_i w^2$$

$$+\sum_{l=0}^{2k} c_l^* \sum_{i=0}^{\infty}\sum_{j=0}^{k-2} b_{n-l-1+2i(k-1)+2j,l}\, p_{n-l-1+2i(k-1)+2j} w^2$$

$$= \sum_{i=0}^{cn} c_i^{**} p_i w^2 + \sum_{l=0}^{2k} c_l^* \sum_{i=dn}^{\infty}\sum_{j=0}^{k-2} b_{n-l-1+2i(k-1)+2j,l}\, p_{n-l-1+2i(k-1)+2j} w^2$$

$$= P_{cn} w^2 + M_{dn}.$$

Now we can estimate the remainder term:

$$|M_{dn}| \le C(k) K^n \sum_{i=d_1 n}^{\infty} \|p_{n_1+ci} w^2\| \delta^i \le C(k) n^{\beta} K^n \left(\delta^d\right)^n \tag{101}$$

and because of the uniform convergence of the sum M_{dn}, we can differentiate it term by term and we get that

$$|M_{dn}^{'}| \le C(k) K^n \sum_{i=d_1 n}^{\infty} \|(p_{n_1+ci} w^2)^{'}\| \delta^i \le C(k) n^{\gamma} K^n \left(\delta^d\right)^n \tag{102}$$

Here β and γ are positive constants depending on the weight function (see [11] Cor 1.4. and [12] Theorems 1.1. and 1.3.), and n_1, d_1 are different from n and d by suitable constants. So if we choose d enough large, then $|M_{dn}|$ and $|M_{dn}^{'}|$ are exponentially small, that is we can find a $\delta_1 \in (0,1)$ as the lemma states.

Proof of Theorem 2.: Now we have to estimate the difference of the integral of a polynomial in question, and it's Hermite interpolatory polynomial in the weighted space.

$$\left| w^2(x) \left(\frac{\int_{-\infty}^{x} q_n w^2}{w^2(x)} - H_{cn}\left(\frac{\int_{-\infty}^{(\cdot)} q_n w^2}{w^2}, \left(\frac{\int_{-\infty}^{(\cdot)} q_n w^2}{w^2} \right)^{'}, x \right) \right) \right|$$

$$\le \left| w^2(x) \left(P_{cn} - H_{cn}\left(P_{cn}, (P_{cn})^{'}, x \right) \right) \right|$$

$$+ \left| w^2(x) \left(\frac{M_{dn}}{w^2(x)} - H_{cn}\left(\frac{M_{dn}}{w^2}, \left(\frac{M_{dn}}{w^2} \right)^{'}, x \right) \right) \right|$$

$$\le 0 + |M_{dn}| + |K_{cn}||M_{dn}| + |\hat{K}_{cn}||M_{dn}^{'}| = O\left(n^{\beta_0} \delta_1^n \right),$$

and multiplied it by an n^{α} the result tends to zero in norm.

Remark:(1)By the modification of the multiplier of n, it is enough to suppose on X that $\|K_n\|, \|\hat{K}_n\| = O\left(c^n\right)$, but we needed only this weekwr form.

(2) In the proof of Lemma 7 this rough estimation with M_j-s will not work for greater k-s. It means, that if $k > 10$, say, than it is not enough to compute with this maximum value, but by the same chain of ideas , with finer estimations one can get a more general theorem on w_k-s.

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