ON INHERITED PROPERTIES AND JULIA SETS OF EXCEPTIONAL JACOBI POLYNOMIALS

Á. P. HORVÁTH

Abstract. Some inherited properties of exceptional Jacobi polynomials are derived and as an application it is shown that similarly to the standard case, the equilibrium measure of Julia sets of exceptional Jacobi polynomials tends to the equilibrium measure of the interval of orthogonality in weak-star sense.

1. Introduction

Notion of exceptional orthogonal polynomials is motivated by problems in quantum mechanics, see e.g. [7]. It has a rather extended literature, see eg. [8], [1], [4] and the references therein. Exceptional orthogonal polynomials are complete systems of polynomials with respect to a positive measure, but the exceptional families have finite codimension in the space of polynomials, cf. [8, Definition 7.4] or [4, Definition 6.1]. Similarly to the classical ones exceptional polynomials are eigenfunctions of Sturm-Liouville-type differential operators but unlike the classical cases, the coefficients of these operators are rational functions. An elegant way of constructing exceptional and general Jacobi polynomials the Wronskian method via (two) partitions, cf. e.g. [4], [1]. Some properties, for instance the behavior of zeros are derived in rather general circumstances, see [1, Theorems 6.5 and 6.6]. Exceptional orthogonal polynomials also possess a Bochner-type characterization as each family can be derived from one of the classical families applying finitely many Darboux transformations, see [8]. Taking into account this characterization, below we investigate some properties which are inherited to exceptional Jacobi polynomials from the classical ones. Besides new results, we give some simple proofs to known ones in exceptional orthogonal Jacobi case.

As an application of the properties proved in the second section, in the third one we give the weak-star limit of the equilibrium measure of Julia sets of exceptional Jacobi polynomials. In the standard case, the sequence of equilibrium measures of Julia sets of polynomials orthonormal to a (probability) measure supported on a compact set (of positive capacity) on the complex plane belongs to the family of measure-sequences which tends to the equilibrium measure of set of orthogonality. The other members of this family are for instance, the normalized counting measure based on the zeros of the orthogonal polynomials in question, see e.g. [14] and the references therein, the weighted reciprocal of the Christoffel functions as the sequence of densities, see e.g. [16], [10], the normalized counting measure based on the eigenvalues of the truncated multiplication operator, see e.g. [22], [20] and the normalized counting measure based on the zeros of the average characteristic

2020 *Mathematics Subject Classification.* 42C05, 37F10.

Key words and phrases. Julia set, equilibrium measure, exceptional Jacobi polynomials.

Supported by the NKFIH-OTKA Grants K128922 and K132097.

polynomials, see e.g. [23], [10], [11]. Similar theorems can be derived to exceptional Jacobi polynomials, see e.g. [1], [13]. The result of the third section fits into this family.

2. Some inherited properties of exceptional Jacobi polynomials

We use the Bochner-type characterization of exceptional polynomials given in [8]. According to [8, Theorem 1.2] each system of exceptional (Jacobi) polynomials can be obtained by applying a finite sequence of Darboux transformations to the classical Jacobi system. Below we derive some properties which are inherited after this procedure. Since for the proofs we need the construction, we summarize it in brief, cf. [8].

2.1. **Recursive construction.** Classical Jacobi polynomials $\{p_k^{(\alpha,\beta)}\}_{k=0}^\infty$ are orthonormal on $I=[-1,1]$ with respect to the weight function $W_0(x)=(1-x)^\alpha(1+x)^\beta$, where $\alpha,\beta>-1$ and they are eigenfunctions of the second order linear differential operator with polynomial coefficients

$$T_0[y]=py''+q_0y'+r_0y, \tag{1}$$

with eigenvalues λ_n, where $p(x)=1-x^2$, $q_0(x)=\beta-\alpha-(\alpha+\beta+2)x$, $r_0=0$, $\lambda_n=-n(n+\alpha+\beta+1)$, see [23, (4.2.1)].

Exceptional (Jacobi) polynomials can be obtained from the classical ones by application of finitely many Darboux transformations to the differential operator T_0 that is applying finitely many appropriate first order differential operators to the classical (Jacobi) polynomials, cf. [8, Theorem 1.2]. This procedure is as follows.

$$T_{k-1}=B_kA_k+\tilde\lambda_k, \quad T_k=A_kB_k+\tilde\lambda_k, \tag{2}$$

where

$$A_k[y]=b_k(y'-w_ky) \quad B_k[y]=\hat b_k(y'-\hat w_ky), \tag{3}$$

i.e. the decomposition of the differential operator of the second order, T_{k-1} leads to the definition of T_k, where

$$T_k[y]=py''+q_ky'+r_ky. \tag{4}$$

w_k is the logarithmic derivative of a quasi-rational eigenfunction of T_{k-1} with eigenvalue $\tilde\lambda_k$, that is w_k is rational and fulfils the Riccati equation

$$p(w_k'+w_k^2)+q_{k-1}w_k+r_{k-1}=\tilde\lambda_k. \tag{5}$$

b_k is a suitable rational function and

$$\hat b_k=\frac{p}{b_k}, \qquad \hat w_k=-w_k-\frac{q_{k-1}}{p}+\frac{b_k'}{b_k}. \tag{6}$$

The coefficients of T_k fulfil

$$q_k=q_{k-1}+p'-2\frac{b_k'}{b_k}p=q_0+kp'-2p\sum_{i=1}^k\frac{b_i'}{b_i}. \tag{7}$$

$$r_k=r_{k-1}+q_{k-1}'+w_kp'-\frac{b_k'}{b_k}(q_{k-1}+p')+\left(2\left(\frac{b_k'}{b_k}\right)^2-\frac{b_k''}{b_k}+2w_k'\right)p \tag{8}$$

$$= \sum_{i=0}^{k-1} q_i' + p' \left(\sum_{i=1}^{k} w_i - \sum_{i=1}^{k} \frac{b_i'}{b_i} \right) - \sum_{i=1}^{k} q_{i-1}' \frac{b_i'}{b_i} + p \left(2 \sum_{i=1}^{k} \left(\frac{b_i'}{b_i} \right)^2 - \sum_{i=1}^{k} \frac{b_i''}{b_i} + 2 \sum_{i=1}^{k} w_i' \right).$$

Fixing $\alpha, \beta > -1$ and denoting by

$$\hat{P}_n^{[0]} = p_n^{(\alpha,\beta)},$$

(2) implies that

$$P_n^{[k]} = A_k \hat{P}_n^{[k-1]} \quad \text{and} \quad T_k P_n^{[k]} = \lambda_n P_n^{[k]}, \tag{9}$$

where

$$\hat{P}_n^{[k-1]} = \frac{P_n^{[k-1]}}{\sigma_n^{k-1}}, \tag{10}$$

that is $\{\hat{P}_n^{[k]}\}_{n=0}^{\infty}$ is orthonormal on $(-1,1)$ with respect to the weight function

$$W_k = c \frac{p W_{k-1}}{b_k^2} = c \frac{p^k W_0}{\prod_{i=1}^{k} b_i^2}, \tag{11}$$

and $\sigma_n^k = \|P_n^{[k]}\|_{2,W_k}$. Since W_k has to possess finite moments (cf. [8, Definition 7.4]), $b_i > 0$ on $(-1,1)$ $(i = 1, \dots, k)$, say. Thus w_i has no poles in $(-1,1)$. Actually in view of [1, Table 1], b_i are polynomials, they are positive on $(-1,1)$ and have at most simple zeros at $x = \pm 1$ and $(b_i w_i)(\pm 1) \neq 0$. Thus we denote by

$$\frac{p}{b_i} =: \frac{\tilde{p}_i}{\tilde{b}_i},$$

where $\tilde{b}_i \neq 0$ on $[-1,1]$. More precisely, $b_i(x) = (1-x)^{\frac{1-\varepsilon_{i1}}{2}} (1+x)^{\frac{1-\varepsilon_{i2}}{2}} \tilde{b}_i(x)$, where $\varepsilon_{ij} = \pm 1$, $j = 1, 2$ and so $W_k = c \frac{(1-x)^{\alpha + \sum_{i=1}^{k} \varepsilon_{i1}} (1+x)^{\beta + \sum_{i=1}^{k} \varepsilon_{i2}}}{\prod_{i=1}^{k} \tilde{b}_i^2}$, that is W_k has finite moments if $\alpha + \sum_{i=1}^{k} \varepsilon_{i1}, \beta + \sum_{i=1}^{k} \varepsilon_{i2} > -1$.
Let $n_i = \deg b_i$, $m_i := \deg \tilde{b}_i$, $M_k := \sum_{i=1}^{k} m_i$. Let $\hat{B}_k := \prod_{i=1}^{k} b_i$, $\tilde{B}_k := \prod_{i=1}^{k} \tilde{b}_i$. The degree of $\tilde{B}_k$, M_k, gives the number of gaps in the sequence of degrees of the corresponding exceptional system. For sake of the third section we assume that $\hat{B}_k$ is monic and we denote by Z_k the set of zeros of $\tilde{B}_k$, furthermore we choose c such that W_k be a probability measure.

2.2. **Inherited properties of $\hat{P}_n^{[k]}$.** Here we concentrate mainly that properties which are necessary to prove the result of the next section. We assume that $\{\hat{P}_n^{[k]}\}_{n=0}^{\infty}$ is derived from classical Jacobi polynomials with finitely many Darboux transformations as it is given above. According to [8, Theorem 1.2] a Sturm-Liouville orthogonal polynomial system, see [8, Definition 7.4] can be obtained this way.

The first property is connected to [1, section 6.5].

Property 1. *$\hat{P}_n^{[k]}$ has simple zeros in $(-1,1)$, and $\hat{P}_n^{[k]}(\pm 1) \neq 0$ for all $n, k \geq 0$.*

Proof. Since $\hat{P}_n^{[0]}$ has n simple zeros in $(-1,1)$, we can prove by induction. Assume indirectly that $\hat{P}_n^{[k-1]}$ has simple zeros in $(-1,1)$ and there is an $x_0 \in (-1,1)$, such that $\hat{P}_n^{[k]}(x_0) = \left(\hat{P}_n^{[k]}\right)'(x_0) = 0$. Then by (2) and (9)

$$B_k P_n^{[k]}(x_0) = (\lambda_n - \tilde{\lambda}_k) \hat{P}_n^{[k-1]}(x_0). \tag{12}$$

Since in view of (3) and (6) $B_kP_n^{[k]} = \frac{p}{b_k}\left(\left(P_n^{[k]}\right)' + \left(w_k + \frac{q_{k-1}}{p} - \frac{b_k'}{b_k}\right)P_n^{[k]}\right)$, by the assumption and by (12), $\hat{P}_n^{[k-1]}(x_0) = 0$. According to (9) and (3)

$$b_k(x_0)\left(\hat{P}_n^{[k-1]}\right)'(x_0) = (b_kw_k)(x_0)\hat{P}_n^{[k-1]}(x_0) + P_n^{[k]}(x_0).$$

Because $b_k(x_0) > 0$, $\left(P_n^{[k-1]}\right)'(x_0) = 0$, that is $P_n^{[k-1]}$ has a double zero at x_0, which contradicts to the assumption.

We deal with the right end-point, say. $p_n^{(\alpha,\beta)}(1) \neq 0$. Assume that $\hat{P}_n^{[k-1]}(1) \neq 0$. If $b_k(1) = 0$, then recalling that $(b_kw_k)(1) \neq 0$, $P_n^{[k]}(1) = (A_k\hat{P}_n^{[k-1]})(1) = 0 - (b_kw_k)(1)\hat{P}_n^{[k-1]}(1) \neq 0$.
If $b_k(1) \neq 0$, assume that $P_n^{[k]}(1) = 0$. As above, considering that pw_k has no pole in 1, we compute
$(B_kP_n^{[k]})(1) = \frac{1}{b_k(1)}\left(p(1)\left(P_n^{[k]}\right)'(1) + \left((pw_k)(1) + q_{k-1}(1) - \frac{pb_k'}{b_k}(1)\right)P_n^{[k]}(1)\right) = 0 = (\lambda_n - \tilde{\lambda}_k)P_n^{[k-1]}(1)$, which is contradiction.

Below we need the next assumption.

$$(w_kpW_{k-1})(\pm 1) = 0, \quad k \geq 1. \tag{13}$$

Since $b_k > 0$ on $(-1,1)$ and has at most simple zeros at $x = \pm 1$ and b_kw_k has no poles in $[-1,1]$, thus (13) fulfils if $\alpha + \frac{\varepsilon_{k1}+1}{2} + \sum_{i=1}^{k-1}\varepsilon_{i1}, \beta + \frac{\varepsilon_{k2}+1}{2} + \sum_{i=1}^{k-1}\varepsilon_{i2} > 0$.
Recalling the notation (10) we have

Property 2. *Supposing* (13),

$$\sigma_n^k = \sqrt{-\lambda_n + C(k)} \sim n,$$

where $C(k)$ is a constant which depends on k.

Proof. $\hat{P}_n^{[0]}$ fulfils this property, see [23, (4.3.4)]. By (9) and (5)

$$\left(\sigma_n^{k+1}\right)^2 = \int_I \left(\left(\hat{P}_n^{[k]}(x)\right)'\right)^2 p(x)W_k(x) - 2\left(\hat{P}_n^{[k]}(x)\right)'\hat{P}_n^{[k]}(x)w_{k+1}(x)p(x)W_k(x)dx$$
$$+ \int_I (\tilde{\lambda}_{k+1} - q_k(x)w_{k+1}(x) - p(x)w_{k+1}'(x))\left(\hat{P}_n^{[k]}(x)\right)^2 W_k(x)dx.$$

Considering (7) and (11)

$$(W_kp)' = W_kq_k. \tag{14}$$

So

$$\left(\sigma_n^{k+1}\right)^2 = \int_I \left(\left(\hat{P}_n^{[k]}(x)\right)'\right)^2 p(x)W_k(x)dx$$
$$- \int_I \left(\left(\hat{P}_n^{[k]}(x)\right)^2 w_{k+1}(x)p(x)W_k(x)\right)'dx + \int_I \tilde{\lambda}_{k+1}\left(\hat{P}_n^{[k]}(x)\right)^2 W_k(x)dx$$
$$= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.$$

By orthonormality $\mathcal{I}_3 = \tilde{\lambda}_{k+1}$ which is independent of n. $\mathcal{I}_2 = 0$, by (13).
Integrating by parts and then considering (9), (4) and (14) we have

$$\mathcal{I}_1 = -\int_I \hat{P}_n^{[k]}(x)\left(\left(\hat{P}_n^{[k]}(x)\right)'p(x)W_k(x)\right)'dx$$

$$= \int_I (r_k(x) - \lambda_n) \left(\hat{P}_n^{[k]}(x) \right)^2 W_k(x) dx = -\lambda_n + \mathcal{I}_4.$$

Observe, that $\|r_k\|_{\infty,I} = A(k)$, where $A(k)$ is a constant depends on k but independent of n and $\|\cdot\|_{\infty,I}$ means the infinite norm on I. In view of (8) r_k has no poles in $(-1,1)$. Although according to (8) it can have a pole of order one at $x = \pm 1$, but considering (7), q_k has no poles at $x = \pm 1$, and since $\hat{P}_n^{[k]}(\pm 1) \neq 0$, by (9) $r_k(\pm 1)$ must be finite. Thus, by orthonormality

$$|\mathcal{I}_4| \leq A(k).$$

Since $|\lambda_n| \sim n^2$, the proof is finished.

Location of zeros of exceptional Jacobi polynomials were investigated by several authors, see e.g. [9], [12]. The next property is proved in a different way in [1, Theorem 6.6].

Property 3. *Supposing* (13) *if n is large enough, $\hat{P}_n^{[k]}$ has M_k exceptional zeros (with multiplicity), that is M_k zeros out of the interval of orthogonality. Moreover the exceptional zeros tend to the zeros of $\tilde{B}_k$, when n tends to infinity.*

Proof. Recalling that $\hat{P}_n^{[0]} = p_n^{(\alpha,\beta)}$, for classical Jacobi polynomials we have

$$p\left(\hat{P}_n^{[0]}\right)' = D_n \hat{P}_{n+1}^{[0]} + E_n \hat{P}_n^{[0]} + F_n \hat{P}_{n-1}^{[0]},$$

where $\lim_{n\to\infty} \frac{D_n}{n} = \frac{1}{2}$, $\lim_{n\to\infty} E_n = \frac{\alpha-\beta}{2}$, $\lim_{n\to\infty} \frac{F_n}{n} = -\frac{1}{2}$, see [23, (4.5.5)], and

$$\lim_{n\to\infty} \frac{\hat{P}_{n-1}^{[0]}(z)}{\hat{P}_n^{[0]}(z)} = z - \sqrt{z^2-1}, \tag{15}$$

where the convergence is locally uniform on $\mathbb{C} \setminus [-1,1]$, see e.g. [17]. (Here and below we take that branch of the square root, which takes positive numbers to positive.) Thus

$$\frac{\hat{P}_n^{[1]}}{\hat{P}_n^{[0]}} = \frac{1}{\sigma_n^1} \left(\frac{pb_1 \left(\hat{P}_n^{[0]}\right)'}{p\hat{P}_n^{[0]}} - b_1 w_1 \right)$$

$$= \frac{n}{\sigma_n^1} \left(\frac{b_1}{p} \left(\frac{D_n}{n} \frac{\hat{P}_{n+1}^{[0]}}{\hat{P}_n^{[0]}} + \frac{E_n}{n} + \frac{F_n}{n} \frac{\hat{P}_{n-1}^{[0]}}{\hat{P}_n^{[0]}} \right) - \frac{b_1 w_1}{n} \right),$$

that is in view of (15) and Property 2

$$\lim_{n\to\infty} \frac{\hat{P}_n^{[1]}(z)}{\hat{P}_n^{[0]}(z)} = \lim_{n\to\infty} \frac{b_1(z)\left(\hat{P}_n^{[0]}(z)\right)'}{\sigma_n^1 \hat{P}_n^{[0]}(z)} = -\frac{b_1(z)}{\sqrt{z^2-1}},$$

locally uniformly on $\mathbb{C} \setminus [-1,1]$. We continue by induction.

$$\frac{\hat{P}_n^{[k]}}{\hat{P}_n^{[k-1]}} = \frac{b_k \left(\hat{P}_n^{[k-1]}\right)'}{\sigma_n^k \hat{P}_n^{[k-1]}} - \frac{b_k w_k}{\sigma_n^k} = \mathcal{Q}_1 + \mathcal{Q}_2,$$

where $\mathcal{Q}_2 \to 0$ locally uniformly on $\mathbb{C}$. As $P_n^{[k-1]} = A_{k-1} \hat{P}_n^{[k-2]}$,

$$\mathcal{Q}_1 = \frac{b_k b_{k-1} p \left(\hat{P}_n^{[k-2]}\right)'' + pb_k \left(\hat{P}_n^{[k-2]}\right)' (b_{k-1}' - b_{k-1}w_{k-1}) - pb_k(b_{k-1}w_{k-1})' \hat{P}_n^{[k-2]}}{\sigma_n^k \sigma_n^{k-1} p \hat{P}_n^{[k-1]}}.$$

Expressing $\left(\hat{P}_n^{[k-2]}\right)''$ from the differential equation we have

$$\mathcal{Q}_1=\frac{b_k\left(\hat{P}_n^{[k-2]}\right)'(pb_{k-1}'-pb_{k-1}w_{k-1}-b_{k-1}q_{k-2})}{\sigma_n^k\sigma_n^{k-1}p\hat{P}_n^{[k-1]}}$$

$$+\frac{\hat{P}_n^{[k-2]}b_k(-b_{k-1}r_{k-2}+\lambda_nb_{k-1}-p(b_{k-1}w_{k-1})')}{\sigma_n^k\sigma_n^{k-1}p\hat{P}_n^{[k-1]}}=\mathcal{Q}_3+\mathcal{Q}_4.$$

We have

$$\mathcal{Q}_3=\frac{\frac{\left(\hat{P}_n^{[k-2]}\right)'}{\sigma_n^{k-1}\hat{P}_n^{[k-2]}}}{p\frac{\hat{P}_n^{[k-1]}}{\hat{P}_n^{[k-2]}}}\frac{b_k(pb_{k-1}'-pb_{k-1}w_{k-1}-b_{k-1}q_{k-2})}{\sigma_n^k}=R_n^kQ_n^k,$$

and

$$\mathcal{Q}_4=\frac{b_k(-b_{k-1}r_{k-2}+\lambda_nb_{k-1}-p(b_{k-1}w_{k-1})')}{\sigma_n^k\sigma_n^{k-1}}\frac{1}{p\frac{\hat{P}_n^{[k-1]}}{\hat{P}_n^{[k-2]}}}=V_n^kS_n^k.$$

Now we assume that $\lim_{n\to\infty}\frac{\hat{P}_n^{[k-1]}(z)}{\hat{P}_n^{[k-2]}(z)}=\lim_{n\to\infty}\frac{b_{k-1}(z)\left(\hat{P}_n^{[k-2]}(z)\right)'}{\sigma_n^{k-1}P_n^{[k-2]}(z)}=-\frac{b_{k-1}(z)}{\sqrt{z^2-1}}$ locally uniformly on $\mathbb{C}\setminus[-1,1]\setminus Z_{k-2}$. Then $R_n^k(z)\to\frac{-(z^2-1)^{-\frac{1}{2}}}{b_{k-1}(z^2-1)^{\frac{1}{2}}}$ on $\mathbb{C}\setminus[-1,1]\setminus Z_{k-1}$ and $Q_n^k(z)\to 0$ on $\mathbb{C}$ locally uniformly, $\mathcal{Q}_3$ tends to zero on $\mathbb{C}\setminus[-1,1]\setminus Z_{k-1}$ locally uniformly.
Since $\lim_{n\to\infty}\frac{-\lambda_n}{\sigma_n^k\sigma_n^{k-1}}=1$, V_n^k tends to $-b_kb_{k-1}$ on the compacts of $\mathbb{C}$, and as above $S_n^k(z)$ tends to $\frac{1}{b_{k-1}(z)\sqrt{z^2-1}}$ on $\mathbb{C}\setminus[-1,1]\setminus Z_{k-1}$ locally uniformly. Thus

$$\lim_{n\to\infty}\mathcal{Q}_4(z)=\lim_{n\to\infty}\frac{\hat{P}_n^{[k]}(z)}{\hat{P}_n^{[k-1]}(z)}=-\frac{b_k(z)}{\sqrt{z^2-1}}$$

locally uniformly on $\mathbb{C}\setminus[-1,1]\setminus Z_{k-1}$. Now take

$$\lim_{n\to\infty}\frac{\hat{P}_n^{[k]}}{\hat{P}_n^{[0]}}(z)=\lim_{n\to\infty}\frac{\hat{P}_n^{[k]}}{\hat{P}_n^{[k-1]}}\frac{\hat{P}_n^{[k-1]}}{\hat{P}_n^{[k-2]}}\dots\frac{\hat{P}_n^{[1]}}{\hat{P}_n^{[0]}}(z)=\frac{(-1)^k\hat{B}_k(z)}{(\sqrt{z^2-1})^k}$$

locally uniformly on $\mathbb{C}\setminus[-1,1]\setminus Z_{k-1}$. Both the left-hand and right-hand sides are regular on $\mathbb{C}\setminus[-1,1]$, so we can extend the domain of convergence there. Finally, considering that the zeros of $\hat{B}_k$ and $\tilde{B}_k$ coincide on $\mathbb{C}\setminus[-1,1]$ and applying Hurwitz's theorem the statement is proved.

Property 4. *Denote by* γ_n^k *the leading coefficient of* $\hat{P}_n^{[k]}$. *Supposing* (13), *we have*

$$\lim_{n\to\infty}(\gamma_n^k)^{\frac{1}{n}}=2. \tag{16}$$

Proof. Let γ_n^0 be the leading coefficient of $\hat{P}_n^{[0]}$. (For simplicity assume, that γ_n^0 and γ_n^k are positive.) It is known that $\lim_{n\to\infty}(\gamma_n^0)^{\frac{1}{n}}=2$, see e.g. [23, (4.21.6), (4.3.4)]. Notice that b_i and b_iw_i are polynomials, cf. (9). Denote the leading coefficient of b_{k+1} and $b_{k+1}w_{k+1}$ by c_{k+1} and d_{k+1}, respectively. If n is large enough, the degree of $\hat{P}_n^{[k]}$ is $n+m_k$, where m_k is independent of n. Recalling that $\sigma_n^{k+1}\hat{P}_n^{[k+1]} = b_{k+1}\left(\hat{P}_n^{[k]}\right)' - b_{k+1}w_{k+1}\hat{P}_n^{[k]}$, we have

$$\frac{\gamma_n^k}{\sigma_n^{k+1}}\min\left\{|c_{k+1}|(n+m_k), |d_{k+1}|\right\} \le \gamma_n^{k+1} \le \frac{\gamma_n^k}{\sigma_n^{k+1}}\max\left\{|c_{k+1}|(n+m_k), |d_{k+1}|\right\}.$$

Thus the result can be derived from the corresponding result with respect to the classical Jacobi polynomials, by induction.

The next property is proved in X_m-case in [9] and in general (with a different proof) in [1, Theorem 6.5]. By the construction we have that the degree of $\hat{P}_n^{[k]}$, $N(n)$, is greater than n. In view of Property 3 (if n is large enough), $\hat{P}_n^{[k]}$ has $N(n) > n - M_k$ regular zeros.

Property 5. *Let $x_{1n}^k, \dots, x_{N(n)n}^k$ be the regular zeros of the exceptional Jacobi polynomials, $\hat{P}_n^{[k]}$. Let $x_{in} = \cos\varphi_{in}$. Supposing (13), for every $[\gamma,\delta] \subset [0,\pi]$*

$$\left|\frac{1}{n}\sum_{\substack{i\\ \gamma\le\varphi_{in}\le\delta}} 1 - \frac{\delta-\gamma}{\pi}\right| \le C_k\sqrt{\frac{\log n}{n}},$$

where C_k is a constant, depends on α; β; k; b_i, b_iw_i, $i=1,\dots,k$ but is independent of n.

The basis of the proof is the next lemma.

Lemma 1. [6] *Let $1 \ge \zeta_{1,n} > \dots > \zeta_{n,n} \ge -1$, $n \in \mathbb{N}_+$ any system of points, and let $\eta_{i,n} \in [0,\pi]$ be defined by $\zeta_{i,n} = \cos\eta_{i,n}$. Let $\omega_n(\zeta) = \prod_{i=1}^n(\zeta - \zeta_{i,n})$. If for all $\zeta \in [-1,1]$*

$$|\omega_n(\zeta)| \le \frac{A(n)}{2^n}$$

holds, then for every subinterval $[\gamma,\delta] \subset [0,\pi]$ we have

$$\left|\sum_{\substack{i\\ \gamma\le\eta_{in}\le\delta}} 1 - \frac{\delta-\gamma}{\pi}n\right| < \frac{8}{\log 3}\sqrt{n\log A(n)}. \tag{17}$$

Proof. $\|\hat{P}_n^{[0]}\|_{\infty,I} \le C_0 n^{q-1}$, see [23, (7.32.2) and (4.3.4)], where $q = \max\{\alpha,\beta\}$. Assume that $\|\hat{P}_n^{[k-1]}\|_{\infty,I} \le c_{k-1}n^{k-1+q-1}$, where, recalling that b_i and b_iw_i are polynomials,
$c_{k-1} := C_0\prod_{i=1}^{k-1}\max\{\|b_i\|_{\infty,I}, \|b_iw_i\|_{\infty,I}\}$. In view of (9), Markov's inequality and Property 2,
$\|\hat{P}_n^{[k]}\|_{\infty,I} \le c_{k-1}\max\{\|b_k\|_{\infty,I}, \|b_kw_k\|_{\infty,I}\}n^{k-1+q-1+2-1} = c_kn^{k+q-1}$. Now let us decompose

$$\hat{P}_n^{[k]} = \gamma_n^k e_n^k r_n^k,$$

where e_n^k and r_n^k are monic polynomials, the zeros of e_n^k are the exceptional zeros of $\hat{P}_n^{[k]}$ and the zeros of r_n^k are the regular ones. According to Property 3 there is an $\epsilon_k > 0$ such that for all $x \in [-1,1]$, $|e_n^k(x)| > \epsilon_k$ and by Property 4, $\gamma_n^k > 2^{n-1}$ if n is large enough. Thus on $[-1,1]$

$$|r_n^k| = \frac{|\hat{P}_n^{[k]}|}{\gamma_n^k|e_n^k|} \le \frac{2c_kn^{k+q-1}}{\epsilon_k 2^n} =: \frac{\hat{c}_kA_n}{2^n}.$$

Thus the previous lemma implies the result with $A(n)=A(k,n)=\hat{c}_kA_n$.

Besides the normalized zero-counting measure the Christoffel function measure tends to the equilibrium measure as well.

Property 6. *Assuming* (13) *for all* $k\in\mathbb{N}$

$$\nu_n^{[k]}\to\mu_e \tag{18}$$

in weak-star sense, where

$$d\nu_n^{[k]}(x)=\frac{1}{n}\sum_{l=0}^{n-1}\left(\hat{P}_l^{[k]}\right)^2(x)W_k(x)dx$$

and μ_e *is the equilibrium measure of* $[-1,1]$.

Proof. Since the twice continuously differentiable functions are dense in $C[-1,1]$ it is enough to show that for any $f\in C^2[-1,1]$, $\nu_n^{[k]}(f)$ tends to $\mu_e(f)$. The result is known for $\{\hat{P}_l^{[0]}\}_{l=0}^\infty$, see e.g. [16]. Let $k\geq 1$. Recalling that $\hat{P}_l^{[k]}=\frac{1}{\sigma_l^k}A_k\hat{P}_l^{[k-1]}$

$$\int_I f(x)d\nu_n^{[k]}(x)=\frac{1}{n}\sum_{l=0}^{n-1}\int_I f(x)\left(\hat{P}_l^{[k]}(x)\right)^2W_k(x)dx=\frac{1}{n}\sum_{l=0}^{n-1}\mathfrak{I}_l^{[k]}.$$

$$\begin{aligned}\mathfrak{I}_l^{[k]}=\frac{1}{\left(\sigma_l^k\right)^2}&\left(\int_I f(x)b_k^2(x)\left(\left(\hat{P}_l^{[k-1]}(x)\right)'\right)^2W_k(x)dx\right.\\&+\int_I f(x)b_k^2(x)w_k^2(x)\left(\hat{P}_l^{[k-1]}(x)\right)^2W_k(x)dx\end{aligned}$$
$$\left.-\int_I f(x)2b_k^2(x)w_k(x)\left(\hat{P}_l^{[k-1]}(x)\right)'\hat{P}_l^{[k-1]}(x)W_k(x)dx\right)=\frac{1}{\left(\sigma_l^k\right)^2}(\mathfrak{I}_{l,1}^{[k]}+\mathfrak{I}_{l,2}^{[k]}-\mathfrak{I}_{l,3}^{[k]}).$$

In view of (11)

$$\mathfrak{I}_{l,2}^{[k]}=\int_I f(x)\left(\hat{P}_l^{[k-1]}(x)\right)^2w_k^2(x)p(x)W_{k-1}(x)dx.$$

By assumption (13) $pW_{k-1}(\pm1)=0$, thus taking into consideration (14) and (4)

$$\begin{aligned}\mathfrak{I}_{l,1}^{[k]}=-\int_I\hat{P}_l^{[k-1]}(x)&\left(\left(\hat{P}_l^{[k-1]}(x)\right)''p(x)f(x)W_{k-1}(x)\right.\\+\left(\hat{P}_l^{[k-1]}(x)\right)'f'(x)p(x)W_{k-1}(x)&+\left.\left(\hat{P}_l^{[k-1]}(x)\right)'f(x)q_{k-1}(x)W_{k-1}(x)\right)dx\\=-\int_I\hat{P}_l^{[k-1]}(x)&\left((\lambda_l-r_{k-1}(x))\hat{P}_l^{[k-1]}(x)f(x)W_{k-1}(x)\right.\\&\left.+\left(\hat{P}_l^{[k-1]}(x)\right)'f'(x)p(x)W_{k-1}(x)\right)dx.\end{aligned}$$

Thus

$$\begin{aligned}\left(\sigma_l^k\right)^2\mathfrak{I}_l^{[k]}&=\int_I f(x)\left(p(x)w_k^2(x)-\lambda_l+r_{k-1}(x)\right)\left(\hat{P}_l^{[k-1]}(x)\right)^2W_{k-1}(x)dx\\-\int_I&\left(\left(\hat{P}_l^{[k-1]}(x)\right)^2\right)'\left(\frac{1}{2}f'(x)+f(x)w_k(x)\right)p(x)W_{k-1}(x)dx=\mathfrak{I}_{l,4}^{[k]}-\mathfrak{I}_{l,5}^{[k]}.\end{aligned}$$

Again by (13)

$$-\mathcal{J}_{l,5}^{[k]} = \int_I \left(\hat{P}_l^{[k-1]}(x)\right)^2 \left(\frac{1}{2}f''(x) + f'(x)w_k(x)\right) p(x)W_{k-1}(x)dx$$

$$+\int_I \left(\hat{P}_l^{[k-1]}(x)\right)^2 \left(p(x)w_k'(x) + q_{k-1}(x)w_k(x)\right) f(x)W_{k-1}(x)dx.$$

So in view of (5)

$$\mathcal{J}_l^{[k]} = \frac{1}{\left(\sigma_l^k\right)^2}\left(\tilde{\lambda}_k - \lambda_l\right)\mathcal{J}_l^{[k-1]}$$

$$+\frac{1}{\left(\sigma_l^k\right)^2}\int_I \left(\hat{P}_l^{[k-1]}(x)\right)^2 \left(\frac{1}{2}f''(x) + f'(x)w_k(x)\right) p(x)W_{k-1}(x)dx.$$

Notice that $|\mathcal{J}_l^{[k]}| \le \|f\|_{\infty,I}$ and $|pw_k|$ is bounded on $[-1,1]$, thus the last integral is also bounded independently of l. Thus considering Property 2 we finish the proof with

$$\int_I f(x)d(\nu_n^{[k-1]} - \nu_n^{[k]})(x) \le c(k,f)\frac{1}{n}\sum_{l=0}^{n-1}\frac{1}{l^2+1}.$$

Let

$$\hat{\Pi}_n^k := \operatorname{span}\{\hat{P}_0^{[k]},\dots,\hat{P}_n^{[k]}\}.$$

The next observation relates to the recurrence relation (see [18, Theorem 1]) which is as follows. If s is a polynomial of degree L such that its derivative is divisible by $\tilde{B}_k$, then $s\hat{P}_n^{[k]} = \sum_{i=-L}^{L} a_{i,n}\hat{P}_{n+i}^{[k]}$, where $a_{i,n}$-s are constants. The fact that $\tilde{B}_k^2 P$ can be expressed as a linear combination of exceptional polynomials $\hat{P}_n^{[k]}$ is proved in [5, Lemma 1.1]. To our purpose a similar statement with any fixed polynomial is enough, but we need that the lengths of the linear combinations does not exceed a fixed number.

Property 7. *There is a fixed s such that*

$$\hat{B}_k^2 P \in \hat{\Pi}_{n+s}^k, \tag{19}$$

where P is an arbitrary polynomial of degree n.

Proof. Actually we prove that $s = s_k = \sum_{i=1}^k (n_i+1)$, provided that $\deg(b_i w_i) = \deg b_i - 1$, $i = 1,\dots,k$, cf. [1, Table 1].

For $k=0$ it is obvious with $s=0$. For $k=1$, expressing $P = \sum_{i=0}^n a_i p_i^{(\alpha,\beta)}$,

$$\int_I b_1^2(x)P(x)\hat{P}_l^{[1]}(x)W_1(x)dx$$

$$= \frac{c}{\sigma_l^1}\int_I b_1^2(x)\left(\sum_{i=0}^n a_i p_i^{(\alpha,\beta)}(x)\right)\Big(b_1(x)\sqrt{l(l+\alpha+\beta+1)}p_{l-1}^{(\alpha+1,\beta+1)}(x)$$

$$-b_1(x)w_1(x)p_l^{(\alpha,\beta)}(x)\Big)\frac{p(x)w^{(\alpha,\beta)}(x)}{b_1^2(x)}dx$$

$$= k_l\sum_{i=0}^n a_i\int_I b_1(x)p_i^{(\alpha,\beta)}(x)p_{l-1}^{(\alpha+1,\beta+1)}(x)w^{(\alpha+1,\beta+1)}(x)dx$$

$$-d_l\sum_{i=0}^n a_i\int_I p(x)b_1(x)w_1(x)p_i^{(\alpha,\beta)}(x)p_l^{(\alpha,\beta)}(x)w^{(\alpha,\beta)}(x)dx,$$

where k_l, d_l are constants depending on l, α and β and $p(x) = 1 - x^2$, cf. (1). Orthogonality and the remark above imply the statement.

For $k > 1$

$$\int_I \hat{B}_k^2(x)P(x)\hat{P}_l^{[k]}(x)W_k(x)dx$$

$$= \frac{c}{\sigma_l^k}\int_I \hat{B}_{k-1}^2(x)P(x)(b_k(x)(\hat{P}_l^{[k-1]}(x))' - b_k(x)w_k(x)\hat{P}_l^{[k-1]}(x))p(x)W_{k-1}(x)dx = (*)$$

Integrating by parts by (14)

$$(*) = \frac{c}{\sigma_l^k}\Bigg(\Big[\hat{P}_l^{[k-1]}(x)P(x)b_k(x)p(x)W_0(x)\Big]_I$$

$$-\int_I \hat{P}_l^{[k-1]}(x)\Big(2\hat{B}_{k-1}(x)\hat{B}_{k-1}'(x)P(x)b_k(x)p(x)W_{k-1}(x)$$

$$+\hat{B}_{k-1}^2(x)(Pb_k)'(x)p(x)W_{k-1}(x) + \hat{B}_{k-1}^2(x)P(x)b_k(x)q_{k-1}(x)W_{k-1}(x)\Big)dx$$

$$-\int_I \hat{B}_{k-1}^2(x)P(x)b_k(x)w_k(x)p(x)\hat{P}_l^{[k-1]}(x)W_{k-1}(x)dx\Bigg).$$

Considering that $\hat{B}_{k-1}' = \hat{B}_{k-1}\sum_{i=1}^{k-1}\frac{b_i'}{b_i}$ and (7)

$$(*) = -\frac{c}{\sigma_l^k}\int_I \hat{P}_l^{[k-1]}(x)\hat{B}_{k-1}^2(x)\left(p(x)(P(x)b_k(x))'\right.$$

$$\left.+P(x)b_k(x)(p(x)w_k(x) + q_0(x) + (k-1)p'(x))\right)W_{k-1}(x)dx.$$

Since the polynomial in the bracket is of degree $n + n_k + 1$, the result is given by induction.

Property 8. *Supposing* (13)

$$\lim_{n\to\infty}\frac{1}{n}\log|\hat{P}_n^{[k]}(z)| = \log|z + \sqrt{z^2-1}| \tag{20}$$

locally uniformly on $\mathbb{C}\backslash[-1,1]\backslash Z_k$, *where that branch of the square root is considered which maps positive numbers to positive numbers.*

Proof. Recalling that $\deg\prod_{i=}^k \tilde{b}_i = M_k$ and denoting by $\{x_{i,r}\}_{i=1}^{N(n)}$, $\{x_{j,e}\}_{j=1}^{M_k}$, $\{x_i\}_{i=1}^n$ the regular zeros of $\hat{P}_n^{[k]}$, the exceptional zeros of $\hat{P}_n^{[k]}$ and the zeros of $p_n^{(\alpha,\beta)}$, respectively, we have

$$\frac{1}{n}\log|\hat{P}_n^{[k]}(z)| = \frac{1}{n}\log\gamma_n^k + \frac{1}{n}\sum_{i=1}^n\log|z - x_{i,r}| + \frac{1}{n}\sum_{j=1}^{M_k}\log|z - x_{j,e}|. \tag{21}$$

Let $z \in \mathbb{C}\setminus[-1,1]$. Property 5 means that the normalized counting measure based on the regular zeros of $\hat{P}_n^{[k]}$ tends to the equilibrium measure of $[-1,1]$ in weak-star sense. Thus

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{N(n)}\log|z - x_{i,r}| = \int_I\log|z-x|d\mu_e(x)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\log|z - x_i| = \log|z + \sqrt{z^2-1}| - \log 2,$$

where we used the corresponding result for classical Jacobi polynomials, and the last equality fulfils by [24, Theorem 1]. Note, that on each compact set $K \subset \mathbb{C}\setminus[-1,1]$

the functions $f_n(z) = \frac{1}{n}\sum_{i=1}^{N(n)} \log|z - x_{i,r}|$ are equicontinuous. Indeed, let $z, w \in K$, $\operatorname{dist}(K, I) = d$ and then

$$|f_n(z) - f_n(w)| \le \frac{1}{n}\sum_{i=1}^{N(n)} \log\left(1 + \frac{|z-w|}{|w - x_{i,r}|}\right) \le \frac{|z-w|}{d}.$$

Since the sequence is pointwise convergent, it is uniformly convergent on K as well.

According to Property 3, the third sum of (21) tends to zero locally uniformly on $\mathbb{C} \setminus [-1,1] \setminus Z_k$. Indeed, as K is compact and has a positive distance from Z_k, if n is large enough, then $|\log|z - x_{j,e}||$ is uniformly bounded on K.
Comparing the first term of the right-hand side of (21) to (16), the proof is finished.

3. Application - Equilibrium measures of Julia sets

Dynamical properties of sequences of orthonormal polynomials given by a Borel probability measure supported on a non-polar compact subset of the complex plane were investigated in [3] and [19]. Although the results are proved in general circumstances for standard orthogonal polynomials in the cited paper, they cannot be applied to exceptional families since they have finite codimension in the space of polynomials. Below we give the extension of the result mentioned above to exceptional families.
For sake of simplicity in this section $j > 0$ is fixed and arbitrary; we denote by $\hat{P}_n := \hat{P}_n^{[j]}$, we omit the index j everywhere, and assume (13). The corresponding weight function is denoted by W.

3.1. Tools.

3.1.1. *Polynomial Dynamics.* The basin of attraction for ∞ for $\hat{P}_n$ is

$$\Omega_n := \{z \in \mathbb{C} : \lim_{k\to\infty} \hat{P}_n^k(z) = \infty\}, \tag{22}$$

where $\hat{P}_n^k = \hat{P}_n \circ \hat{P}_n \circ \cdots \circ \hat{P}_n$, that is composition k times. (Similarly $\hat{P}_n^{-k}$ denotes the inverse.)

$$K_n := \mathbb{C} \setminus \Omega_n, \quad J_n := \partial\Omega_n = \partial K_n$$

are the filled Julia set and the Julia set for $\hat{P}_n$, respectively.

First we note that for each polynomial $p(z) = \gamma z^n + \cdots + c$ of degree $n > 1$ there is an R_p such that for all z with $|z| > R_p$ $|p(z)| > 2|z|$. Thus Ω_p (cf. (22)) can be expressed as

$$\Omega_p := \{z \in \mathbb{C} : \lim_{k\to\infty} p^k(z) = \infty\} = \cup_{k\ge 0} p^{-k}(\mathbb{C} \setminus \overline{D(0, R_p)}). \tag{23}$$

So the filled Julia set of $\hat{P}_n$ is

$$K_n = \cap_{k\ge 0} \hat{P}_n^{-k}(\overline{D(0, R_n)}). \tag{24}$$

As in general, J_n and K_n are compact, completely invariant sets, i.e. $\hat{P}_n^{-1}(J_n) = J_n = \hat{P}_n(J_n)$.

3.1.2. *Potential Theory.* μ is a compactly supported Borel probability measure on the complex plane denoted by $\mu \in \mathcal{M}$. The (logarithmic) potential function of μ is

$$U^\mu(z) = \int_{\mathbb{C}} \log \frac{1}{|w-z|} d\mu(w).$$

The energy of μ is

$$E(\mu) = \int_{\mathbb{C}} U^\mu(z) d\mu(z).$$

Let $K \subset \mathbb{C}$ compact.

$$V(K) = \inf\{E(\mu) : \mu \in \mathcal{M}, \operatorname{supp}\mu \subset K\}.$$

The capacity of K is

$$\operatorname{cap} K = e^{-V(K)}.$$

3.2. **Limit of equilibrium measures of Julia sets.** The next theorem is the main result of this section.

Theorem 1. *With the notation above, for $n \in \mathbb{N}$ let J_n be the Julia set of the exceptional orthonormal Jacobi polynomial $\hat{P}_n$, and let μ_n and μ_e be the equilibrium measure of J_n and $[-1,1]$, respectively. Then*

$$\mu_n \to \mu_e \tag{25}$$

in weak-star sense.

We remark here that the equilibrium measure of the Julia set of a polynomial is the unique measure of maximal entropy with respect to the polynomial, cf. [2, Theorem 17.1] and [15, Theorem 9].

To prove Theorem 1 we need some lemmas. At first let us consider some corollaries of Property 8.

Corollary 1. *(of Property 8) There is an $\tilde{R}$, such that $K_n \subset D(0,\tilde{R})$ for all $n \in \mathbb{N}$.*

Proof. As K_n is compact for each n, it is enough to show that there is an $\tilde{R}$ such that $|\hat{P}_n(z)| > 2|z|$ if $|z| > \tilde{R}$ and n is large enough, cf. (24). The required inequality is ensured by (20).

Similarly to [3, Lemma 2.2] one can derive

Corollary 2. *(of Property 8) There exist $R > 1$ and $N \in \mathbb{N}$ such that for all $n > N$*

$$K_n \subset \hat{P}_n^{-1}(\overline{D(0,R)}) \subset D(0,R). \tag{26}$$

Proof. Let $R > \max\{1, \tilde{R}\}$ such that $D(0,R) \supset Z_k$. Let $\varepsilon := \inf_{|z|=R} \log|z + \sqrt{z^2-1}|$, which is positive. By Property 8 $\frac{1}{n}\log|\hat{P}_n(z)| \geq \frac{\varepsilon}{2}$ if $n > N$ and $|z| = R$. We can choose N so large that $\log R < N\frac{\varepsilon}{2}$. Thus the generalized minimum principle implies that

$$\hat{P}_n(\mathbb{C} \setminus D(0,R)) \subset \mathbb{C} \setminus \overline{D(0,R)}, \quad \forall\ n > N,$$

and so

$$\hat{P}_n^{-1}(\overline{D(0,R)}) \subset D(0,R). \tag{27}$$

Comparing (24), Corollary 1 and (27) the proof is complete.

Proposition 1. *Let $K \subset \mathbb{C}$ compact, $K \cap [-1,1] = \emptyset$. Then there is an $M \in \mathbb{N}$ (depending on K, but independent of n) such that for all $\hat{P}_n$, $\deg \hat{P}_n > 0$, and any $w \in K_n$*

$$\text{card}\left(\hat{P}_n^{-1}(w) \cap K\right) < M. \tag{28}$$

Proof. In view of Corollary 1 we can take $|w| < R$ with an arbitrary fixed R rather than $w \in K_n$. We can also assume, that M is large enough. Let $d = d(K, I)$ be the distance of K and I, and define $a := \frac{1}{\sqrt{1+\frac{d^2}{4}}}$. Let $\|\hat{B}\|_{\infty,I} =: A$, and choose $c := \frac{1}{\kappa(1+R)A^2}$. Here κ is a constant ensured by (16) such that $\frac{\gamma_{n+u}}{\gamma_n} < \kappa$ with a fixed u (will be given later), for all n. We assume that M is so large that $a^M < c$. Now suppose indirectly that for all M (large enough) and N there is an $n > N$ such that $\hat{P}_n(z) = w$ has at least M solutions, $z_1, \dots, z_M$ in K.
Choosing $y_1, \dots, y_M$ to be the nearest points from I to $z_1, \dots, z_M$, respectively, one can define the rational function

$$r(z) := \prod_{j=1}^{M} \frac{z - y_j}{z - z_j}, \quad \|r\|_{\infty,I} \le a^M < c, \tag{29}$$

cf. [21, Lemma I.3.2] and [19, Lemma 3.3]. With this rational function we define the monic polynomial of degree $\deg \hat{P}_n + s_0$, where $s_0 = 2\sum_{i=1}^{j} \deg b_i$ as

$$Q(z) = \frac{1}{\gamma_n} r\hat{B}^2(\hat{P}_n(z) - w).$$

Taking $u = s_0 + s$, in view of (29)

$$\|Q\|_{2,W} \le \frac{1}{\gamma_n}\|r\|_{\infty,[-1,1]}\|\hat{B}^2\|_{\infty,[-1,1]}\|\hat{P}_n^(z) - w\|_{2,W}$$

$$\le \frac{1}{\gamma_n \kappa(1+R)}\|\hat{P}_n^{[k]}(z) - w\|_{2,W}.$$

In that case, when $\deg \hat{P}_0 > 0$, let $1 = \sum_{i=0}^{\infty} e_i \hat{P}_i$, in norm. By orthonormality

$$\|\hat{P}_n(z) - w\|_{2,W} = \sqrt{1 + |w|^2 - 2\Re w e_n} \le \sqrt{1 + R^2 + 2R|e_n|}.$$

Note, that if $\deg \hat{P}_0 = 0$, the right-hand side above is $1 + R^2$. As $1 \in L^2_W$, e_n tends to zero thus if n is large enough $\frac{\|\hat{P}_n(z)-w\|_{2,W}}{1+R} < 1$.

According to (19)

$$Q = \frac{1}{\gamma_{n+s_0+s}}\hat{P}_{n+s_0+s} + \sum_{i=0}^{n+s_0+s-1} a_i \hat{P}_i^{[k]}. \tag{30}$$

and choosing κ as above we have

$$\|Q\|_{2,W} < \frac{\gamma_{n+s_0+s}}{\kappa\gamma_n}\frac{1}{\gamma_{n+s_0+s}}1 < \frac{1}{\gamma_{n+s_0+s}}\|\hat{P}_{n+s_0+s}\|_{2,W}.$$

Comparing to (30) it is impossible.

Proof. (of Theorem 1) In view of Corollary 1 the supports of equilibrium measures of the Julia sets, μ_n, are uniformly bounded. Furthermore applying the Möbius transform $w_1 = \frac{w}{(\gamma_n)^{\frac{1}{n-1}}}$, $z_1 = \frac{z}{(\gamma_n)^{\frac{1}{n-1}}}$ to $w = \hat{P}_n(z) = \gamma_n z^n + \dots a_0$, and denoting by $F(w) := \frac{1}{n}\sum_{z,\ \hat{P}_n(z)=w} f(z)$ for an f measurable, and observing that $\frac{1}{n^i}\sum_{z,\ z=\hat{P}_n^{-i}(0)} f(z) = \frac{1}{n^{i-1}}\sum_{w,\ w=\hat{P}_n^{-(i-1)}(0)} F(w)$, [2, Theorem 16.1] reads as

$$\int_{\mathbb{C}} f(z)d\mu_n(z) = \int_{\mathbb{C}} \frac{1}{n} \sum_{z,\ \hat{P}_n(z)=w} f(z)d\mu_n(w). \tag{31}$$

If $K \subset \mathbb{C}$ such that K is compact and disjoint from I, then applying (31) to the indicator function of K and considering Proposition 1 one can derive that $\mu_n(K)$ tends to zero. Thus, if a subsequence μ_{n_i} has a weak-star limit, say ν, its support is contained by I.

According to [2, Lemma 15.1] and (16), lower semicontinuity of the energy implies that $I(\nu) \le \liminf_i I(\mu_{n_i}) = I(\mu_e)$. That is ν must be the unique equilibrium measure for any subsequence and so the proof is finished.

References

[1] N. Bonneux, Exceptional Jacobi polynomials, *J. Approx. Theory*, **239** (2019), 72-112.

[2] H. Brolin, Invariant sets under iteration of rational functions, *Arkiv f. Math.* **6** (1965) 103-144.

[3] J. S. Christiansen, C. Henriksen, H. L. Pedersen, C. L. Petersen, Julia sets of orthogonal polynomials, *Potent. Anal.* **50** (2019) 401-413.

[4] A.J. Durán, Exceptional Hahn and Jacobi orthogonal polynomials, *J. Approx. Theory* **214** (2017) 9-48.

[5] A. Durán, Corrigendum to the papers on Exceptional orthogonal polynomials: J. Approx. Theory 182 (2014) 29-58, 184 (2014) 176-208 and 214 (2017) 9-48, *J. Approx. Theory* **253** (2020) 105349.

[6] P. Erdős, P. Turán, On the uniformly-dense distribution of certain sequences of points, *Ann. of Math.* **41** (1940), 162-173.

[7] D. Gómez-Ullate, N. Kamran, R. Milson, An extended class of orthogonal polynomials defined by a Sturm-Liouville problem, *J. Math. Analysis and Appl.*, **359** (2009), 352-367.

[8] M. Á. García-Ferrero, D. Gómez-Ullate, R. Milson, A Bochner type classification theorem for exceptional orthogonal polynomials, *J. Math. Anal. Appl.*, **472** (1), (2019) 584-626.

[9] D. Gómez-Ullate, F. Marcellán, R. Milson, Asymptotic and interlacing properties of zeros of exceptional Jacobi and Laguerre polynomials, *J. Math. Anal. Appl.* **399** (2013) 480-495.

[10] A. Hardy, Average characteristic polynomials of determinantal point processes, *Ann. de l'Institut Henri Poincaré - Probabilités et Statistiques*, **51** (1) (2015), 283-303.

[11] A. Hardy, Polynomial ensembles and recurrence coefficients, *Constr. Approx.* **48** (2018), 137-162.

[12] Á. P. Horváth, The Electrostatic Properties of Zeros of Exceptional Laguerre and Jacobi Polynomials and Stable Interpolation, *J. Approx. Theory*, **194** (2015), 87-107.

[13] Á. P. Horváth, Asymptotics for recurrence coefficients of X_1-Jacobi exceptional polynomials and Christoffel function, *Integral Transforms Spec. Funct.* **31** (2020) 87-106.

[14] A. B. J. Kuijlaars, W. Van Assche, The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients, *J. Approx. Theory*, **99** (1999), 167-197.

[15] M. Lyubich, Entropy properties of rational endomorphisms of the Riemann sphere, *Ergod. Th. Dynam. Sys.* **3** (1983) 351-385.

[16] A. Máté, P. Nevai and V. Totik, Szegő's extremum problem on the unit circle, *Ann. of Math.*, **134**, (2) (1991), 433-453.

[17] A. Máté, P. Nevai, W. Van Assche, The supports of measures associated with orthogonal polynomials and the spectra of the related self-adjoint operators, *Rocky Mountain J. Math.* **21** (1991) 501-527.

[18] S. Odake, Recurrence Relations of the Multi-Indexed Orthogonal Polynomials : III, *Journal of Mathematical Physics* **57** (2016), 023514.
[19] C. L. Petersen, E. Uhre, Weak limits of the measures of maximal entropy for orthogonal polynomials, *Potent. Anal.* (2020) https://doi.org/10.1007/s11118-019-09824-5.
[20] B. Simon, The Christoffel-Darboux kernel, Proceedings of Symposia in Pure Mathematics (Perspectives in Partial Differential Equations, Harmonic Analysis and Applications) Ed. D. Mitrea, E, Mitrea *79* (2008) 295-335.
[21] H. Stahl, V. Totik, General Orthogonal Polynomials, Encyclopedia of Mathematics and its Applications, Cambridge University Press (1992)
[22] M. H. Stone, Linear Transformations in Hilbert Space and their Applications to Analysis, Amer. Math. Soc. Coll. Publ. **15.** Amer. Math. Soc. New York, 1932.
[23] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ. **23.** Amer. Math. Soc. Providence RI, 1975 (fourth edition)
[24] J. L.Ullman. A survey of exterior asymptotics for orthogonal polynomials associated with a finite interval and a study of the case of general weight measures. In Approximation Theory and Spline Functions, eds. S. P. Singh et al., pp. 467-78. D. Reidel, Dordrecht, 1984.

Department of Analysis,
Budapest University of Technology and Economics

g.horvath.agota@renyi.hu