On topological relaxations of chromatic conjectures

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Fractional chromatic relaxations

Some famous conjectures of combinatorics give an upper bound to the chromatic number of a graph.

- Hadwiger's conjecture
- Behzad-Vizing conjecture on total coloring
- Hedetniemi's conjecture

One can get a weaker statement from a conjecture like these by asking to bound the fractional chromatic number from above instead. Such weaker statements have been proven for all three of the problems mentioned.

Reed and Seymour showed in 1998 such a fractional relaxation of the Hadwiger conjecture: if a graph G does not have K_{m+1} as a minor, then $\chi_f(G) \leq 2m$.

B. Reed, P. D. Seymour, Fractional coloring and Hadwiger's conjecture, *J. Combin. Theory Ser. B*, **74** (1998), 147–152.

Topological relaxations

László Lovász' proof of the Kneser conjecture in 1978 gives a way to bound the chromatic number from below without also bounding the fractional chromatic number. We can think of these bounds as a general graph parameter. (More than one different parameters in fact.)

We get different relaxations of the conjectures if we put this parameter in place of the chromatic number.

Simonyi and Tardos proved such a relaxation of the Hadwiger conjecture in 2006: if a graph G does not contain K_{m+1} as a minor, then $2 + \mathrm{ind}(B(G)) < 2m$.

J. Matoušek, *Using the Borsuk-Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry*, Universitext, Springer-Verlag, Heidelberg, 2003.

J. Matoušek, G. M. Ziegler, Topological lower bounds for the chromatic number: A hierarchy, *Jahresber. Deutsch. Math.-Verein.*, **106** (2004), no. 2, 71–90.

G. Simonyi, G. Tardos, Local chromatic number, Ky Fan's theorem, and circular colorings, *Combinatorica*, **26** (2006), 587–626.

A necessary condition for the topological bound

If a graph G does not contain K_{m+1} as a minor, then $2 + \mathrm{ind}(B(G)) < 2m$. How to prove?

Csorba, Lange, Schurr, Waßmer proved a simple necessary condition for the topological bound. If $t \leq 2 + \mathrm{ind}(B(G))$, then the graph G contains a $K_{k,l}$ complete bipartite graph as a subgraph for every k, l such that $t = k + l$.

Apply this for $m + 1 = k = l$ to get the theorem.

P. Csorba, C. Lange, I. Schurr, A. Waßmer, Box complexes, neighbourhood complexes, and chromatic number, *J. Combin. Theory Ser. A*, **108** (2004), 159–168.

Odd Hadwiger conjecture

We say that graph G has K_{m+1} as an odd minor if we can select m vertex-disjoint tree subgraphs from it and give a 2-coloring of the vertices such that the coloring is proper on any one of the m chosen trees but there is a monocolored edge joining any two of the trees. Gerards and Seymour conjectured that if G has no odd K_{m+1} minor then $\chi(G) \leq m$.

Kawarabayashi and Reed have proved a fractional relaxation of this conjecture (within a factor of 2).

Topological relaxation not known.

We have examined some well-known families of graphs where the fractional chromatic number is low: Kneser graphs and Schrijver graphs (for certain parameters), and generalized Mycielski graphs, and found $K_{\chi(G)}$ odd minors in them.

K-i. Kawarabayashi, B. Reed, Fractional coloring and the odd Hadwiger's conjecture, *European J. Combin.*, **29** (2008), 411–417.

G. Simonyi, A. Zsbán, On topological relaxations of chromatic conjectures. To appear in *European Journal of Combinatorics*, 2010.

Behzad-Vizing conjecture

The vertices of the total graph $T(G)$ of a graph G are the vertices and edges of G; elements adjacent or incident in G are joined with an edge in $T(G)$.

The Behzad-Vizing conjecture states that $\chi(T(G)) \leq 2 + \Delta(G)$ (where Δ means the maximal degree).

Kilakos and Reed proves fractional analog in 1993: $\chi_f(T(G)) \leq 2 + \Delta(G)$.

We prove a topological relaxation: $2 + \mathrm{ind}(B(T(G))) \leq 2 + \Delta(G)$.

K. Kilakos, B. Reed, Fractionally coloring total graphs, *Combinatorica*, **13** (1993), 435–440.

Hedetniemi's conjecture

Hedetniemi's conjecture concerns the categorical product of graphs: it states that $\min(\chi(F), \chi(G)) = \chi(F \times G)$ for any two graphs.

Tardif proved a fractional relaxation in 2001:
$\min(\chi_f(F), \chi_f(G)) \leq 2\chi(F \times G)$. He also proved another analog in 2005:
$\min(\chi_f(F), \chi_f(G)) \leq 4\chi_f(F \times G)$.

Hell in 1977 and Dochtermann in 2006 prove a topological analog.

We gave an analog for a different topological bound:
$\mathrm{coind}(B(F \times G)) = \min(\mathrm{coind}(B(F)), \mathrm{coind}(B(G)))$. The same is basically proved by Dochtermann and Schultz in 2009.

C. Tardif, Hedetniemi's conjecture, 40 years later, *Graph Theory Notes N. Y.*, **54** (2008), 46–57.

P. Hell, An introduction to the category of graphs, *Topics in graph theory* Ann. New York Acad. Sci., **328** (1979), New York Acad. Sci., 120–136,

A. Dochtermann, Hom complexes and homotopy theory in the category of graphs, *Europ. J. Combin.*, **30** (2009), 490-509.

A. Dochtermann, C. Schultz, Topology of Hom complexes and test graphs for bounding chromatic number, 2009, to appear in *Israel J. Math.*

Thanks for your attention

About the proof

How does the proof of our topological analog of Hedetniemi's conjecture work?

Bipartite graphs can be characterized as either graphs G for which there exists a $G \to K_2$ graph homomorphism, or where there is no $C_{2k+1} \to G$ homomorphism for any $1 \leq k$.

Graphs G with an m-coloring are those graphs for which there exists a $G \to K_m$ homomorphism. Is there also a sequence of test graphs such that G has an m-coloring iff there is no $T_k \to G$ homomorphism for any k, such that these test graphs admit $T_1 \leftarrow T_2 \leftarrow \cdots \leftarrow T_{k-1} \leftarrow T_k \leftarrow \dots$ homomorphisms? The existence of such a sequence of test graphs is equivalent to Hedetniemi's conjecture.

The similar equivalence holds for the topological analogs. Simonyi and Tardos in 2006 essentially gives such a sequence of test graphs, and so does Dochtermann and Schultz in 2009.