On the curvature of the quantum state space with pull-back metrics

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Structure:

– Space of classical probability distributions: $\mathcal{P}_n$
– Geometry on $\mathcal{P}_n$
– A Conjecture about the scalar curvature of $\mathcal{P}_n$
– Scalar curvature computation of $\mathcal{P}_n$
– Pull-back metric of the quantum mechanical state space: $(\mathcal{M}_n, g_f)$
– Scalar curvature computation of $\mathcal{M}_n$
– Some result about the Conjecture

Classical discrete distributions

Base set: $X_n = \{1, 2, \dots, n\}$ $(n \in \mathbb{N})$.

Space of distributions on X_n is

$$\mathcal{P}_n = \left\{ (\vartheta_1, \dots, \vartheta_n) \in \mathbb{R}^n \;\middle|\; \forall k \in \{1, \dots, n\} : \vartheta_k > 0,\ \sum_{k=1}^{n} \vartheta_k = 1 \right\}.$$

Majorization:

$a = (a_1, \dots, a_n) \in \mathcal{P}_n$ is said to be majorized by $b = (b_1, \dots, b_n) \in \mathcal{P}_n$ (denoted by $a \prec b$) if for their decreasingly ordered set of parameters $(a_i^{\downarrow})_{i=1,\dots,n}$ and $(b_i^{\downarrow})_{i=1,\dots,n}$

$$\sum_{l=1}^{k} a_l^{\downarrow} \leq \sum_{l=1}^{k} b_l^{\downarrow}$$

holds for all $1 \leq k < n$.

Tangent space of the Riemannian manifold (M, g) at $p \in M$ will be denoted by $\mathrm{T}_p\, M$, and the **tangent bundle**

$$\mathrm{T}\, M = \bigcup_{p \in M} p \times \mathrm{T}_p\, M.$$

Riemannian metric is a function

$$g \in \prod_{p \in M} \mathrm{Lin}(\mathrm{T}_p\, M^2, \mathbb{R})$$

such that:

1. $\forall p \in M \;\; g(p) : \mathrm{T}_p\, M \times \mathrm{T}_p\, M \to \mathbb{R}$ is a scalar product,
2. the "function $g(p)$ is continuous in p".

Riemannian manifold is a pair (M, g).

The canonical Riemannian metric g_c of the spaces $M = \mathbb{R}^n, \mathbb{R}^n_+$ at every point $p \in M$ for every tangent vectors $x, y \in \mathrm{T}_p\, M$ is

$$g_c(p)(x, y) = \sum_{i=1}^{n} x_i y_i.$$

`Pull-back metric:` Assume that N is a manifold and (M, g) is a Riemannian space and $\phi : N \to M$ is smooth function. For every point $p \in N$ we have a map

$$\phi_{*p} : \mathrm{T}_p N \to \mathrm{T}_{\phi(p)} M$$

which connects the tangent spaces. We can define a Riemannian metric on N: for a point $p \in N$ and for tangent vectors $X, Y \in \mathrm{T}_p N$

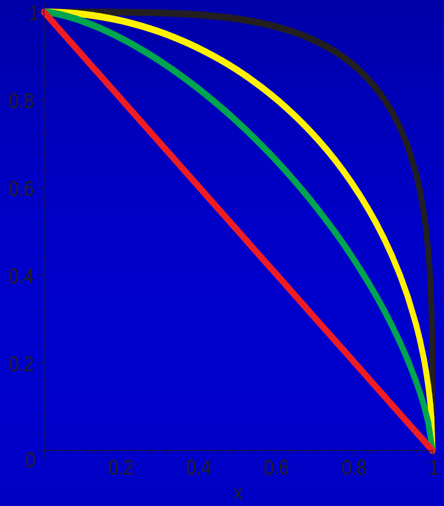
$$\phi^*(g)(p)(X, Y) := g\big(\phi(p)\big)\big(\phi_{*p}(X), \phi_{*p} Y\big).$$

For every parameter $\alpha \in \mathbb{R}$ the α-**`geometry of`** $\mathcal{P}_n$ is the pull-back geometry of the Riemannian manifold $(\mathbb{R}^n, g_c)$ induced by the map

$$\phi_{\alpha,n} : \mathcal{P}_n \to \mathbb{R}^n \quad (\vartheta_1, \dots, \vartheta_n) \mapsto \begin{cases} \frac{2}{1-\alpha}\left(\vartheta_1^{\frac{1-\alpha}{2}}, \dots, \vartheta_n^{\frac{1-\alpha}{2}}\right), & \text{if} \quad \alpha \neq 1 \\ (\log \vartheta_1, \dots, \log \vartheta_n), & \text{if} \quad \alpha = 1. \end{cases}$$

This Riemannian space is denoted by $(\mathcal{P}_n, g_\alpha)$.

The space $(\mathcal{P}_2, g_\alpha)$:



$\alpha = -1$: red line
$\alpha = -\dfrac{1}{3}$: green line
$\alpha = 0$: yellow line
$\alpha = \dfrac{1}{2}$: black line

It is widely believed that the differential geometrical properties of the space $\mathcal{P}_n$ has statistical interpretation. For example the `scalar curvature` measures the `average statistical uncertainty`.

It is reasonable to expect that more mixed states are less distinguishable than less mixed states. It means mathematically that in this case the scalar curvature of a Riemann structure should have the following monotonicity property:

If D_1 is more mixed than D_2 then $\mathrm{Scal}(D_2)$ should be smaller then $\mathrm{Scal}(D_1)$. (This is known as `Petz conjecture`, if the state space is endowed with the Kubo–Mori metric.)

`Conjecture`*(Gibilisco and Isola)*: On the spaces $(\mathcal{P}_n, g_\alpha)$ and $(\mathcal{M}_n, g_\alpha)$ the scalar curvature is monotonously increasing, with respect to the majorization relation if $\alpha \in\ ]-1, 0[$ and it is monotonously decreasing if $\alpha \in\ ]0, 1[$.

`Extended Riemannian space` is $(\tilde{\mathcal{P}}_n, \tilde{g}_\alpha)$, where $\tilde{\mathcal{P}}_n = \mathbb{R}^n_+$ and $\tilde{g}$ is the pull-back geometry of $(\mathbb{R}^n_+, g_c)$ metric induced by the map

$$\tilde{\phi}_{\alpha,n} : \tilde{\mathcal{P}}_n \to \mathbb{R}^n \quad (\vartheta_1, \dots, \vartheta_n) \mapsto \begin{cases} \frac{2}{1-\alpha}\left(\vartheta_1^{\frac{1-\alpha}{2}}, \dots, \vartheta_n^{\frac{1-\alpha}{2}}\right), & \text{if} \quad \alpha \neq 1 \\ (\log \vartheta_1, \dots, \log \vartheta_n), & \text{if} \quad \alpha = 1. \end{cases}$$

`Theorem:` Assume that (M, g) is an n dimensional submanifold of the $n+1$ dimensional Riemannian space $(\tilde{M}, \tilde{g})$. The Levi–Civita covariant derivative is $\tilde{\nabla}$ and the Riemannian curvature tensor is $\tilde{R}$. The normal vector field of M is $N : M \to \mathrm{T}\,\tilde{M}$. For every tangent vector $X, Y \in \mathrm{T}\,\tilde{M}$ let us define the following map.

$$S(X, Y) : M \to \mathbb{R} \qquad p \mapsto -\tilde{g}(\tilde{\nabla}_X N, Y)$$

For every point $p \in M$ if $(A_t)_{t=1,\dots,n}$ is an orthonormal basis in $\mathrm{T}_p\,M$ (that is $g(A_t, A_s) = \delta_{ts}$) then the scalar curvature of M at a point p is $\mathrm{Scal}(p) =$

$$\sum_{t,s=1}^{n} \tilde{g}(\tilde{R}(A_t, A_s)A_s, A_t) + S(A_s, A_s)S(A_t, A_t) - S(A_t, A_s)S(A_s, A_t). \tag{1}$$

QP 2006

Geometry of $(\mathcal{P}_n, g_\alpha)$

At a point $\vartheta \in \mathcal{P}_n$:
Riemannian metric: $\tilde{g}_{ij} = \tilde{g}(\partial_i, \partial_j) = \delta_{ij}\vartheta_i^{-1-\alpha}$.
The inverse of the metric tensor is $\tilde{g}^{ij} = \delta_{ij}\vartheta_i^{1+\alpha}$.

The Christoffel symbol: $\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n} \Gamma_{ij}^{..k}\partial_k$.

$$\tilde{\Gamma}_{ij}^{..k} = \frac{1}{2}\sum_{m=1}^{n} \tilde{g}^{km}(\partial_i\tilde{g}_{jm} + \partial_j\tilde{g}_{im} - \partial_m\tilde{g}_{ij}) = -\frac{1+\alpha}{2}\vartheta_k^{-1}\delta_{ij}\delta_{jm}$$

The space $\tilde{\mathcal{P}}_n$ is diffeomorphic to $\mathbb{R}^n$, so $\tilde{R}_{ijk}^{...i} = 0$ holds.
The normal vector field of the submanifold $\mathcal{P}_n$ is

$$N(\vartheta) = \frac{1}{c(\vartheta)}\sum_{i=1}^{n} \vartheta_i^{1+\alpha}\partial_i, \quad \text{where} \quad c(\vartheta) = \sqrt{\sum_{i=1}^{n} \vartheta_i^{1+\alpha}},$$

since

$$\tilde{g}(N, N) = 1 \quad \text{and} \quad \tilde{g}(N, \partial_i - \partial_n) = 0.$$

$$S(\partial_i,\partial_j) = -\tilde{g}(\tilde{\nabla}_{\partial_i}N,\partial_j) = \frac{\beta}{2c(\vartheta)}\delta_{ij}\vartheta_i^{-1} - \frac{\beta}{c(\vartheta)^3}\vartheta_i^{-\beta-1}.$$

For every point $\vartheta \in \mathcal{P}_n$ if $(A_t)_{t=1,\dots,n-1}$ is an ONB in $\mathrm{T}_\vartheta\,\mathcal{P}_n$, $(A_t)_{t=1,\dots,n-1} \cup N(\vartheta)$ is an ONB in $\mathrm{T}_\vartheta\,\tilde{\mathcal{P}}_n$. The scalar curvature is

$$\mathrm{Scal}(\vartheta) = \sum_{t,s=1}^{n}\big(S(B_s,B_s)S(B_t,B_t) - S(B_t,B_s)S(B_s,B_t)\big)$$
$$-\,2\sum_{t=1}^{n}\big(S(N(\vartheta),N(\vartheta))S(B_t,B_t) - S(B_t,N(\vartheta))S(N(\vartheta),B_t)\big).$$

where $(B_t)_{t=1,\dots,n}$ is an orthonormal basis in $\mathrm{T}_\vartheta\,\tilde{\mathcal{P}}_n$. We have

$$S(\partial_i,N(\vartheta)) = 0 \quad \text{and} \quad S(N(\vartheta),N(\vartheta)) = 0.$$

The set $\left(\vartheta_t^{-\frac{\beta}{2}}\partial_t\right)_{t=1,\dots,n}$ form an ONB in $\mathrm{T}_\vartheta\,\tilde{\mathcal{P}}_n$, therefore

$$\mathrm{Scal}(\vartheta) = \sum_{t,s=1}^{n}\vartheta_s^{-\beta}S(\partial_s,\partial_s)\vartheta_t^{-\beta}S(\partial_t,\partial_t) - \vartheta_s^{-\beta}\vartheta_t^{-\beta}S(\partial_t,\partial_s)S(\partial_s,\partial_t).$$

Discrete distributions
Geometry of (P_n, g_α)
Geometry of M_n
Conjecture
Title page
Go back
Full Screen
Close
Quit

Theorem: The scalar curvature of the space $(\mathcal{P}_n, g_\alpha)$ at a point $\vartheta \in \mathcal{P}_n$ is

$$\mathrm{Scal}(\vartheta) = \frac{(1+\alpha)^2}{4c(\vartheta)^2} \sum_{\substack{t,s=1 \\ t\neq s}}^{n} \vartheta_t^\alpha \vartheta_s^\alpha \left(1 - \frac{\vartheta_t^{\alpha+1} + \vartheta_s^{\alpha+1}}{c(\vartheta)^2}\right),$$

where $c(\vartheta) = \sqrt{\sum_{k=1}^{n} \vartheta_k^{\alpha+1}}$.

– The parameter $\alpha = -1$ corresponds to the case, when $\phi_{\alpha,n} : \mathcal{P}_n \to \mathbb{R}^n$ is the natural embedding. In this case $\mathcal{P}_n$ is a part of an $n-1$ dimensional hyperplane, so its scalar curvature is zero.

– At the parameter value $\alpha = 0$ the function $\phi_{\alpha,n}$ maps $\mathcal{P}_n$ to the surface of the Euclidean ball with radius $R = 2$. In this case the scalar curvature formula gives

$$\mathrm{Scal}(\vartheta) = \frac{(n-1)(n-2)}{4} = \frac{\dim(\mathcal{P}_n)(\dim(\mathcal{P}_n) - 1)}{R^2}$$

which is just the scalar curvature of the $\dim(\mathcal{P}_n)$ dimensional sphere with radius R.

Pull-back geometry of the state space

State space: For every n let us denote by $\mathcal{M}_n$ the set of positive states, that is

$$\mathcal{M}_n = \{D \in M_n \mid D = D^*,\ D > 0,\ \operatorname{Tr} D = 1\}.$$

Some concepts of the classical probability theory can be extended to the noncommutative case. One of them is the majorization relation.

The state $D_1 \in \mathcal{M}_n$ is said to be **majorized** by the state $D_2 \in \mathcal{M}_n$, denoted by $D_1 \prec D_2$, if the relation $\mu_1 \prec \mu_2$ holds for their set of eigenvalues μ_1 and μ_2.

In the classical case we have defined only a special kind of pull-back metrics, in that case the function was a power function or a logarithmic one. In this quantum setting we consider those $f :]0,1[\to \mathbb{R}$ functions, which have an analytic extension to a neighborhood of the interval $]0,1[$ and $f'(x) \neq 0$ for every $x \in]0,1[$. We call such functions **admissible functions**.

The set of real or complex self-adjoint matrices will be denoted by $\mathcal{M}_n^{\mathrm{sa}}$, and geometrically it will be considered as a Riemannian space $(\mathbb{R}^d, g_c)$, where $d_{\mathbb{R}} = \frac{(n-1)(n+2)}{2}$ for real matrices and $d_{\mathbb{C}} = n^2 - 1$ for complex ones and g_E is the **canonical Riemannian metric on** $\mathcal{M}_n^{\mathrm{sa}}$. That is, at every point $D \in \mathcal{M}_n^{\mathrm{sa}}$ for every vectors $X, Y \in \mathcal{M}_n^{\mathrm{sa}}$ in the tangent space at D the metric is

$$g_c(D)(X,Y) = \operatorname{Tr} XY.$$

Assume that $f :]0,1[\to \mathbb{R}$ is an admissible function. The pull back of the Riemannian metric $(\mathcal{M}_n^{\mathrm{sa}}, g_c)$ to the space $\mathcal{M}_n$ induced by the map

$$\phi_{f,n} : \mathcal{M}_n \to M_n^{\mathrm{sa}} \qquad D \mapsto f(D).$$

is called the **pull-back geometry of** $\mathcal{M}_n$ and it is denoted by g_f. This Riemannian space will be denoted by $(\mathcal{M}_n, g_f)$.

Since the function f has an analytic extension to a neighborhood of the interval $]0,1[$ we have by the Riesz–Dunford operator calculus for every $D \in \mathcal{M}_n$

$$f(D) = \frac{1}{2\pi\,\mathrm{i}} \oint_\gamma f(z)(z\,\mathrm{id} - D)^{-1}\,\mathrm{d}\,z,$$

where id denotes the identity matrix and γ is a smooth curve winding once around the spectrum of D counterclockwise. The derivative of f at $D \in \mathcal{M}_n$ for $X \in \mathrm{T}_D\,\mathcal{M}_n$ is

$$df(D)(X) = \frac{1}{2\pi\,\mathrm{i}} \oint_\gamma f(z)(z\,\mathrm{id} - D)^{-1} X (z\,\mathrm{id} - D)^{-1}\,\mathrm{d}\,z.$$

Let $D \in \mathcal{M}_n$ and choose a basis of $\mathbb{R}^n$ such that $D = \sum_{i=1}^{n} \lambda_i E_{ii}$ is diagonal, where $(E_{ij})_{1 \le i,j \le n}$ is the usual system of matrix units. Let us define the following self-adjoint matrices.

$$\begin{aligned} F_{ij} &= E_{ij} + E_{ji}, & 1 \le i \le j \le n; \\ H_{ij} &= \mathrm{i}\,E_{ij} - \mathrm{i}\,E_{ji}, & 1 \le i < j \le n. \end{aligned}$$

The set of matrices $(F_{ij})_{1\le i\le j\le n} \cup (H_{ij})_{1\le i<j\le n}$ form a basis of $T_D\tilde{\mathcal{M}}_n$ for complex matrices and $(F_{ij})_{1\le i\le j\le n}$ form a basis for real ones. Using the equation

$$g(D)(X,Y) = \mathrm{Tr}(df(D)(X)df(D)(Y))$$

for the pull-back metric we have the following theorem.

`Theorem:` On the Riemannian space $(\mathcal{M}_n, g_f)$ for a state $D \in \mathcal{M}_n$ choose a basis of $\mathbb{R}^n$ where $D = \sum_{i=1}^{n} \lambda_i E_{ii}$. Then we have for the metric

$$\text{if } \ 1 \le i < j \le n, 1 \le k < l \le n: \begin{cases} g(D)(H_{ij}, H_{kl}) = \delta_{ik}\delta_{jl} 2M_{ij}^2 \\ g(D)(F_{ij}, F_{kl}) = \delta_{ik}\delta_{jl} 2M_{ij}^2 \\ g(D)(H_{ij}, F_{kl}) = 0, \end{cases}$$

$$\text{if } \ 1 \le i < j \le n, 1 \le k \le n: \quad g(D)(H_{ij}, F_{kk}) = g(D)(F_{ij}, F_{kk}) = 0,$$

$$\text{if } \ 1 \le i \le n, 1 \le k \le n: \quad g(D)(F_{ii}, F_{kk}) = \delta_{ik} 4M_{ii}^2,$$

where

$$M_{ij} = \begin{cases} \dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} & \text{if } \lambda_i \neq \lambda_j \\ f'(\lambda_i) & \text{if } \lambda_i = \lambda_j. \end{cases}$$

The Christoffel symbol can be computed from the derivative of the Riemannian metric $g(D)(\Gamma(D)(X,Y),Z) =$

$$\frac{1}{2}(dg(D)(X)(Y,Z) + dg(D)(Y)(X,Z) - dg(D)(X,Y)).$$

Since the derivative of the Riemannian metric is $dg(D)(Z)(X,Y) =$

$$\mathrm{Tr}(d^2f(D)(Z)(X)df(D)(Y) + df(D)(X)d^2f(D)(Z)(Y))$$

we have the following expression for the Christoffel symbol

$$\Gamma(D)(X,Y) = (df(D))^{-1}(d^2f(D)(X,Y)).$$

From the Riesz–Dunford operator calculus the second derivative of the matrix-valued function f is

$$d^2f(D)(E_{ij})(E_{kl}) = \delta_{jk}M_{ilj}E_{il} + \delta_{il}M_{kji}E_{kj},$$

where

$$M_{ijk} = \frac{1}{2\pi\,\mathrm{i}}\oint_\gamma \frac{f(z)}{(z-\lambda_i)(z-\lambda_j)(z-\lambda_k)}\,\mathrm{d}\,z.$$

Combining these results together the Christoffel symbol is the following.

$$\Gamma(D)(F_{ij})(F_{kl}) = F_{il}\delta_{jk}\frac{M_{ilk}}{M_{il}} + F_{kj}\delta_{il}\frac{M_{ijk}}{M_{jk}} + F_{ik}\delta_{jl}\frac{M_{ikl}}{M_{ik}} + F_{lj}\delta_{ik}\frac{M_{ijl}}{M_{lj}}$$

$$\Gamma(D)(H_{ij})(H_{kl}) = -F_{il}\delta_{jk}\frac{M_{ilk}}{M_{il}} - F_{kj}\delta_{il}\frac{M_{ijk}}{M_{jk}} + F_{ik}\delta_{jl}\frac{M_{ikl}}{M_{ik}} + F_{lj}\delta_{ik}\frac{M_{ijl}}{M_{lj}}$$

$$\Gamma(D)(H_{ij})(F_{kl}) = H_{il}\delta_{jk}\frac{M_{ilk}}{M_{il}} + H_{kj}\delta_{il}\frac{M_{ijk}}{M_{jk}} + H_{ik}\delta_{jl}\frac{M_{ikl}}{M_{ik}} + H_{lj}\delta_{ik}\frac{M_{ijl}}{M_{lj}}$$

The normal vector field of the submanifold $\mathcal{M}_n$ is

$$N(D) = \frac{1}{c(D)}(f'(D))^{-2}, \quad \text{where} \quad c(D) = \sqrt{\mathrm{Tr}(f'(D))^{-2}}$$

since

$$g(D)(N(D), N(D)) = \frac{1}{c(D)^2}\sum_{i,j=1}^{n} g(D)\left(\frac{1}{M_{ii}^2}E_{ii}, \frac{1}{M_{jj}^2}E_{jj}\right) = 1$$

$$g(D)(N(D), E_{ii} - E_{nn}) = 0.$$

In this setting the definition of the map S, is

$$S(D)(X,Y) = -g(D)(\Gamma(D)(X)(N),Y).$$

First we note that

$$\Gamma(D)(X)(N) = dN(D)(X) + \Gamma(D)(X)(N(D)).$$

Using the $h = (f')^{-2}$ notation the normal vector field is

$$N(D) = \frac{1}{\sqrt{\operatorname{Tr} h(D)}} h(D)$$

and its derivative is $dN(D)(X) =$

$$-\frac{1}{2}\frac{1}{(\operatorname{Tr} h(D))^{\frac{3}{2}}} \operatorname{Tr}(dh(D)(X))h(D) + \frac{1}{\sqrt{\operatorname{Tr} h(D)}} dh(D)(X).$$

After some computation we have the function S:

If $1 \le i < j \le n, 1 \le k < l \le n$: $\begin{cases} S(D)(H_{ij}, H_{kl}) = -\frac{2}{c(D)}\delta_{ik}\delta_{jl}\rho_{ij}M_{ij}^2 \\ S(D)(F_{ij}, F_{kl}) = -\frac{2}{c(D)}\delta_{ik}\delta_{jl}\rho_{ij}M_{ij}^2 \\ S(D)(H_{ij}, F_{kl}) = 0. \end{cases}$

If $1 \le i < j \le n, 1 \le k \le n$: $\begin{cases} S(D)(H_{ij}, F_{kk}) = S(D)(F_{kk}, H_{ij}) = 0 \\ S(D)(F_{ij}, F_{kk}) = S(D)(F_{kk}, F_{ij}) = 0. \end{cases}$

If $1 \le i \le n, 1 \le k \le n$: $S(D)(F_{ii}, F_{kk}) = -\dfrac{8}{c(D)^3}\dfrac{M_{iii}}{M_{ii}^3} + \dfrac{8}{c(D)}\delta_{ik}\dfrac{M_{iii}}{M_{ii}}$.

where

$$M_{ijj} = \begin{cases} \dfrac{f(\lambda_i) - f(\lambda_j)}{(\lambda_i - \lambda_j)^2} + \dfrac{f'(\lambda_j)}{\lambda_j - \lambda_i} & \text{if } \lambda_i \neq \lambda_j \\ \dfrac{1}{2}f''(\lambda_i) & \text{if } \lambda_i = \lambda_j \end{cases}$$

$$\rho_{ij} = \begin{cases} -\dfrac{1}{f'(\lambda_i)f'(\lambda_j)}\dfrac{f'(\lambda_i) - f'(\lambda_j)}{f(\lambda_i) - f(\lambda_j)} & \text{if } \lambda_i \neq \lambda_j \\ -\dfrac{f''(\lambda_i)}{f'(\lambda_i)^3} & \text{if } \lambda_i = \lambda_j. \end{cases}$$

The basis of the scalar curvature computation is Equation (1), where summation runs on an orthonormal basis of the tangent space of the submanifold. Fortunately it is no matter if we add the normal vector field to this summation or not, as in the classical case, its summand is 0 since

$$S(D)(F_{ij}, N(D)) = 0,$$
$$S(D)(H_{ij}, N(D)) = 0$$

$$\begin{aligned} S(D)(F_{ii}, N(D)) &= \frac{1}{2c(D)} \sum_{k=1}^{n} \frac{S(D)(F_{ii}, F_{kk})}{M_{kk}^2} \\ &= \frac{4}{c(D)^2} \frac{M_{iii}}{M_{ii}^3} \left[\sum_{k=1}^{n} \left(\frac{-1}{c(D)^2 M_{kk}^2} \right) + 1 \right] = 0, \end{aligned}$$

$$S(D)(N(D), N(D)) = \frac{1}{2c(D)} \sum_{k=1}^{n} \frac{1}{M_{kk}^2} S(D)(F_{kk}, N(D)) = 0.$$

It means that at a given point $D \in \mathcal{M}_n$ for an orthonormal basis $(A_t)_{t\in I}$ in $\mathrm{T}_D\,\tilde{\mathcal{M}}_n$ the scalar curvature is

$$\mathrm{Scal}(D) = \sum_{t\in I, s\in I} S(A_s, A_s)S(A_t, A_t) - S(A_t, A_s)S(A_s, A_t).$$

At a point $D \in \mathcal{M}_n$ the set of matrices

$$\left\{\frac{1}{2M_{ii}}F_{ii}\right\}_{1\le i\le n} \bigcup \left\{\frac{1}{\sqrt{2}M_{ij}}F_{ij}\right\}_{1\le i<j\le n} \bigcup \left\{\frac{1}{\sqrt{2}M_{ij}}H_{ij}\right\}_{1\le i<j\le n}$$

form an orthonormal basis in $\mathrm{T}_D\,\tilde{\mathcal{M}}_n$ in the case of complex matrices.

It means that we have three kinds of basis elements: diagonal, off-diagonal real and off-diagonal complex ones.

Theorem: The scalar curvature of the real and complex state space $(\mathcal{M}_n, g_f)$ for an admissible function f at a point $D \in \mathcal{M}_n$ with eigenvalues $(\lambda_i)_{i=1,\dots,n}$ is

$$\mathrm{Scal}(D)_{\mathbb{R}} = \frac{4}{c(D)^4}\left[\sum_{i\neq k}^{n}\frac{M_{iii}M_{kkk}}{M_{ii}^3M_{kk}^3}\left(c(D)^2-\frac{1}{M_{ii}^2}-\frac{1}{M_{kk}^2}\right)\right.$$
$$\left.-\left(\sum_{k}^{n}\frac{M_{kkk}}{M_{kk}^3}\left(c(D)^2-\frac{1}{M_{kk}^2}\right)\right)\left(\sum_{i<j}^{n}\rho_{ij}\right)\right]+\frac{1}{c(D)^2}\left(\sum_{i<j}^{n}\rho_{ij}\right)^2$$
$$-\frac{1}{c(D)^2}\sum_{i<j}^{n}\rho_{ij}^2$$

$$\mathrm{Scal}(D)_{\mathbb{C}} = \frac{4}{c(D)^4}\left[\sum_{i\neq k}^{n}\frac{M_{iii}M_{kkk}}{M_{ii}^3M_{kk}^3}\left(c(D)^2-\frac{1}{M_{ii}^2}-\frac{1}{M_{kk}^2}\right)\right.$$
$$\left.-2\left(\sum_{k}^{n}\frac{M_{kkk}}{M_{kk}^3}\left(c(D)^2-\frac{1}{M_{kk}^2}\right)\right)\left(\sum_{i<j}^{n}\rho_{ij}\right)\right]+\frac{4}{c(D)^2}\left(\sum_{i<j}^{n}\rho_{ij}\right)^2$$
$$-\frac{2}{c(D)^2}\sum_{i<j}^{n}\rho_{ij}^2.$$

Symbols in the scalar curvature formula

$$M_{ii} = f'(\lambda_i), \quad M_{iii} = \frac{f''(\lambda_i)}{2}, \quad c(D) = \sqrt{\sum_{k=1}^{n} \frac{1}{f'(\lambda_k)^2}},$$

$$\rho_{ij} = \begin{cases} -\dfrac{1}{f'(\lambda_i) f'(\lambda_j)} \dfrac{f'(\lambda_i) - f'(\lambda_j)}{f(\lambda_i) - f(\lambda_j)} & \text{if } \lambda_i \neq \lambda_j \\ -\dfrac{f''(\lambda_i)}{f'(\lambda_i)^3} & \text{if } \lambda_i = \lambda_j. \end{cases}$$

We can test the theorem in three different cases:

1. If we restrict ourselves to the functions of the form

$$f(x) = \frac{2}{1-\alpha} x^{\frac{1-\alpha}{2}}$$

and to diagonal matrices, then we get back the scalar curvature of the classical α-geometry.

2. If we consider the full real or complex state space $\mathcal{M}_n$ and the function is $f(x) = 2\sqrt{x}$ then the pull-back metric is the `Wigner-Yanase metric`. In this case we map the state space to the surface of an Euclidean ball with radius $R = 2$. For a given state $D \in \mathcal{M}_n$ with eigenvalues $(\lambda_1, \ldots, \lambda_n)$ we have for the scalar curvatures:

$$\mathrm{Scal}_{\mathbb{R}}(D) = \frac{d_{\mathbb{R}}(d_{\mathbb{R}} - 1)}{R^2}$$

$$\mathrm{Scal}_{\mathbb{C}}(D) = \frac{d_{\mathbb{C}}(d_{\mathbb{C}} - 1)}{R^2}$$

which are just the well-known scalar curvatures of the Euclidean spheres in dimensions $d_{\mathbb{R}}$ and $d_{\mathbb{C}}$ with radius R.

3. Finally if we use the $f(x) = x$ function, then we map the state space into the flat Euclidean space, so the scalar curvature is 0.

Monotonicity conjecture

`Conjecture`*(Gibilisco and Isola)*: On the spaces $(\mathcal{P}_n, g_\alpha)$ and $(\mathcal{M}_n, g_\alpha)$ the scalar curvature is monotonously increasing, with respect to the majorization relation if $\alpha \in]-1, 0[$ and it is monotonously decreasing if $\alpha \in]0, 1[$.

Gibilisco and Isola proved a similar statement only for the curvature of the space $(\mathcal{P}_2, g_\alpha)$.

A linear map T on $\mathbb{R}^n$ is a T-**`transform`** if there exists $0 \le t \le 1$ and indices k, l such that $T(x_1, \ldots, x_n)$ is equal to

$$(x_1, \ldots, x_{k-1}, tx_k + (t-1)x_l, x_{k+1}, \ldots, x_{l-1}, (1-t)x_k + tx_l, x_{l+1}, \ldots, x_n).$$

For every $a \in \mathcal{P}_n$ and for every T transform $T(a) \prec a$.

For given $a, b \in \mathcal{P}_n$ if $a \prec b$, then we can go continuously from a to b using only T-transformations.

Theorem: Assume that we have $a, b \in \mathcal{P}_n$ with decreasingly ordered elements $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. The following statements are equivalent.

1. The distribution a is more mixed than b.
2. One can find a sequence $(c_z)_{z=1,\ldots,d}$ between them such that for all $z = 1, \ldots, d$: $c_z \in \mathcal{P}_n$,

   $$a = c_1 \prec c_2 \prec \cdots \prec c_d = b$$

   holds and the set of values of c_z and c_{z-1} is the same except two elements.
3. The set $(a_1, \ldots, a_n)$ can be obtained from $(b_1, \ldots, b_n)$ by a finite number of T-transforms.

According to this Theorem in order to prove the monotonicity of the scalar curvature with respect to the majorization, it is enough to consider those distributions which have only two different elements.

Corollary: The scalar curvature of the space $(\mathcal{P}_3, g_\alpha)$ at $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3) \in \mathcal{P}_3$ is

$$\mathrm{Scal}(\vartheta) = \frac{(1+\alpha)^2}{2} \frac{\vartheta_1^\alpha \vartheta_2^\alpha \vartheta_3^\alpha}{\left(\vartheta_1^{\alpha+1} + \vartheta_2^{\alpha+1} + \vartheta_3^{\alpha+1}\right)^2}.$$

Corollary: To prove Gibilisco's and Isola's Conjecture for the space $(\mathcal{P}_3, g_\alpha)$ it is enough to show that for every distribution $(a_1, a_2, a_3) \in \mathcal{P}_3$ if $a_1 > a_2$ then the function

$$\left[0, \frac{a_1 - a_2}{2}\right] \to \mathbb{R} \qquad x \mapsto \frac{(a_1 - x)^\alpha (a_2 + x)^\alpha}{\left((a_1 - x)^{\alpha+1} + (a_2 + x)^{\alpha+1} + a_3^{\alpha+1}\right)^2}$$

is decreasing if $\alpha \in \left]-1, 0\right[$ and increasing if $\alpha \in \left]0, 1\right[$.

Corollary: The scalar curvature of the real and complex state space $(\mathcal{M}_2, g_f)$ for an admissible function f at a point $D \in \mathcal{M}_2$ with eigenvalues λ_1 and λ_2 are

$$\mathrm{Scal}(D)_{\mathbb{R}} = 2x_2 \qquad \mathrm{Scal}(D)_{\mathbb{C}} = 4x_2 + 2x_4,$$

where

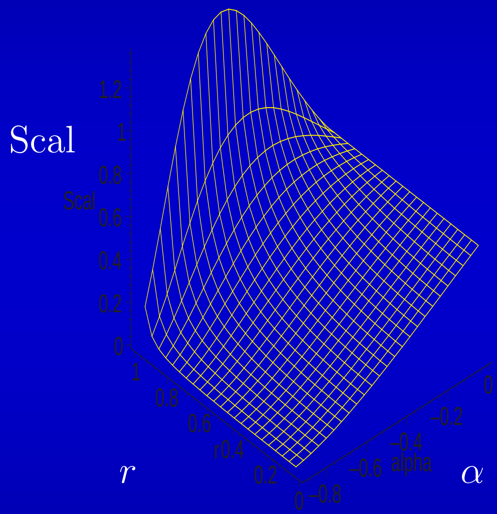
$$x_2 = \frac{f'(\lambda_1)f'(\lambda_2)}{(f'(\lambda_1)^2 + f'(\lambda_2)^2)^2}\left(\frac{f''(\lambda_1)}{f'(\lambda_1)} + \frac{f''(\lambda_2)}{f'(\lambda_2)}\right)\frac{f'(\lambda_1) - f'(\lambda_2)}{f(\lambda_1) - f(\lambda_2)}$$

$$x_4 = \frac{1}{f'(\lambda_1)^2 + f'(\lambda_2)^2}\left(\frac{f'(\lambda_1) - f'(\lambda_2)}{f(\lambda_1) - f(\lambda_2)}\right)^2.$$

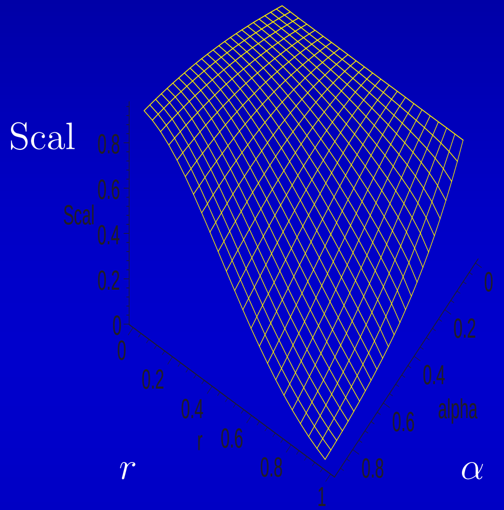
We write the eigenvalues of a state $D \in \mathcal{M}_2$ as $\frac{r+1}{2}$ and $\frac{r-1}{2}$, where r is the interval $]0,1[$. Using this parameter, for states $D_1, D_2 \in \mathcal{M}_2$ the relation $D_1 \prec D_2$ holds if and only if $r_1 \le r_2$. Numerically we computed the scalar curvature of the state space $(\mathcal{M}_2, g_\alpha)$ using Maple.

The scalar curvature of the real state space can be seen on the following graphs.

$\alpha \in]-1, 0[$:



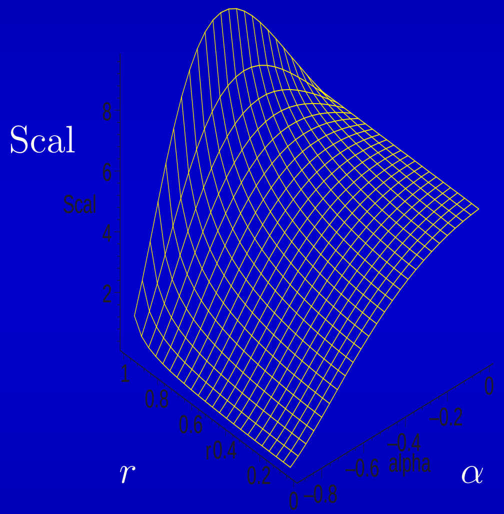
$\alpha \in \,]0,1[$:



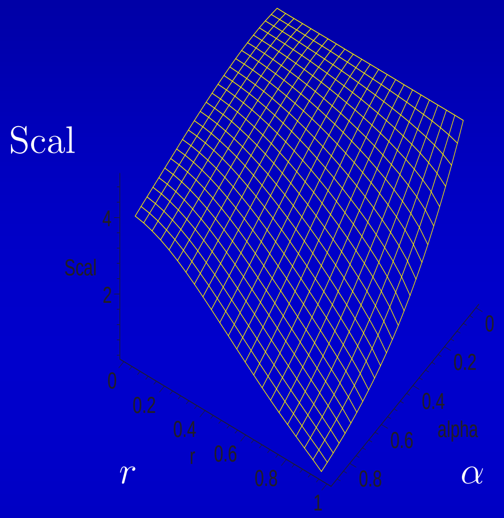
It seems that the scalar curvature is increasing with respect to the majorization if $\alpha \in \,]-1,0[$ and decreasing for parameters $\alpha \in \,]0,1[$.

The following graphs are about the scalar curvature of the complex state space $(\mathcal{M}_2, g_\alpha)$.

$\alpha \in \,]-1, 0[$:



$\alpha \in]0,1[$:



We can check again that the foreseen properties of the scalar curvature function seems to be true.

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