On the volume of statistical manifolds

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Outline:

– Quantum mechanical state space
– Hilbert–Schmidt measure
– Computing the volume of the state space:
    Lemmas and notations
    An Example
– Monotone metrics
– Volume of the space of Qubits
– Summary

Space of distributions on a finite set is

$$\mathcal{P}_n = \left\{ (p_1, \ldots, p_n) \in \mathbb{R}^n \;\middle|\; p_k > 0,\ \sum_{k=1}^{n} p_k = 1 \right\}$$

$$\mathcal{P}_n \ni (p_1, \ldots, p_n) \quad \Leftrightarrow \quad \begin{pmatrix} p_1 & 0 & \ldots & 0 \\ 0 & p_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_n \end{pmatrix}$$

$$\bigcap$$

$$\mathcal{M}_n \ni D \quad \Leftrightarrow \quad \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

$\mathcal{M}_n$: Set of $n \times n$ self-adjoint, positive definit, trace 1 matrices with real, complex or quaternionic entries.

(*State space.*)

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The Hilbert-Schmidt measure on $\mathcal{M}_n$ is defined by the Euclidean metric

$$d(D_1, D_2) = \sqrt{\mathrm{Tr}(D_1 - D_2)^2}\ .$$

We can consider $\mathcal{M}_n$ as a manifold with metric

$$g_D(X, Y) = \mathrm{Tr}(XY) \quad D \in \mathcal{M}_n \quad X, Y \in \mathrm{T}_D\,\mathcal{M}_n\ .$$

Induces the flat, Euclidean geometry on the set of states. The invariant volume measure is

$$\rho(D) = \sqrt{\det g_D} = 1\ .$$

(Which is the most simple prior on $\mathcal{M}_n$.) The volume of the state space is

$$\text{Volume} = \int_{\mathcal{M}_n} 1 \;\mathrm{d}D\ ,$$

where

$$\mathrm{d}D = \mathrm{d}a_{11}\,\mathrm{d}a_{12} \ldots \mathrm{d}a_{22}\,\mathrm{d}a_{23} \ldots \mathrm{d}a_{n-1,n}\ .$$

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This volume was computed by Zyczkowski and Sommers
(J. Phys. A 36, 10115–10130):

– using some integral formulas from random matrix theory
– and volume formulas for special flag manifolds.

They used the following decompostion of the measure:

$$\mathrm{d}V = \mathrm{d}\mu(\underbrace{\lambda_1, \dots, \lambda_n}_{\text{Eigenvalues}}) \times \mathrm{d}\nu_{\text{Haar}}\left(\begin{matrix}\text{orthogonal-}\\ \text{unitary-}\end{matrix}\ \text{group}\right)$$

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– More simple calculations
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Consider the simplex

$$\Delta_n = \left\{ (x_1, \ldots, x_n) \in \,]0,1[^n \;\middle|\; \sum_{k=1}^{n} x_k = 1 \right\}$$

$$\int_{\Delta_n} \left(\prod_{i=1}^{n} x_i^k \right) \; \mathrm{d}x_1 \ldots \mathrm{d}x_n =$$

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– using the beta integral

$$\int\limits_0^1 x^a (1-x)^b \, \mathrm{d}x = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}$$

integrate: $\int\limits_0^{1-(x_1+\cdots+x_{n-2})} \ldots \mathrm{d}x_{n-1}, \quad \int\limits_0^{1-(x_1+\cdots+x_{n-3})} \ldots \mathrm{d}x_{n-2}$, ...

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– collect the terms and simplify.

Lemma: Consider the simplex

$$\Delta_n = \left\{ (x_1, \dots, x_n) \in \,]0,1[^n \;\middle|\; \sum_{k=1}^{n} x_k = 1 \right\} \text{ then}$$

$$\boxed{\int\limits_{\Delta_n} \left(\prod_{i=1}^{n} x_i^k \right) \, \mathrm{d}x_1 \dots \mathrm{d}x_n = \frac{\Gamma(k+1)^n}{\Gamma(n(k+1))}}$$

Proof :

– Substitute $x_n = 1 - (x_1 + \dots + x_{n-1})$

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– collect the terms and simplify.

A simple consequence of the beta integral:

$$G_{a,b} := \int_0^1 r^a (1-r^2)^b \,\mathrm{d}x = \frac{1}{2} \frac{\Gamma(b+1)\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a}{2}+b+\frac{3}{2}\right)} \,.$$

The surface F_{n-1} of the unit sphere in an n dimensional space:

$$F_{n-1} = \frac{n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \,.$$

Assume that T is an $n \times n$, symmetric positive definite matrix, $\rho > 0$ and $\underline{x} = (x_1, \ldots, x_n)$. (The eigenvalues of T are $(\mu_i)_{i=1,\ldots,n}$.) Then we have

$$\int\limits_{\rho - \langle \underline{x}, T\underline{x} \rangle \geq 0} (\rho - \langle \underline{x}, T\underline{x} \rangle)^k \,\mathrm{d}\underline{x} =$$

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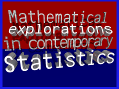
$$= \int_0^1 F_{n-1} r^{n-1} (1-r^2)^k \, \mathrm{d}r \times \sqrt{\frac{\rho^n}{\det T}} = F_{n-1} G_{n-1,k} \frac{\rho^{n/2+k}}{\sqrt{\det T}}$$



Introduction
HS measure
Lemmas...
... notations
An Example
The volume
qFisher metrics
Qubit case
Summary

In the real case:

$$\int\limits_{\rho-\langle \underline{x}, T\underline{x}\rangle \geq 0} (\rho - \langle \underline{x}, T\underline{x}\rangle)^k \, \mathrm{d}\underline{x} = \pi^{n/2} \frac{\Gamma(k+1)}{\Gamma(n+k+1)} \times \frac{\rho^{n/2+k}}{(\det T)^{1/2}}$$

If d is the dimension of the field ($\mathbb{R}$: $d=1$, $\mathbb{C}$: $d=2$, ...)

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Some Notations:

$$A_4 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12}^* & a_{22} & a_{23} & a_{24} \\ a_{13}^* & a_{23}^* & a_{33} & a_{34} \\ a_{14}^* & a_{24}^* & a_{34}^* & a_{44} \end{pmatrix}$$

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Lemma: $\det A_n = a_{nn}(\det A_{n-1}) - \langle \underline{x}_{n-1}, T_{n-1}\underline{x}_{n-1} \rangle.$

Let us compute the integral of the $\sqrt{\det}$ function on 4×4, complex density matrices ($\mathcal{M}_4$).

$$A_4 \in \mathcal{M}_4 \quad \Longleftrightarrow \quad \begin{cases} \det A_4 > 0, \det A_3 > 0 \\ \det A_2 > 0, \det A_1 > 0 \\ a_{11} + a_{22} + a_{33} + a_{44} = 1 \end{cases}$$

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Assume that the diagonal elements and submatrix A_3 are fixed, then the condition for $\underline{x}_3$ is:

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$$\begin{aligned} V(A_3) &= \int\limits_{a_{44} \det A_3 - \langle \underline{x}_3, T_3 \underline{x}_3 \rangle > 0} \sqrt{\det A_4} \, \mathrm{d}\underline{x}_3 \\ &= \int\limits_{a_{44} \det A_3 - \langle \underline{x}_3, T_3 \underline{x}_3 \rangle > 0} \left(a_{44} \det A_3 - \langle \underline{x}_3, T_3 \underline{x}_3 \rangle \right)^{1/2} \mathrm{d}\underline{x}_3 \end{aligned}$$

$$\int\limits_{\rho - \langle \underline{x}, T\underline{x} \rangle \geq 0} (\rho - \langle \underline{x}, T\underline{x} \rangle)^k \, \mathrm{d}\underline{x} = \frac{\pi^{dn/2} \Gamma(k+1)}{\Gamma\left(\frac{dn}{2} + k + 1\right)} \times \frac{\rho^{dn/2+k}}{(\det T)^{d/2}}$$

$$V(A_3) = \frac{8\pi^3}{105} a_{44}^{7/2} \times (\det A_3)^{3/2}$$

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$$= \frac{8\pi^3}{105} a_{44}^{7/2} \int\limits_{a_{33} \det A_2 - \langle \underline{x}_2, T_2 \underline{x}_2 \rangle > 0} (a_{33} \det A_2 - \langle \underline{x}_2, T_2 \underline{x}_2 \rangle)^{3/2} \, \mathrm{d}\underline{x}_2$$

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$$V(A_2) = \frac{8\pi^3}{105} a_{44}^{7/2} \times \frac{4\pi^2}{35} a_{33}^{7/2} \times (\det A_2)^{5/2}$$

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$$\begin{aligned} V(A_2) &= \int\limits_{a_{33} \det A_2 - \langle \underline{x}_2, T_2 \underline{x}_2 \rangle > 0} \frac{8\pi^3}{105} a_{44}^{7/2} (\det A_3)^{3/2} \, \mathrm{d}\underline{x}_2 \\ &= \frac{8\pi^3}{105} a_{44}^{7/2} \int\limits_{a_{33} \det A_2 - \langle \underline{x}_2, T_2 \underline{x}_2 \rangle > 0} (a_{33} \det A_2 - \langle \underline{x}_2, T_2 \underline{x}_2 \rangle)^{3/2} \, \mathrm{d}\underline{x}_2 \end{aligned}$$

$$\int\limits_{\rho - \langle \underline{x}, T\underline{x} \rangle \geq 0} (\rho - \langle \underline{x}, T\underline{x} \rangle)^k \, \mathrm{d}\underline{x} = \frac{\pi^{dn/2} \Gamma(k+1)}{\Gamma\left(\frac{dn}{2} + k + 1\right)} \times \frac{\rho^{dn/2+k}}{(\det T)^{d/2}}$$

$$V(A_2) = \frac{8\pi^3}{105} a_{44}^{7/2} \times \frac{4\pi^2}{35} a_{33}^{7/2} \times (\det A_2)^{5/2}$$

Assume that the diagonal elements are fixed, then the condition for $\underline{x}_1$ is:

$$a_{22}a_{11} - \langle \underline{x}_1, T_1 \underline{x}_1 \rangle > 0 \ .$$

$$V(A_1) = \int\limits_{a_{22}a_{11} - \langle \underline{x}_1, T_1 \underline{x}_1 \rangle > 0} \frac{32\pi^5}{3675} (a_{33}a_{44})^{7/2} (\det A_2)^{5/2} \, \mathrm{d}\underline{x}_1$$

$$= \frac{32\pi^5}{3675} (a_{33}a_{44})^{7/2} \int\limits_{a_{22}a_{11} - \langle \underline{x}_1, T_1 \underline{x}_1 \rangle > 0} (a_{22}a_{11} - \langle \underline{x}_1, T_1 \underline{x}_1 \rangle)^{5/2} \, \mathrm{d}\underline{x}_1$$

$$V(a_{11}, a_{22}, a_{33}, a_{44}) = \frac{64\pi^6}{25725} (a_{11}a_{22}a_{33}a_{44})^{7/2}$$

Now we integrate with respect to the diagonal elements:

$$V = \frac{64\pi^6}{25725} \int_{\Delta_4} \left(\prod_{i=1}^{4} x_{ii}^{7/2} \right) \mathrm{d}x_{11} \dots \mathrm{d}x_{44}$$

$$\int_{\Delta_n} \left(\prod_{i=1}^{n} x_i^k \right) \mathrm{d}x_1 \dots \mathrm{d}x_n = \frac{\Gamma(k+1)^n}{\Gamma(n(k+1))}$$

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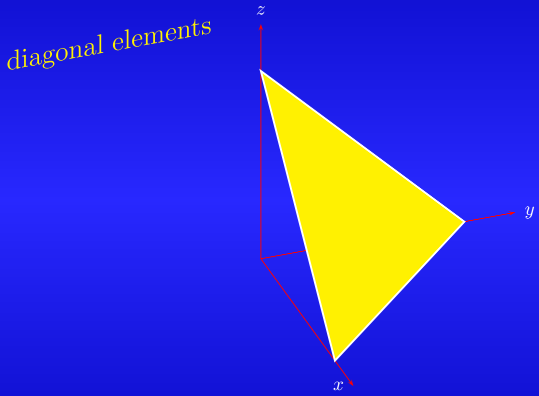
Decomposition of the state space: 3×3 real case:



Mathematical explorations in contemporary Statistics

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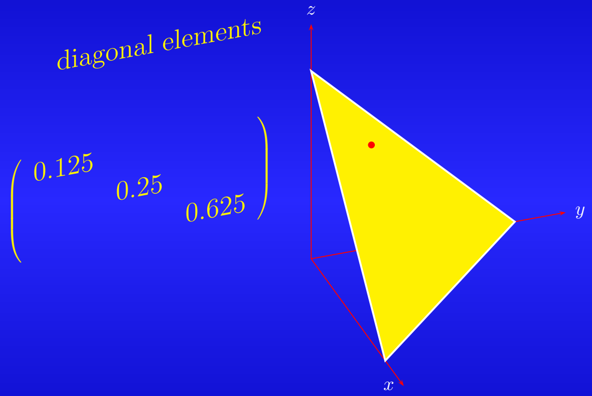

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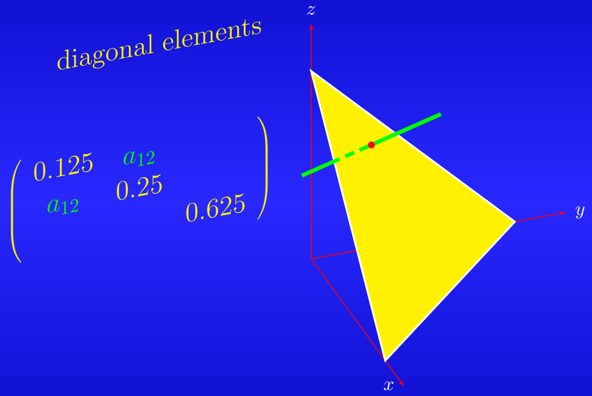
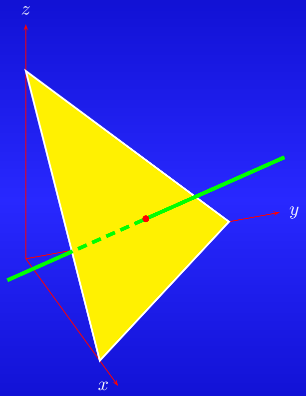


Decomposition of the state space: 3×3 real case:

diagonal elements

$$\begin{pmatrix} 0.25 & a_{12} & \\ a_{12} & 0.5 & \\ & & 0.25 \end{pmatrix}$$



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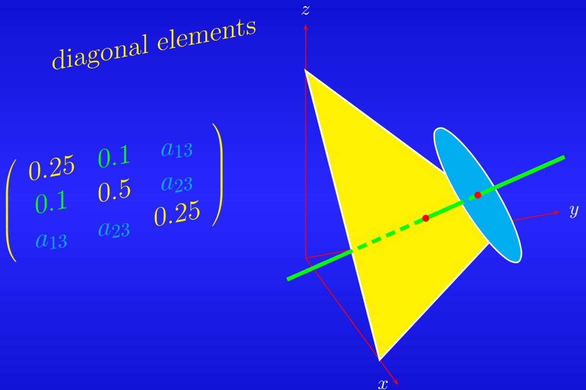

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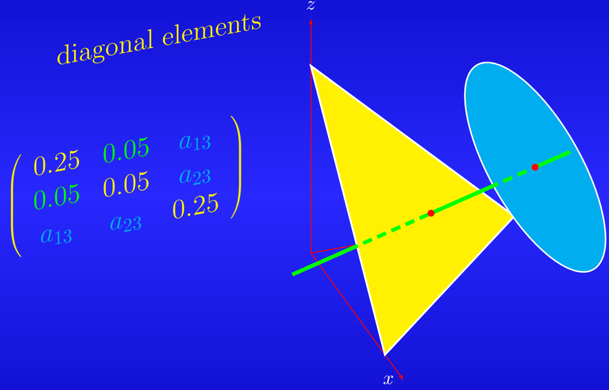
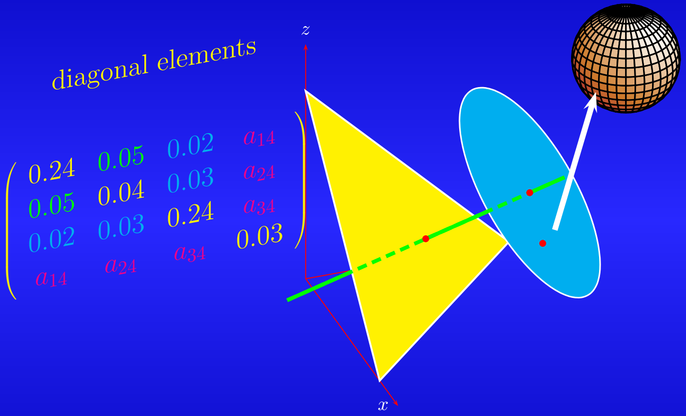


Decomposition of the state space: 4×4 real case:



Theorem: For every $n \in \mathbb{N}$ the volume of the state space $\mathcal{M}_n$ is

$$V(\mathcal{M}_n) = \frac{\pi^{dn(n-1)/4}}{\Gamma\left(d\frac{n(n-1)}{2} + n\right)} \prod_{i=1}^{n-1} \Gamma\left(\frac{id}{2} + 1\right)$$

and the integral of the function $\det^{\alpha}$ with respect to the normalized Hilbert–Schmidt measure is

$$\int\limits_{\mathcal{M}_n} \det{}^{\alpha} = \frac{\Gamma\left(\frac{dn(n-1)}{2} + n\right)}{\Gamma\left(\frac{dn(n-1)}{2} + n + n\alpha\right)} \prod_{i=1}^{n} \frac{\Gamma\left(d\frac{i-1}{2} + 1 + \alpha\right)}{\Gamma\left(d\frac{i-1}{2} + 1\right)} \, .$$

An operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}$ with the property $f(x) = xf(x^{-1})$ generates a monotone (under stochastic maps) Riemannian metric on $\mathcal{M}_n$ ($D \in \mathcal{M}_n$, $X, Y \in \mathrm{T}_D \mathcal{M}_n$)

$$g_D^{(f)}(X,Y) = \mathrm{Tr}\left(X \left(R_{n,D}^{\frac{1}{2}} f(L_{n,D} R_{n,D}^{-1}) R_{n,D}^{\frac{1}{2}} \right)^{-1} (Y) \right),$$

where $L_{n,D}(X) = DX$, $R_{n,D}(X) = XD$.

$g^{(f)}$ is considered as noncommutative Fisher information.

The Jeffreys' prior $\rho^{(f)}(D) \simeq \sqrt{\det g_D^{(f)}}$.

The volume of the state space with respect to these metrics:

$$V^{(f)} = \int_{\mathcal{M}_n} \rho^{(f)}(D) \, \mathrm{d} D = ???$$

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$$V^{(f)} = \int_{\mathcal{M}_n} \rho^{(f)}(D)\, \mathrm{d}D = ???$$

In the space of Qubits we use the Stokes parametrization:

$$D = \frac{1}{2} \begin{pmatrix} 1+x & y+\mathrm{i}\,z \\ y+\mathrm{i}\,z & 1-x \end{pmatrix} \; .$$

$\mathcal{M}_2$ can be identified with the unit ball in $\mathbb{R}^3$ and $\mathbb{R}^2$.
The Riemannian metric $g^{(f)}$ in this coordinate system is

$$g_f(x,y,z) = \frac{1}{2} \begin{pmatrix} \frac{1}{2\lambda_1\lambda_2} & 0 & 0 \\ 0 & \frac{1}{\lambda_1 f\left(\frac{\lambda_2}{\lambda 1}\right)} & 0 \\ 0 & 0 & \frac{1}{\lambda_1 f\left(\frac{\lambda_2}{\lambda 1}\right)} \end{pmatrix}$$

$$g_f(x,y) = \frac{1}{2} \begin{pmatrix} \frac{1}{2\lambda_1\lambda_2} & 0 \\ 0 & \frac{1}{\lambda_1 f\left(\frac{\lambda_2}{\lambda 1}\right)} \end{pmatrix} \; .$$

The volume is an integral on the unit ball, which is in spherical and polar coordinates

$$V\left(\mathcal{M}_2^{(\mathbb{C})}\right) = 4\pi \int_0^1 \frac{r^2}{\sqrt{1-r^2}(1+r) f\left(\frac{1-r}{1+r}\right)}\, \mathrm{d}r \ ,$$

$$V\left(\mathcal{M}_2^{(\mathbb{R})}\right) = 2\pi \int_0^1 \frac{r}{\sqrt{1-r}(1+r)\sqrt{f\left(\frac{1-r}{1+r}\right)}}\, \mathrm{d}r \ .$$

In another form:

$$V\left(\mathcal{M}_2^{(\mathbb{C})}\right) = 2\pi \int_0^1 \left(\frac{1-t}{1+t}\right)^2 \frac{1}{\sqrt{t} f(t)}\, \mathrm{d}t$$

$$V\left(\mathcal{M}_2^{(\mathbb{R})}\right) = \sqrt{2}\pi \int_0^1 \frac{1-t}{1+t} \frac{1}{\sqrt{t+t^2}\sqrt{f(t)}}\, \mathrm{d}t \ .$$

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Some operator monotone functions and the volumes:

$f(x)$:	$V\left(\mathcal{M}_2^{(\mathbb{C})}\right)$:	$V\left(\mathcal{M}_2^{(\mathbb{R})}\right)$:
$\frac{1+x}{2}$	π^2	2π
$\frac{2x}{1+x}$	∞	∞
$\frac{x-1}{\log x}$	$2\pi^2$	~ 8.298
$\sqrt{x}$	∞	4π
$(\sqrt{x}+1)^2/4$	$4\pi(\pi-2)$	$4\pi(2-\sqrt{2})$
$\frac{2\sqrt{x}(x-1)}{(1+x)\log x}$	∞	~ 19.986
$\frac{2(x-1)^2}{(1+x)(\log x)^2}$	$\pi^4/2$	~ 11.51
$\frac{2(\beta x+1-\beta)((1-\beta)x+\beta)}{x+1}$	$\pi^2\frac{1-2\sqrt{\beta-\beta^2}}{(1-2\beta)^2\sqrt{\beta-\beta^2}}$	$? < \infty$

Parameter: $\beta \in \left]0, \frac{1}{2}\right[$.

Lemma: $V\left(\mathcal{M}_2^{(\mathbb{R})}\right) < V\left(\mathcal{M}_2^{(\mathbb{C})}\right)$

Proof:

$$V\left(\mathcal{M}_2^{(\mathbb{R})}\right)^2 = \left| \int_0^1 \frac{1-t}{1+t} \, \frac{1}{\sqrt[4]{t}\sqrt{f(t)}} \times \frac{\sqrt{2}\pi\sqrt[4]{t}}{\sqrt{t+t^2}} \, \mathrm{d}t \right|^2$$

$$\leq \int_0^1 \left(\frac{1-t}{1+t}\right)^2 \frac{1}{\sqrt{t}f(t)} \, \mathrm{d}t \times \int_0^1 \frac{2\pi^2\sqrt{t}}{t+t^2} \, \mathrm{d}t$$

$$= V\left(\mathcal{M}_2^{(\mathbb{C})}\right) \times \frac{\pi^2}{2} \quad \Rightarrow V\left(\mathcal{M}_2^{(\mathbb{R})}\right)^2 \leq \frac{\pi^2}{2} V\left(\mathcal{M}_2^{(\mathbb{C})}\right)$$

$$2\pi \leq V\left(\mathcal{M}_2^{(\mathbb{R})}\right) \quad \Rightarrow \boxed{V\left(\mathcal{M}_2^{(\mathbb{R})}\right) \leq \frac{\pi}{4} V\left(\mathcal{M}_2^{(\mathbb{C})}\right)}$$

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$$\leq \int_0^1 \left(\frac{1-t}{1+t}\right)^2 \frac{1}{\sqrt{t}f(t)} \,\mathrm{d}t \times \int_0^1 \frac{2\pi^2\sqrt{t}}{t+t^2} \; \mathrm{d}t$$

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Define:

$$I_{\mathbb{R}} = \int_0^1 \frac{1}{\sqrt{t f(t)}}\ \mathrm{d}t \qquad I_{\mathbb{C}} = \int_0^1 \frac{1}{\sqrt{t f(t)}}\ \mathrm{d}t$$

Lemma: $$\frac{\pi}{2} I_{\mathbb{R}} - \frac{(4-\sqrt{2})\pi}{6} \le V\left(\mathcal{M}_2^{(\mathbb{R})}\right) \le \sqrt{2}\pi I_{\mathbb{R}}$$

$$\frac{\pi}{2} I_{\mathbb{C}} - \frac{\pi}{2}\left(\frac{16}{3} - \pi\right) \le V\left(\mathcal{M}_2^{(\mathbb{C})}\right) \le 2\pi I_{\mathbb{C}}$$

– The volume is equiconvergent with the integrals $I_{\mathbb{R}}$, $I_{\mathbb{C}}$.
– If the volume is infinite then it is concentrated at the boundary of the state space.

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Summary:

– The volume of the state space was computed.
– The volume of the space of qubits was studied.

Some open questions:

– The volume of the state space with respect to these metrics:

[illegible]

– The volume of separable states?

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Thank you for your attention!

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