The Mackey-formalism of Quantum Mechanics and Relativity Theories

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Outline:

– Brief introdution to the Mackey-model.
– Some results for the Galilei and the Poincare relativity.
– Problems with the general relativity in this model.

The Makey model

This model is about Quantum *mechanics*.
(Not about electromagnetic fields, not about interactions.)

Arguments for and against this model:

Against:
– Just about mechanics.
– Not about interactions of particles.
– One can argue against the axioms of the model.

For:
– Mathematically exact.
– Everything is based on axioms and theorems.
– Uses simple axioms.
– Very powerful tool.

The general Mackey-model: $\langle F, G, L, Q, T, \mathcal{M} \rangle$

– F the set of *events*, (problem for physicists)
– G transformation group (Galilei or Poincare group), such that $\forall g \in G$ there exists a

$$g : F \to F \quad E \mapsto gE$$

map,
– L is an orthomodular lattice,
– $T : F \to L \quad E \to T(E)$ map, (problem for physicists)
– $Q : G \to \mathrm{Aut}(L)$ representation of the group G, such that

$$Q_g(T(E)) = T(gE) \quad \forall E \in F, \, \forall g \in G$$

holds,
– $\mathcal{M}$ set of probability distributions on L, (*state space*).

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In the most studied model:

– G: Galilei or Poincare group,

– L: is the orthomodular lattice of projections of a separable complex Hilbert-space,
($\sim$ the lattice of closed subspaces),

– Q: is the projective representation of G.
(*Wigner Theorem*)

– M: set of self-adjoint, positive, trace one operators
(*Gleason Theorem*).

+ Topology:

– G is a *topological group*,

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Elementary Particles

Widely believed physical interpretation:

Elementary particles. $\Longleftrightarrow$ Irreducible Representations.

I. Simplification: (G must be a Lie-group)

Projective Representations. $\Longleftarrow$ Unitary Representations.

$$\text{Proj. Repr. of } G \Longleftarrow \text{Unit. Repr. of } (\tilde{G})_{i \in I}$$

II. Theory of Induced Unitary Representation:

Universal method to get every irreducible (continuous unitary) representation of *all locally compact* group.

The Galilei group:

– $\mathcal{G}_n = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times SO(n)$ as a set, the multiplication rule is

$$(t_1, x_1, v_1, R_1)(t_2, x_2, v_2, R_2) = (t_1 + t_2, x_1 + R_1 x_2 + v_1 t_2, v_1 + R_1 v_2, R_1 R_2).$$

– This is induced by the natural action:

$$(t, x, v, R) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \quad (\tau, \xi) \mapsto (t + \tau, A\xi + tv + x).$$

– The topology comes from the topology of $\mathbb{R}^m$.

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The Poincare group:

Define

$$G := \mathrm{Diag}(1, \underbrace{-1, \ldots, -1}_{n-1}).$$

The Lorentz group is

$$\mathcal{L}_n = \left\{ A \in \mathrm{Mat}_n \middle| \; A^t G A = G \right\}.$$

The Poincare group is $\mathcal{P}_n = \mathcal{L}_n \times \mathbb{R}^n$ as a set, and the multiplication rule is

$$(A_1, x_1)(A_2, x_2) = (A_1 A_2, x_1 + A_1 x_2).$$

– The topology comes from the topology of $\mathbb{R}^m$.

$\mathcal{P}_n$ is a Lie group.

Projective Representation of the $3+1$ Galilei group

I. Simplification:

$$\text{Proj. Repr. of } \mathcal{G}_3.$$
$$\Uparrow$$
$$\text{Unit. Repr. of } (\tilde{\mathcal{G}}_{3,m})_{m\in]0,\infty[}.$$

II. Theory of Induced Unitary Representation:
The unitary representations of the group $\tilde{\mathcal{G}}_{3,m}$ can be indexed by $j \in \frac{1}{2}\mathbb{N}$.

Particles are characterized by:

$$\boxed{\textit{mass } m \in]0,\infty[\text{ and } \textit{spin } j \in \tfrac{1}{2}\mathbb{N}.}$$

Moreover we have:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi$$

which is slightly different from the Schrödinger equation

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Projective Representation of the $3{+}1$ Poincare group

I. Simplification:

$$\text{Proj. Repr. of } \mathcal{P}_4.$$
$$\Uparrow$$
$$\text{Unit. Repr. of } \mathcal{P}_4.$$

II. Theory of Induced Unitary Representation:

(i) $v < c$,

(ii) $v = c$,

(iii) $v > c$.

(i) Slower than light particles are characterized by:

$$\boxed{\textit{mass } m \in]0, \infty[\text{ and } \textit{spin } j \in \tfrac{1}{2}\mathbb{N}.}$$

For given m, and $j = 1/2$ we have the free Dirac-equation:

$$\sum_{k=0}^{3} \gamma_k \frac{\partial}{\partial x_k} \psi = m\psi$$

One can deduce similarly the other wave equations for given spin.

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(ii) Light like particles:

There three type of particles:

– (a) which are characterized by the *spin* $j \in \mathbb{Z}$,
– (b) which are characterized by the continuous *spin* $j \in]0, \infty[$,
– (c) which are characterized by the continuous *spin* $j \in]0, \infty[$.

The direct sum of the spin 1 and the spin -1 representations of the particle type (a) is called *photon*.

One can deduce for photons the Maxwell wave equation:

$$\Box \Psi = 0.$$

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(iii) Faster than Light particles:

Theorem [Andai]: Three type of particles:

– (a) which are characterized by a *mass* $m \in]0, \infty[$ and a continuous *spin* $j \in \mathbb{R}$,

– (b) which are characterized by a *mass* $m \in]0, \infty[$ and a continuous *spin* $j \in]0, \infty[$,

– (c) which are characterized by a *mass* $m \in]0, \infty[$ and a discreet *spin* $j \in \frac{1}{2}\mathbb{Z}$.

The wave equation of these particles is still an open problem.

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The wave equation of these particles is still an open problem.

Projective Representation of the $2+1$ Galilei group

I. Simplification:

$$\text{Proj. Repr. of } \mathcal{G}_2.$$
$$\Uparrow$$
$$\text{Unit. Repr. of } (\tilde{\mathcal{G}}_{2,m,f,j})_{m,f,j\in[0,\infty[}.$$

II. Theory of Induced Unitary Representation:
The unitary representations of the group $\tilde{\mathcal{G}}_{2,m,f,j}$ gives many elementary particles.

List of the particles:

– (a) The particles with $m \neq 0$, $j = 0$, can be characterized by $\rho \in \mathbb{R}$, $n_0 \in \mathbb{Z} \setminus \{0\}$, $n_2 \in \mathbb{Z}$.

– (b) The particles with $m = 0$, $j \neq 0$, can be characterized by $n_0 \in \mathbb{Z} \setminus \{0\}$, $n_2 \in \mathbb{Z}$.

– (c) The particles with $m = 0$, $j \neq 0$, can be characterized by $\rho \in]0, \infty[$, $n_0, n_2 \in \mathbb{Z}$.

– (d) The particles with $m = 0$, $j = 0$, can be characterized by $\rho \in \mathbb{R}$, $n_0, n_2 \in \mathbb{Z}$.

– (e) The particles with $m = 0$, $j = 0$, can be characterized by $\rho \in]0, \infty[$, $n_0, n_2 \in \mathbb{Z}$.

– (f) The particles with $m \neq 0$, $j \neq 0$, can be characterized by $n_0, n_2 \in \mathbb{Z} \setminus \{0\}$.

Deeper analysis: f: something like "magnetic force".

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Projective Representation of the $n+1$ Galilei group

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$$\text{Unit. Repr. of } (\tilde{\mathcal{G}}_{n,m})_{m\in]0,\infty[}.$$

II. Theory of Induced Unitary Representation:

Future plan.

Towards General Relativity

In this case G is the diffeomorphism group of the space-time.

Then G is an infinite dimensional Lie group.

I. Simplification:
This step maybe could be done, but the result is unknown.

II. Theory of Induced Unitary Representation:
Since G is *not* locally compact, this is still problem for mathematicians!

References

Foundation of the Theory (among others):
1. G. W. Mackey: Induced representations.
2. V. Bargmann: Unitary ray representations, Group Theoretic wave equations.
3. C. C. Moore: Cohomology Theory of locally compact groups.
4. M. H. Stone: Linear Transformations in Hilbert spaces.
5. V. S. Varadarajan: Geometry of Quantum Theory.
6. J. Wigner: Unitary Representations.

Nowadays (among others):
D. R. Grigore: Repr. of the Galilei, Poincare group.
S. K. Bose: Repr. of the Galilei, Poincare group.

Thank you.