Convergence Theorems for Integrals through Product Measure

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Our intuition captures the integral as the area under the function. In Measure Theory, integrals are defined for measurable functions. In the book "The Lebesgue Integral," J.C. Burkill provided a characterisation of measurability of a function $f : \mathbb{R} \to \mathbb{R}$ through measurability of the set of points (x, y) in the $\mathbb{R}^2$ plane for which $0 \leq y < f(x)$ holds [1]. We shall call this set a shading of the function $f(x)$. Indeed, the measure of the shading of $f(x)$ agrees with its corresponding integral. Furthermore, it is indifferent for the integral whether the points on the curve are included in the shading. Consequently, many convergence theorems for integrals can be proven by the continuity of measure. Burkill's approach exploits the σ-finite nature of the measure space on $\mathbb{R}^2$. This thesis aims to extend the result to integrals of functions on a measure space not necessarily σ-finite. To the best of our knowledge, we are not aware of any publications done on the topic.

Without the σ-finite property, one loses the privilege to extend results from finite measure space. To worsen the situation, the product measure may not be unique. This raises a question whether we can define a shading of a measurable function as a set in the product σ-algebra whose measure is independent of the product measure used. We will see that this is indeed the case. It is indifferent, for the product measure constructed by the Carathéodory's extension theorem, whether the points on the curve are included. This result will be sufficient for proving convergence theorems for integral by the continuity of measure. To facilitate the proof, we introduce the shady isomorphism whose role highlights highlight the relationships between (i) integral and product measure, (i) function and its shading, (iii) union and supremum, and (iv) intersection and infimum.

[1] John Burkill, *The Lebesgue Integral*, Cambridge University Press, London, 1958.