19. Határozatlan integrál 1. megoldása

I.

1. $$\int \frac{1+x}{\sqrt{x}}\,\mathrm{d}\,x = \int x^{-\frac{1}{2}} + x^{\frac{1}{2}}\,\mathrm{d}\,x = 2\sqrt{x} + \frac{2}{3}x^{\frac{3}{2}} + C$$
2. $$\int \frac{1}{x^2}\,\mathrm{d}\,x = -\frac{1}{x} + C$$
3. $$\int 1 + \mathrm{e}^{x-1}\,\mathrm{d}\,x = x + \mathrm{e}^{x-1} + C$$
4. $$\int \frac{x^2-1}{x^2+1}\,\mathrm{d}\,x = \int \frac{(x^2+1)-2}{x^2+1}\,\mathrm{d}\,x = x - 2\operatorname{arctg} x + C$$
5. $$\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}\,x = \operatorname{arsh} x + C$$
6. $$\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}\,x = \operatorname{arch} x + C$$
7. $$\int \frac{1}{\sin^2} = -\operatorname{ctg} + C$$
8. $$\int \frac{1}{\cos^2} = \operatorname{tg} + C$$
9. $$\int \frac{1}{\operatorname{sh}^2} = -\operatorname{cth} + C$$
10. $$\int \frac{1}{\operatorname{ch}^2} = \operatorname{th} + C$$
11. $$\int \operatorname{sh} = \operatorname{ch} + C$$
12. $$\int \operatorname{ch} = \operatorname{sh} + C$$

II. A parciális integrálás segítségével határozzuk meg az alábbi integrálokat, ahol $a, b \in \mathbb{R} \setminus \{0\}$ és $k \in \mathbb{N}$.

1. $\displaystyle\int xe^{ax} \,\mathrm{d}\, x = \frac{1}{a^2}\,\mathrm{e}^{ax}(ax-1) + C$
2. $\displaystyle\int x^2\,\mathrm{e}^{-ax} \,\mathrm{d}\, x = -\frac{1}{a^3}\,\mathrm{e}^{-ax}(a^2x^2 + 2ax + 2) + C$
3. $\displaystyle\int x\sin x \,\mathrm{d}\, x = \sin x - x\cos x + C$
4. $\displaystyle\int \mathrm{e}^{ax}\sin bx \,\mathrm{d}\, x = \frac{\mathrm{e}^{ax}}{a^2+b^2}(a\sin(bx) - b\cos(bx)) + C$
5. $\displaystyle\int \mathrm{e}^{x}\cos x \,\mathrm{d}\, x = \frac{\mathrm{e}^{x}}{2}(\sin x + \cos x) + C$
6. $\displaystyle\int \sqrt{1-x^2} \,\mathrm{d}\, x = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$
7. $\displaystyle\int \sqrt{1+x^2} \,\mathrm{d}\, x = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\operatorname{arsh} x + C$
8. $\displaystyle\int \sqrt{x^2-1} \,\mathrm{d}\, x = \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\operatorname{arch} x + C$
9. $\displaystyle\int \arcsin x \,\mathrm{d}\, x = x\arcsin x + \sqrt{1-x^2} + C$
10. $\displaystyle\int \operatorname{arctg} x \,\mathrm{d}\, x = x\operatorname{arctg} x - \frac{1}{2}\ln(1+x^2) + C$
11. $\displaystyle\int x\operatorname{arctg} ax \,\mathrm{d}\, x = \frac{x^2}{2}\operatorname{arctg}(ax) + \frac{1}{2a^2}\operatorname{arctg}(ax) - \frac{x}{2a} + C$
12. $\displaystyle\int x^3\ln^2 x \,\mathrm{d}\, x = \frac{x^4}{4}\ln^2 x - \frac{x^4}{8}\ln x + \frac{x^4}{32} + C$

III. A következőkben a racionális törtfüggvényekre vonatkozó integrálási szabályt alkalmazzuk.

1. $\displaystyle\int \frac{1}{1-x^2}\,\mathrm{d}\,x = \int \frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{2}\cdot\frac{1}{x-1}\,\mathrm{d}\,x = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C$
2. $\displaystyle\int \frac{1}{x^2-2x-3}\,\mathrm{d}\,x = \int \frac{1}{4}\cdot\frac{1}{x-3} - \frac{1}{4}\cdot\frac{1}{x+1}\,\mathrm{d}\,x = \frac{1}{4}\ln|x-3| - \frac{1}{4}\ln|x+1| + C$
3. $\displaystyle\int \frac{1}{x^2+2x+6}\,\mathrm{d}\,x = \frac{1}{\sqrt{5}}\operatorname{arctg}\left(\frac{x+1}{\sqrt{5}}\right) + C$
4. $\displaystyle\int \frac{x^2-1}{(x+2)^3}\,\mathrm{d}\,x = \int \frac{1}{x+2} - \frac{4}{(x+2)^2} + \frac{3}{(x+2)^3}\,\mathrm{d}\,x =$
   $\displaystyle= \ln|x+2| + \frac{4}{x+2} - \frac{3}{2(x+2)^2} + C$
5. $\displaystyle\int \frac{1}{x^3+1}\,\mathrm{d}\,x = \int \frac{1}{3}\cdot\frac{1}{x+1} - \frac{1}{3}\cdot\frac{x-2}{x^2-x+1}\,\mathrm{d}\,x =$
   $\displaystyle= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left|x^2-x+1\right| + \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right) + C$
6. $\displaystyle\int \frac{x^4}{(x-2)(x-3)(x-4)}\,\mathrm{d}\,x = \int x + 9 + \frac{8}{x-2} - \frac{81}{x-3} + \frac{128}{x-4}\,\mathrm{d}\,x =$
   $\displaystyle= \frac{x^2}{2} + 9x + 8\ln|x-2| - 81\ln|x-3| + 128\ln|x-4| + C$
7. $\displaystyle\int \frac{16x^2+4x}{x^4+4}\,\mathrm{d}\,x = \int \frac{4x+1}{x^2-2x+2} - \frac{4x+1}{x^2+2x+2}\,\mathrm{d}\,x =$
   $\displaystyle= 2\ln\left|x^2-2x+2\right| + 5\operatorname{arctg}(x-1) - 2\ln\left|x^2+2x+5\right| + 3\operatorname{arctg}(x+1) + C$
8. $\displaystyle\int \frac{x^3}{(x^2+1)^2}\,\mathrm{d}\,x = \int \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\,\mathrm{d}\,x = \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C$
9. $\displaystyle\int \frac{x^4+4}{x^3-1}\,\mathrm{d}\,x = \int x + \frac{5}{3}\cdot\frac{1}{x-1} - \frac{5x+7}{3(x^2+x+1)}\,\mathrm{d}\,x =$
   $\displaystyle= \frac{x^2}{2} + \frac{5}{3}\ln|x-1| - \frac{5}{6}\ln\left|x^2+x+1\right| - \sqrt{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$
10. $\displaystyle\int \frac{1}{(x^2+x+1)^2}\,\mathrm{d}\,x = \frac{2x+1}{3(x^2+x+1)} + \frac{4}{3\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$
11. $\displaystyle\int \frac{x}{(x^2+2x+2)^2}\,\mathrm{d}\,x = -\frac{x+2}{2(x^2+2x+2)} - \frac{1}{2}\operatorname{arctg}(x+1) + C$
12. $\displaystyle\int \frac{1}{(x^2+2x+2)^2}\,\mathrm{d}\,x = \frac{x+1}{2(x^2+2x+2)} + \frac{1}{2}\operatorname{arctg}(x+1) + C$