20. Határozatlan integrál 2. megoldása

I.

1. $t = x+2$ $\quad \displaystyle\int \frac{x^3}{(x+2)^4}\,\mathrm{d}\,x = \int \frac{(t-2)^3}{t^4}\,\mathrm{d}\,t = \ln|x+2| + \frac{6}{x+2} - \frac{6}{(x+2)^2} + \frac{8}{3(x+2)^3} + C$

2. $t = \sqrt{1+x}$ $\quad \displaystyle\int \frac{1}{\sqrt{1+x}+(\sqrt{1+x})^3}\,\mathrm{d}\,x = \int \frac{2}{1+t^2}\,\mathrm{d}\,t = 2\,\mathrm{arctg}\,\sqrt{1+x} + C$

3. $t = \sqrt[4]{x-1}$ $\quad \displaystyle\int x\sqrt[4]{x-1}\,\mathrm{d}\,x = \int 4t^4 + 4t^8\,\mathrm{d}\,t = \frac{4}{5}(x-1)^{\frac{5}{4}} + \frac{4}{9}(x-1)^{\frac{9}{4}} + C$

4. $t = \mathrm{e}^x$ $\quad \displaystyle\int \frac{\mathrm{e}^{4x}}{1+\mathrm{e}^x}\,\mathrm{d}\,x = \int \frac{t^3}{1+t}\,\mathrm{d}\,t = \frac{\mathrm{e}^{3x}}{3} - \frac{\mathrm{e}^{2x}}{2} + \mathrm{e}^x - \ln(1+\mathrm{e}^x) + C$

5. $t = \sqrt{\mathrm{e}^x - 1}$ $\quad \displaystyle\int \sqrt{\mathrm{e}^x - 1}\,\mathrm{d}\,x = \int \frac{2t^2}{t^2+1}\,\mathrm{d}\,t = 2\sqrt{\mathrm{e}^x - 1} - 2\,\mathrm{arctg}(\sqrt{\mathrm{e}^x - 1}) + C$

6. $t = \sqrt{x}$ $\quad \displaystyle\int \sqrt{x}\,\mathrm{e}^{\sqrt{x}}\,\mathrm{d}\,x = \int 2t^2\,\mathrm{e}^t\,\mathrm{d}\,t = 2\,\mathrm{e}^{\sqrt{x}}(x - 2\sqrt{x} + 2) + C$

7. $t = \arcsin x$ $\quad \displaystyle\int \sqrt{1-x^2}\,\mathrm{d}\,x = \int \cos^2 t\,\mathrm{d}\,t = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$

8. $t = \sin x$ $\quad \displaystyle\int \sin^k x \cdot \cos x\,\mathrm{d}\,x = \int t^k\,\mathrm{d}\,t = \frac{\sin^{k+1} x}{k+1} + C$

9. $t = \cos x$ $\quad \displaystyle\int \cos^k x \cdot \sin x\,\mathrm{d}\,x = \int -t^k\,\mathrm{d}\,t = -\frac{\cos^{k+1} x}{k+1} + C$

10. $t = \sin x$ $\quad \displaystyle\int \mathrm{ctg}\,x\,\mathrm{d}\,x = \int \frac{1}{t}\,\mathrm{d}\,t = \ln|\sin x| + C$

11. $t = \cos x$ $\quad \displaystyle\int \mathrm{tg}\,x\,\mathrm{d}\,x = \int -\frac{1}{t}\,\mathrm{d}\,t = -\ln|\cos x| + C$

12. $t = \mathrm{tg}\,x$ $\quad \displaystyle\int \mathrm{tg}^2\,x\,\mathrm{d}\,x = \int \frac{t^2}{1+t^2}\,\mathrm{d}\,t = \mathrm{tg}\,x - x + C$

13. $t = \mathrm{tg}\,x$ $\quad \displaystyle\int \mathrm{tg}^4\,x\,\mathrm{d}\,x = \int \frac{t^4}{1+t^2}\,\mathrm{d}\,t = \frac{\mathrm{tg}^3\,x}{3} - \mathrm{tg}\,x + x + C$

14. $t = \mathrm{e}^x$ $\quad \displaystyle\int \frac{2}{\mathrm{e}^{3x} - \mathrm{e}^x}\,\mathrm{d}\,x = \int \frac{2}{t^4 - t^2}\,\mathrm{d}\,t = \ln|\mathrm{e}^x - 1| - \ln|\mathrm{e}^x + 1| + 2\,\mathrm{e}^{-x} + C$

15. $t = \sqrt{x-1}$ $\quad \displaystyle\int \frac{1}{x + \sqrt{x-1} - 1}\,\mathrm{d}\,x = \int \frac{2}{t+1}\,\mathrm{d}\,t = 2\ln(\sqrt{x-1} + 1) + C$

II. A helyettesítés $t = \operatorname{tg} x$.

1. $$\int \frac{1}{\operatorname{tg} x - 1}\,\mathrm{d}\,x = \int \frac{1}{(t-1)(t^2+1)}\,\mathrm{d}\,t = \int \frac{1}{2}\cdot\frac{1}{t-1} - \frac{1}{2\cdot}\frac{t+1}{t^2+1}\,\mathrm{d}\,t =$$
$$= \frac{1}{2}\ln|\operatorname{tg} x - 1| - \frac{1}{4}\ln(1+\operatorname{tg}^2 x) - \frac{x}{2} + C$$

2. $$\int \frac{1}{\sin^2 x + \sin 2x}\,\mathrm{d}\,x = \int \frac{1}{t(t+2)}\,\mathrm{d}\,t = \int \frac{1}{2t} - \frac{1}{2}\cdot\frac{1}{t+2}\,\mathrm{d}\,t = \frac{1}{2}\ln|\operatorname{tg} x| - \frac{1}{2}\ln|2+\operatorname{tg} x| + C$$

3. $$\int \frac{1+\operatorname{tg}^2 x}{1-\operatorname{tg}^2 x}\,\mathrm{d}\,x = \int \frac{1}{1-t^2}\,\mathrm{d}\,t = \int \frac{1}{2}\cdot\frac{1}{t+1} - \frac{1}{2}\cdot\frac{1}{t-1}\,\mathrm{d}\,t == \frac{1}{2}\ln\left|\frac{\operatorname{tg} x+1}{\operatorname{tg} x-1}\right| + C$$

4. $$\int \frac{1}{1+3\cos^2 x}\,\mathrm{d}\,x = \int \frac{1}{t^2+4}\,\mathrm{d}\,t = \frac{1}{2}\operatorname{arctg}\left(\frac{1}{2}\operatorname{tg} x\right) + C$$

III.

1. $$\int x\,\mathrm{e}^{-x^2}\,\mathrm{d}\,x = -\frac{1}{2}\,\mathrm{e}^{-x^2} + C$$

2. $$\int \frac{3^x}{1+9^x}\,\mathrm{d}\,x = \frac{1}{\ln 3}\operatorname{arctg} 3^x + C$$

3. $$\int \frac{\mathrm{e}^x}{\sqrt{1-\mathrm{e}^{2x}}}\,\mathrm{d}\,x = \arcsin \mathrm{e}^x + C$$

4. $$\int \frac{\mathrm{e}^x}{\sqrt[3]{1+\mathrm{e}^x}}\,\mathrm{d}\,x = \frac{3}{2}(1+\mathrm{e}^x)^{\frac{2}{3}} + C$$

5. $$\int \frac{\cos\ln x}{x}\,\mathrm{d}\,x = \sin\ln|x| + C$$

6. $$\int \frac{\cos x}{1+\sin^2 x}\,\mathrm{d}\,x = \operatorname{arctg}\sin x + C$$

IV.

1. $$\int \cos 2x\cos 5x\,\mathrm{d}\,x = \int \frac{1}{2}\cos(3x) + \frac{1}{2}\cos(7x)\,\mathrm{d}\,x = \frac{1}{6}\sin(3x) + \frac{1}{14}\sin(7x) + C$$

2. $$\int \cos^5 x\sin^2 x\,\mathrm{d}\,x = \int \cos x(\sin^2 x - 2\sin^4 x + \sin^6 x)\,\mathrm{d}\,x = \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$$

3. $$\int \cos^5 x\,\mathrm{d}\,x = \int \cos x(1 - 2\sin^2 x + \sin^4 x)\,\mathrm{d}\,x = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

4. $$\int \cos^5 x\sin^3 x\,\mathrm{d}\,x = \int \cos x(\sin^3 x - 2\sin^5 x + \sin^7 x)\,\mathrm{d}\,x = \frac{1}{4}\sin^4 x - \frac{1}{3}\sin^6 x + \frac{1}{8}\sin^8 x + C$$

5. $$\int \sin^4 x\,\mathrm{d}\,x = \int \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\,\mathrm{d}\,x = \frac{3}{8}x - \frac{1}{4}\cos(2x) + \frac{1}{32}\sin(4x) + C$$

6. $$\int \cos^5 x\sin^5 x\,\mathrm{d}\,x = \int \cos x(\sin^5 x - 2\sin^7 x + \sin^9 x)\,\mathrm{d}\,x = \frac{1}{6}\sin^6 x - \frac{1}{4}\sin^8 x + \frac{1}{10}\sin^{10} x + C$$