Határozott integrál alkalmazásainak megoldása

I. Az integrálformulákból közvetlenül adódnak a formulák.

II. 1. Ivhosszszámítás. $L_f = \int_I \sqrt{1+f'(x)^2}\,\mathrm{d}\,x$

a. $L_f = \int_0^b \sqrt{1+a^2}\,\mathrm{d}\,x = \left[x\sqrt{1+a^2}\right]_0^b = b\sqrt{1+a^2}$

b. $L_f = \int_0^b \sqrt{1+4a^2x^2}\,\mathrm{d}\,x = \left[\frac{x\sqrt{1+4a^2x^2}}{2} + \frac{\operatorname{arsh}(2ax)}{4a}\right]_0^b = \frac{b\sqrt{1+4a^2b^2}}{2} + \frac{\operatorname{arsh}(2ab)}{4a}$

c. $L_f = \int_0^b \sqrt{1+a^2\,\mathrm{e}^{2ax}}\,\mathrm{d}\,x = \left[\frac{\sqrt{1+a^2\,\mathrm{e}^{2ax}}}{a} - \frac{1}{a}\cdot\operatorname{arth}\left(\frac{1}{\sqrt{1+a^2\,\mathrm{e}^{2ax}}}\right)\right]_0^b =$

$= \frac{\sqrt{1+a^2\,\mathrm{e}^{2ab}} - \sqrt{1+a^2}}{a} - \frac{1}{a}\cdot\operatorname{arth}\left(\frac{1}{\sqrt{1+a^2\,\mathrm{e}^{2ab}}}\right) + \frac{1}{a}\cdot\operatorname{arth}\left(\frac{1}{\sqrt{1+a^2}}\right)$

d. $L_f = \int_0^b \operatorname{ch} x\,\mathrm{d}\,x = [\operatorname{sh} x]_0^b = \operatorname{sh} b$

2. Felszínszámítás. $F_f = 2\pi\int_I f(x)\sqrt{1+f'(x)^2}\,\mathrm{d}\,x$

a. $F_f = 2\pi\int_0^b ax\sqrt{1+a^2}\,\mathrm{d}\,x = \pi\left[ax^2\sqrt{1+a^2}\right]_0^b = \pi ab^2\sqrt{1+a^2}$

b. $F_f = 2\pi\int_0^b ax^2\sqrt{1+4a^2x^2}\,\mathrm{d}\,x \overset{t=\operatorname{arsh}(2ax)}{=} \frac{\pi}{4a^2}\int_0^{\operatorname{arsh}(2ab)} \operatorname{sh}^2 t\operatorname{ch}^2 t\,\mathrm{d}\,t = \frac{\pi}{16a^2}\int_0^{\operatorname{arsh}(2ab)} \operatorname{sh}^2(2t)\,\mathrm{d}\,t =$

$= \left[\frac{\pi}{32a^2}\left(\frac{\operatorname{sh}(4t)}{4} - t\right)\right]_0^{\operatorname{arsh}(2ab)} = \left[\frac{\pi}{32a^2}\left(\operatorname{sh} t\operatorname{ch} t(1+2\operatorname{sh}^2 t) - t\right)\right]_0^{\operatorname{arsh}(2ab)} =$

$= \frac{\pi b(1+8a^2b^2)\sqrt{1+4a^2b^2}}{16a} - \frac{\pi\operatorname{arsh}(2ab)}{32a^2}$

c. $F_f = 2\pi\int_0^b \mathrm{e}^{ax}\sqrt{1+a^2\,\mathrm{e}^{2ax}}\,\mathrm{d}\,x \overset{t=a\,\mathrm{e}^{ax}}{=} \frac{2\pi}{a^2}\int_1^{a\,\mathrm{e}^{ab}} \sqrt{1+t^2}\,\mathrm{d}\,t =$

$= \frac{\pi}{a^2}\left[\operatorname{arsh} t + t\sqrt{1+t^2}\right]_1^{a\,\mathrm{e}^{ab}} = \frac{\pi}{a^2}\left(\operatorname{arsh}(a\,\mathrm{e}^{ab}) - \operatorname{arsh} 1 + a\,\mathrm{e}^{ab}\sqrt{1+a^2\,\mathrm{e}^{2ab}} - \sqrt{2}\right)$

d. $F_f = 2\pi\int_0^b \operatorname{ch}^2 x\,\mathrm{d}\,x = [\pi(x+\operatorname{sh} x\operatorname{ch} x)]_0^b = \pi b + \pi\operatorname{sh} b\operatorname{ch} b$

3. Térfogatszámítás. $V_f = \pi \int_I f^2(x)\,\mathrm{d}\,x$

a. $V_f = \pi \int_0^b a^2x^2\,\mathrm{d}\,x = \left[\frac{\pi a^2 x^3}{3}\right]_0^b = \frac{\pi a^2 b^3}{3}$

b. $V_f = \pi \int_0^b a^2x^4\,\mathrm{d}\,x = \left[\frac{\pi a^2 x^5}{5}\right]_0^b = \frac{\pi a^2 b^5}{5}$

c. $V_f = \pi \int_0^b \mathrm{e}^{2ax}\,\mathrm{d}\,x = \left[\frac{\pi\,\mathrm{e}^{2ax}}{2a}\right]_0^b = \frac{\pi}{2a}\left(\mathrm{e}^{2ab}-1\right)$

d. $V_f = \pi \int_0^b \mathrm{ch}^2(x)\,\mathrm{d}\,x = \left[\frac{\pi}{2}\,\mathrm{sh}\,x\,\mathrm{ch}\,x + \frac{\pi x}{2}\right]_0^b = \frac{\pi}{4}\,\mathrm{sh}(2b) + \frac{\pi b}{2}$

4. Görbe súlypontja. $x_s(L_f) = \frac{1}{L_f}\int_I x\sqrt{1+f'(x)^2}\,\mathrm{d}\,x$, $y_s(L_f) = \frac{1}{L_f}\int_I f(x)\sqrt{1+f'(x)^2}\,\mathrm{d}\,x$ (Vegyük észre, hogy $y_s = \frac{F_f}{2\pi L_f}$ teljesül.)

a. $x_s = \frac{1}{L_f}\int_0^b x\sqrt{1+a^2}\,\mathrm{d}\,x = \frac{1}{L_f}\left[\frac{x^2\sqrt{1+a^2}}{2}\right]_0^b = \frac{b}{2}$

$y_s = \frac{a}{2}$

b. $x_s = \frac{1}{L_f}\int_0^b x\sqrt{1+4a^2x^2}\,\mathrm{d}\,x = \frac{1}{L_f}\left[\frac{\left(1+4a^2x^2\right)^{3/2}}{12a^2}\right]_0^b = \frac{\left(1+4a^2b^2\right)^{3/2}-1}{12a^2L_f}$

$y_s = \frac{2ab(1+8a^2b^2)\sqrt{1+4a^2b^2} - \mathrm{arsh}(2ab)}{32a^2b\sqrt{1+4a^2b^2} + 16a\,\mathrm{arsh}(2ab)}$

d. $x_s = \frac{1}{L_f}\int_0^b x\,\mathrm{ch}\,x\,\mathrm{d}\,x = \frac{1}{L_f}\left[x\,\mathrm{sh}\,x - \mathrm{ch}\,x\right]_0^b = b - \frac{\mathrm{ch}\,b - 1}{\mathrm{sh}\,b}$

$y_s = \frac{b + \mathrm{sh}\,b\,\mathrm{ch}\,b}{2\,\mathrm{sh}\,b}$

5. Síkbeli alakzat súlypontja. $x_s(T_f) = \frac{\int_I xf(x)\,\mathrm{d}\,x}{\int_I f(x)\,\mathrm{d}\,x}$, $y_s(T_f) = \frac{\int_I f(x)^2\,\mathrm{d}\,x}{2\int_I f(x)\,\mathrm{d}\,x}$ (Vegyük észre, hogy $y_s = \frac{V_f}{2\pi\int_I f(x)\,\mathrm{d}\,x}$ teljesül.)

a. $x_s = \frac{\int_0^b ax^2\,\mathrm{d}\,x}{\int_0^b ax\,\mathrm{d}\,x} = \frac{2b}{3}$ $\qquad y_s = \frac{ab}{3}$

b. $x_s = \frac{\int_0^b ax^3\,\mathrm{d}\,x}{\int_0^b ax^2\,\mathrm{d}\,x} = \frac{3b}{4}$ $\qquad y_s = \frac{3ab^2}{10}$

c. $x_s = \frac{\int_0^b x\,\mathrm{e}^{ax}\,\mathrm{d}\,x}{\int_0^b \mathrm{e}^{ax}\,\mathrm{d}\,x} = \frac{\mathrm{e}^{ab}(ab-1)+1}{a(\mathrm{e}^{ab}-1)}$ $\qquad y_s = \frac{1}{4}\cdot\frac{\mathrm{e}^{2ab}-1}{\mathrm{e}^{ab}-1}$

d. $x_s = \frac{\int_0^b x\,\mathrm{ch}\,x\,\mathrm{d}\,x}{\int_0^b \mathrm{ch}\,x\,\mathrm{d}\,x} = \frac{b\,\mathrm{sh}\,b - \mathrm{ch}\,b + 1}{\mathrm{sh}\,b}$ $\qquad y_s = \frac{\mathrm{sh}\,b\,\mathrm{ch}\,b + b}{4\,\mathrm{sh}\,b}$

6. Forgástest súlypontja. $x_s(V_f) = \frac{1}{V_f}\pi \int_I x f(x)^2 \, \mathrm{d}\, x$

$$\text{a.} \quad x_s = \frac{\pi}{V_f}\int_0^b a^2 x^3 \, \mathrm{d}\, x = \frac{3b}{4}$$

$$\text{b.} \quad x_s = \frac{\pi}{V_f}\int_0^b a^2 x^5 \, \mathrm{d}\, x = \frac{5b}{6}$$

$$\text{c.} \quad x_s = \frac{\pi}{V_f}\int_0^b x\, \mathrm{e}^{2ax} \, \mathrm{d}\, x = \frac{b}{1-\mathrm{e}^{-2ab}} - \frac{1}{2a}$$

$$\text{d.} \quad x_s = \frac{\pi}{V_f}\int_0^b x \operatorname{ch}^2 x \, \mathrm{d}\, x = \frac{b \operatorname{sh}(2b) + b^2 + 1 - \operatorname{ch}^2 b}{\operatorname{sh}(2b) + 2b}$$