

Kalkulus 1, 14. hét

Határozott integrál alkalmazásai

I. Az integrálformulákból közvetlenül adódnak a formulák.

II. 1. Ivhosszszámítás. $L_f = \int_I \sqrt{1 + f'(x)^2} \, dx$

$$\text{a. } L_f = \int_0^b \sqrt{1 + a^2} \, dx = \left[x \sqrt{1 + a^2} \right]_0^b = b \sqrt{1 + a^2}$$

$$\text{b. } L_f = \int_0^b \sqrt{1 + 4a^2 x^2} \, dx = \left[\frac{x \sqrt{1 + 4a^2 x^2}}{2} + \frac{\operatorname{arsh}(2ax)}{4a} \right]_0^b = \frac{b \sqrt{1 + 4a^2 b^2}}{2} + \frac{\operatorname{arsh}(2ab)}{4a}$$

$$\begin{aligned} \text{c. } L_f &= \int_0^b \sqrt{1 + a^2 e^{2ax}} \, dx = \left[\frac{\sqrt{1 + a^2 e^{2ax}}}{a} - \frac{1}{a} \cdot \operatorname{arth} \left(\frac{1}{\sqrt{1 + a^2 e^{2ax}}} \right) \right]_0^b = \\ &= \frac{\sqrt{1 + a^2 e^{2ab}} - \sqrt{1 + a^2}}{a} - \frac{1}{a} \cdot \operatorname{arth} \left(\frac{1}{\sqrt{1 + a^2 e^{2ab}}} \right) + \frac{1}{a} \cdot \operatorname{arth} \left(\frac{1}{\sqrt{1 + a^2}} \right) \end{aligned}$$

$$\text{d. } L_f = \int_0^b \operatorname{ch} x \, dx = [\operatorname{sh} x]_0^b = \operatorname{sh} b$$

2. Felszínszámítás. $F_f = 2\pi \int_I f(x) \sqrt{1 + f'(x)^2} \, dx$

$$\text{a. } F_f = 2\pi \int_0^b ax \sqrt{1 + a^2} \, dx = \pi \left[ax^2 \sqrt{1 + a^2} \right]_0^b = \pi ab^2 \sqrt{1 + a^2}$$

$$\begin{aligned} \text{b. } F_f &= 2\pi \int_0^b ax^2 \sqrt{1 + 4a^2 x^2} \, dx \stackrel{t = \operatorname{arsh}(2ax)}{=} \frac{\pi}{4a^2} \int_0^{\operatorname{arsh}(2ab)} \operatorname{sh}^2 t \operatorname{ch}^2 t \, dt = \frac{\pi}{16a^2} \int_0^{\operatorname{arsh}(2ab)} \operatorname{sh}^2(2t) \, dt = \\ &= \left[\frac{\pi}{32a^2} \left(\frac{\operatorname{sh}(4t)}{4} - t \right) \right]_0^{\operatorname{arsh}(2ab)} = \left[\frac{\pi}{32a^2} (\operatorname{sh} t \operatorname{ch} t (1 + 2 \operatorname{sh}^2 t) - t) \right]_0^{\operatorname{arsh}(2ab)} = \\ &= \frac{\pi b(1 + 8a^2 b^2) \sqrt{1 + 4a^2 b^2}}{16a} - \frac{\pi \operatorname{arsh}(2ab)}{32a^2} \end{aligned}$$

$$\begin{aligned} \text{c. } F_f &= 2\pi \int_0^b e^{ax} \sqrt{1 + a^2 e^{2ax}} \, dx \stackrel{t = a e^{ax}}{=} \frac{2\pi}{a^2} \int_1^{a e^{ab}} \sqrt{1 + t^2} \, dt = \\ &= \frac{\pi}{a^2} \left[\operatorname{arsh} t + t \sqrt{1 + t^2} \right]_1^{a e^{ab}} = \frac{\pi}{a^2} \left(\operatorname{arsh}(a e^{ab}) - \operatorname{arsh} 1 + a e^{ab} \sqrt{1 + a^2 e^{2ab}} - \sqrt{2} \right) \end{aligned}$$

$$\text{d. } F_f = 2\pi \int_0^b \operatorname{ch}^2 x \, dx = [\pi(x + \operatorname{sh} x \operatorname{ch} x)]_0^b = \pi b + \pi \operatorname{sh} b \operatorname{ch} b$$

3. Térfogatszámítás. $V_f = \pi \int_I f^2(x) \, dx$

$$\text{a. } V_f = \pi \int_0^b a^2 x^2 \, dx = \left[\frac{\pi a^2 x^3}{3} \right]_0^b = \frac{\pi a^2 b^3}{3}$$

$$\text{b. } V_f = \pi \int_0^b a^2 x^4 \, dx = \left[\frac{\pi a^2 x^5}{5} \right]_0^b = \frac{\pi a^2 b^5}{5}$$

$$\text{c. } V_f = \pi \int_0^b e^{2ax} \, dx = \left[\frac{\pi e^{2ax}}{2a} \right]_0^b = \frac{\pi}{2a} (e^{2ab} - 1)$$

$$\text{d. } V_f = \pi \int_0^b \operatorname{ch}^2(x) \, dx = \left[\frac{\pi}{2} \operatorname{sh} x \operatorname{ch} x + \frac{\pi x}{2} \right]_0^b = \frac{\pi}{4} \operatorname{sh}(2b) + \frac{\pi b}{2}$$

4. Görbe súlypontja. $x_s(L_f) = \frac{1}{L_f} \int_I x \sqrt{1 + f'(x)^2} \, dx$, $y_s(L_f) = \frac{1}{L_f} \int_I f(x) \sqrt{1 + f'(x)^2} \, dx$

(Vegyük észre, hogy $y_s = \frac{F_f}{2\pi L_f}$ teljesül.)

$$\text{a. } x_s = \frac{1}{L_f} \int_0^b x \sqrt{1 + a^2} \, dx = \frac{1}{L_f} \left[\frac{x^2 \sqrt{1 + a^2}}{2} \right]_0^b = \frac{b}{2}$$

$$y_s = \frac{a}{2}$$

$$\text{b. } x_s = \frac{1}{L_f} \int_0^b x \sqrt{1 + 4a^2 x^2} \, dx = \frac{1}{L_f} \left[\frac{(1 + 4a^2 x^2)^{3/2}}{12a^2} \right]_0^b = \frac{(1 + 4a^2 b^2)^{3/2} - 1}{12a^2 L_f}$$

$$y_s = \frac{2ab(1 + 8a^2 b^2) \sqrt{1 + 4a^2 b^2} - \operatorname{arsh}(2ab)}{32a^2 b \sqrt{1 + 4a^2 b^2} + 16a \operatorname{arsh}(2ab)}$$

$$\text{d. } x_s = \frac{1}{L_f} \int_0^b x \operatorname{ch} x \, dx = \frac{1}{L_f} [x \operatorname{sh} x - \operatorname{ch} x]_0^b = b - \frac{\operatorname{ch} b - 1}{\operatorname{sh} b}$$

$$y_s = \frac{b + \operatorname{sh} b \operatorname{ch} b}{2 \operatorname{sh} b}$$

5. Síkbeli alakzat súlypontja. $x_s(T_f) = \frac{\int_I x f(x) \, dx}{\int_I f(x) \, dx}$, $y_s(T_f) = \frac{\int_I f(x)^2 \, dx}{2 \int_I f(x) \, dx}$ (Vegyük észre, hogy

$y_s = \frac{V_f}{2\pi \int_I f(x) \, dx}$ teljesül.)

$$\text{a. } x_s = \frac{\int_0^b a x^2 \, dx}{\int_0^b a x \, dx} = \frac{2b}{3}$$

$$y_s = \frac{ab}{3}$$

$$\text{b. } x_s = \frac{\int_0^b a x^3 \, dx}{\int_0^b a x^2 \, dx} = \frac{3b}{4}$$

$$y_s = \frac{3ab^2}{10}$$

$$\text{c. } x_s = \frac{\int_0^b x e^{ax} \, dx}{\int_0^b e^{ax} \, dx} = \frac{e^{ab}(ab - 1) + 1}{a(e^{ab} - 1)}$$

$$y_s = \frac{1}{4} \cdot \frac{e^{2ab} - 1}{e^{ab} - 1}$$

$$\text{d. } x_s = \frac{\int_0^b x \operatorname{ch} x \, dx}{\int_0^b \operatorname{ch} x \, dx} = \frac{b \operatorname{sh} b - \operatorname{ch} b + 1}{\operatorname{sh} b}$$

$$y_s = \frac{\operatorname{sh} b \operatorname{ch} b + b}{4 \operatorname{sh} b}$$

6. Forgástest súlypontja. $x_s(V_f) = \frac{1}{V_f} \pi \int_I x f(x)^2 \, dx$

a. $x_s = \frac{\pi}{V_f} \int_0^b a^2 x^3 \, dx = \frac{3b}{4}$

b. $x_s = \frac{\pi}{V_f} \int_0^b a^2 x^5 \, dx = \frac{5b}{6}$

c. $x_s = \frac{\pi}{V_f} \int_0^b x e^{2ax} \, dx = \frac{b}{1 - e^{-2ab}} - \frac{1}{2a}$

d. $x_s = \frac{\pi}{V_f} \int_0^b x \operatorname{ch}^2 x \, dx = \frac{b \operatorname{sh}(2b) + b^2 + 1 - \operatorname{ch}^2 b}{\operatorname{sh}(2b) + 2b}$