

Kalkulus 1, 12. hét

Határozatlan integrál, megoldások

I. Számítsuk ki a következő integrálokat.

1. $\int \frac{1+x}{\sqrt{x}} \, dx = \int x^{-\frac{1}{2}} + x^{\frac{1}{2}} \, dx = 2\sqrt{x} + \frac{2}{3}x^{\frac{3}{2}} + C$
2. $\int \frac{1}{x^2} \, dx = -\frac{1}{x} + C$
3. $\int 1 + e^{x-1} \, dx = x + e^{x-1} + C$
4. $\int \frac{x^2-1}{x^2+1} \, dx = \int \frac{(x^2+1)-2}{x^2+1} \, dx = x - 2 \operatorname{arctg} x + C$
5. $\int \frac{1}{\sqrt{1+x^2}} \, dx = \operatorname{arsh} x + C$
6. $\int \frac{1}{\sqrt{x^2-1}} \, dx = \operatorname{arch} x + C$
7. $\int \frac{1}{\sin^2} = -\operatorname{ctg} + C$
8. $\int \frac{1}{\cos^2} = \operatorname{tg} + C$
9. $\int \frac{1}{\operatorname{sh}^2} = -\operatorname{cth} + C$
10. $\int \frac{1}{\operatorname{ch}^2} = \operatorname{th} + C$
11. $\int \operatorname{sh} = \operatorname{ch} + C$
12. $\int \operatorname{ch} = \operatorname{sh} + C$

II. A parciális integrálás segítségével határozzuk meg az alábbi integrálokat, ahol $a, b \in \mathbb{R} \setminus \{0\}$ és $k \in \mathbb{N}$.

1. $\int x e^{ax} \, dx = \frac{1}{a^2} e^{ax} (ax - 1) + C$
2. $\int x^2 e^{-ax} \, dx = -\frac{1}{a^3} e^{-ax} (a^2 x^2 + 2ax + 2) + C$
3. $\int x \sin x \, dx = \sin x - x \cos x + C$
4. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin(bx) - b \cos(bx)) + C$
5. $\int e^x \cos x \, dx = \frac{e^x}{2} (\sin x + \cos x) + C$
6. $\int \sqrt{1-x^2} \, dx = \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \arcsin x + C$

7. $\int \sqrt{1+x^2} \, dx = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2} \operatorname{arsh} x + C$
8. $\int \sqrt{x^2-1} \, dx = \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2} \operatorname{arch} x + C$
9. $\int \arcsin x \, dx = x \arcsin x + \sqrt{1-x^2} + C$
10. $\int \operatorname{arctg} x \, dx = x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + C$
11. $\int x \operatorname{arctg} ax \, dx = \frac{x^2}{2} \operatorname{arctg}(ax) + \frac{1}{2a^2} \operatorname{arctg}(ax) - \frac{x}{2a} + C$
12. $\int x^3 \ln^2 x \, dx = \frac{x^4}{4} \ln^2 x - \frac{x^4}{8} \ln x + \frac{x^4}{32} + C$

III. A következőkben a racionális törtfüggvényekre vonatkozó integrálási szabályt alkalmazzuk.

1. $\int \frac{1}{1-x^2} \, dx = \int \frac{1}{2} \cdot \frac{1}{x+1} - \frac{1}{2} \cdot \frac{1}{x-1} \, dx = \frac{1}{2} \ln|x+1| - \frac{1}{2} \ln|x-1| + C$
2. $\int \frac{1}{x^2-2x-3} \, dx = \int \frac{1}{4} \cdot \frac{1}{x-3} - \frac{1}{4} \cdot \frac{1}{x+1} \, dx = \frac{1}{4} \ln|x-3| - \frac{1}{4} \ln|x+1| + C$
3. $\int \frac{1}{x^2+2x+6} \, dx = \frac{1}{\sqrt{5}} \operatorname{arctg}\left(\frac{x+1}{\sqrt{5}}\right) + C$
4. $\int \frac{x^2-1}{(x+2)^3} \, dx = \int \frac{1}{x+2} - \frac{4}{(x+2)^2} + \frac{3}{(x+2)^3} \, dx =$
 $= \ln|x+2| + \frac{4}{x+2} - \frac{3}{2(x+2)^2} + C$
5. $\int \frac{1}{x^3+1} \, dx = \int \frac{1}{3} \cdot \frac{1}{x+1} - \frac{1}{3} \cdot \frac{x-2}{x^2-x+1} \, dx =$
 $= \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln|x^2-x+1| + \frac{1}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right) + C$
6. $\int \frac{x^4}{(x-2)(x-3)(x-4)} \, dx = \int x+9 + \frac{8}{x-2} - \frac{81}{x-3} + \frac{128}{x-4} \, dx =$
 $= \frac{x^2}{2} + 9x + 8 \ln|x-2| - 81 \ln|x-3| + 128 \ln|x-4| + C$
7. $\int \frac{16x^2+4x}{x^4+4} \, dx = \int \frac{4x+1}{x^2-2x+2} - \frac{4x+1}{x^2+2x+2} \, dx =$
 $= 2 \ln|x^2-2x+2| + 5 \operatorname{arctg}(x-1) - 2 \ln|x^2+2x+2| + 3 \operatorname{arctg}(x+1) + C$
8. $\int \frac{x^3}{(x^2+1)^2} \, dx = \int \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2} \, dx = \frac{1}{2} \ln(x^2+1) + \frac{1}{2(x^2+1)} + C$
9. $\int \frac{x^4+4}{x^3-1} \, dx = \int x + \frac{5}{3} \cdot \frac{1}{x-1} - \frac{5x+7}{3(x^2+x+1)} \, dx =$
 $= \frac{x^2}{2} + \frac{5}{3} \ln|x-1| - \frac{5}{6} \ln|x^2+x+1| - \sqrt{3} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$
10. $\int \frac{1}{(x^2+x+1)^2} \, dx = \frac{2x+1}{3(x^2+x+1)} + \frac{4}{3\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

$$\begin{aligned}
11. \quad & \int \frac{x}{(x^2 + 2x + 2)^2} dx = -\frac{x+2}{2(x^2 + 2x + 2)} - \frac{1}{2} \operatorname{arctg}(x+1) + C \\
12. \quad & \int \frac{1}{(x^2 + 2x + 2)^2} dx = \frac{x+1}{2(x^2 + 2x + 2)} + \frac{1}{2} \operatorname{arctg}(x+1) + C
\end{aligned}$$

IV. A következő integrálok kiszámításához alkalmazzunk megfelelő helyettesítést. ($k \in \mathbb{N}$)

$$\begin{aligned}
1. \quad t = x + 2 \quad & \int \frac{x^3}{(x+2)^4} dx = \int \frac{(t-2)^3}{t^4} dt = \ln|x+2| + \frac{6}{x+2} - \frac{6}{(x+2)^2} + \frac{8}{3(x+2)^3} + C \\
2. \quad t = \sqrt{1+x} \quad & \int \frac{1}{\sqrt{1+x} + (\sqrt{1+x})^3} dx = \int \frac{2}{1+t^2} dt = 2 \operatorname{arctg} \sqrt{1+x} + C \\
3. \quad t = \sqrt[4]{x-1} \quad & \int x \sqrt[4]{x-1} dx = \int 4t^4 + 4t^8 dt = \frac{4}{5}(x-1)^{\frac{5}{4}} + \frac{4}{9}(x-1)^{\frac{9}{4}} + C \\
4. \quad t = e^x \quad & \int \frac{e^{4x}}{1+e^x} dx = \int \frac{t^3}{1+t} dt = \frac{e^{3x}}{3} - \frac{e^{2x}}{2} + e^x - \ln(1+e^x) + C \\
5. \quad t = \sqrt{e^x-1} \quad & \int \sqrt{e^x-1} dx = \int \frac{2t^2}{t^2+1} dt = 2\sqrt{e^x-1} - 2 \operatorname{arctg}(\sqrt{e^x-1}) + C \\
6. \quad t = \sqrt{x} \quad & \int \sqrt{x} e^{\sqrt{x}} dx = \int 2t^2 e^t dt = 2e^{\sqrt{x}}(x - 2\sqrt{x} + 2) + C \\
7. \quad t = \arcsin x \quad & \int \sqrt{1-x^2} dx = \int \cos^2 t dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2} \arcsin x + C \\
8. \quad t = \sin x \quad & \int \sin^k x \cdot \cos x dx = \int t^k dt = \frac{\sin^{k+1} x}{k+1} + C \\
9. \quad t = \cos x \quad & \int \cos^k x \cdot \sin x dx = \int -t^k dt = -\frac{\cos^{k+1} x}{k+1} + C \\
10. \quad t = \sin x \quad & \int \operatorname{ctg} x dx = \int \frac{1}{t} dt = \ln|\sin x| + C \\
11. \quad t = \cos x \quad & \int \operatorname{tg} x dx = \int -\frac{1}{t} dt = -\ln|\cos x| + C \\
12. \quad t = \operatorname{tg} x \quad & \int \operatorname{tg}^2 x dx = \int \frac{t^2}{1+t^2} dt = \operatorname{tg} x - x + C \\
13. \quad t = \operatorname{tg} x \quad & \int \operatorname{tg}^4 x dx = \int \frac{t^4}{1+t^2} dt = \frac{\operatorname{tg}^3 x}{3} - \operatorname{tg} x + x + C \\
14. \quad t = e^x \quad & \int \frac{2}{e^{3x}-e^x} dx = \int \frac{2}{t^4-t^2} dt = \ln|e^x-1| - \ln|e^x+1| + 2e^{-x} + C \\
15. \quad t = \sqrt{x-1} \quad & \int \frac{1}{x+\sqrt{x-1}-1} dx = \int \frac{2}{t+1} dt = 2\ln(\sqrt{x-1}+1) + C
\end{aligned}$$

V. A következő integrálok kiszámításához gondoljunk az elemi függvények ($\sin$, tg , $\exp$, arctg , $\arcsin$,...) deriváltjára és a deriválásnál megismert láncszabályra.

$$\begin{aligned}
1. \quad & \int x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} + C \\
2. \quad & \int \frac{3^x}{1+9^x} dx = \frac{1}{\ln 3} \operatorname{arctg} 3^x + C
\end{aligned}$$

3. $\int \frac{e^x}{\sqrt{1-e^{2x}}} dx = \arcsin e^x + C$
4. $\int \frac{e^x}{\sqrt[3]{1+e^x}} dx = \frac{3}{2}(1+e^x)^{\frac{2}{3}} + C$
5. $\int \frac{\cos \ln x}{x} dx = \sin \ln |x| + C$
6. $\int \frac{\cos x}{1+\sin^2 x} dx = \arctg \sin x + C$

VI. Trigonometrikus azonosságok segítségével számoljuk ki az alábbi határozatlan integrálokat.

1. $\int \cos 2x \cos 5x dx = \int \frac{1}{2} \cos(3x) + \frac{1}{2} \cos(7x) dx = \frac{1}{6} \sin(3x) + \frac{1}{14} \sin(7x) + C$
2. $\int \cos^5 x \sin^2 x dx = \int \cos x (\sin^2 x - 2 \sin^4 x + \sin^6 x) dx = \frac{1}{3} \sin^3 x - \frac{2}{5} \sin^5 x + \frac{1}{7} \sin^7 x + C$
3. $\int \cos^5 x dx = \int \cos x (1 - 2 \sin^2 x + \sin^4 x) dx = \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C$
4. $\int \cos^5 x \sin^3 x dx = \int \cos x (\sin^3 x - 2 \sin^5 x + \sin^7 x) dx = \frac{1}{4} \sin^4 x - \frac{1}{3} \sin^6 x + \frac{1}{8} \sin^8 x + C$
5. $\int \sin^4 x dx = \int \frac{3}{8} - \frac{1}{2} \cos(2x) + \frac{1}{8} \cos(4x) dx = \frac{3}{8} x - \frac{1}{4} \cos(2x) + \frac{1}{32} \sin(4x) + C$
6. $\int \cos^5 x \sin^5 x dx = \int \cos x (\sin^5 x - 2 \sin^7 x + \sin^9 x) dx = \frac{1}{6} \sin^6 x - \frac{1}{4} \sin^8 x + \frac{1}{10} \sin^{10} x + C$